Fixed Point Theorems for $\alpha$-$\psi$-Contractive Type Mappings in Metric Spaces

Seong-Hoon Cho

Department of Mathematics
Hanseo University, Seosan
Chungnam, 356-706, South Korea

Abstract

Some new fixed point theorems for $\alpha$-$\psi$-contractive type mappings are established, and some examples are given.

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1 Introduction and preliminaries

Recently, the authors [5] introduced the notion of $\alpha$-$\psi$ contractive mapping and gave some fixed point theorems for such mappings. The author [2] introduced the notion of weakly $\alpha$-contractive mappings and obtained a generalization of the result of [5].

In [3], the authors extended the result of [5] to the case of multifunctions. Also, they generalized the result of [5] as follows:

Theorem 1.1. Let $(X, d)$ be a complete metric space, and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$. Suppose that a mapping $T : X \to X$ satisfies the following conditions:
(1) \( \alpha(x, y)d(Tx, Ty) \leq \psi(m(x, y)) \) for all \( x, y \in X \),
where \( m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\} \);

(2) for each \( x, y \in X \), \( \alpha(x, y) \geq 1 \) implies \( \alpha(Tx, Ty) \geq 1 \);

(3) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \);

(4) if \( \{x_n\} \) is a sequence with \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and
\[ \lim_{n \to \infty} x_n = x \in X, \] then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \).

Then \( T \) has a fixed point in \( X \).

In [4], the authors proved the following theorem.

**Theorem 1.2.** Let \((X, d)\) be a complete metric space, and let \( \alpha : X \times X \to [0, \infty) \) be a function. Let \( \psi : [0, \infty) \to [0, \infty) \) be a continuous and nondecreasing function such that \( \psi(t) < t \) for each \( t > 0 \). Suppose that \( T : X \to X \) is a mapping such that
\[
\alpha(x, y)d(Tx, Ty) \leq \psi(m(x, y))
\]
for all \( x, y \in X \). Assume that there exists \( x_0 \in X \) such that \( \alpha(T^ix_0, T^jx_0) \geq 1 \) for all \( i, j \in \mathbb{N} \cup \{0\} \) with \( i \neq j \). Suppose that either \( T \) is continuous or \( \alpha(T^ix_0, x) \geq 1 \) for all \( i \in \mathbb{N} \cup \{0\} \) whenever \( \lim_{n \to \infty} T^ix_0 = x \).

Then \( T \) has a fixed point in \( X \).

The purpose of the paper is to generalize the above theorem and results in the literature. We give some examples to support the main result.

2 Fixed point theorems

Let \( \Psi \) be the family of all nondecreasing function \( \psi : [0, \infty) \to [0, \infty) \) such that
\[
\lim_{n \to \infty} \psi^n(t) = 0
\]
for all \( t > 0 \).

**Lemma 2.1.** If \( \psi \in \Psi \), then the following are satisfied.

(a) \( \psi(t) < t \) for all \( t > 0 \);

(b) \( \psi(0) = 0 \);

(c) \( \psi \) is right continuous at \( t = 0 \).
Remark 2.1. (1) If \( \psi : [0, \infty) \to [0, \infty) \) is nondecreasing such that \( \sum_{n=1}^{\infty} \psi^n(t) < \infty \) for each \( t > 0 \), then \( \psi \in \Psi \).

(2) If \( \psi : [0, \infty) \to [0, \infty) \) is upper semicontinuous such that \( \psi(t) < t \) for all \( t > 0 \), then \( \lim_{n \to \infty} \psi^n(t) = 0 \) for all \( t > 0 \).

Let \((X, d)\) be a metric space. We denotes by \( \Lambda \) the family of all functions \( \alpha : X \times X \to [0, \infty) \).

**Theorem 2.2.** Let \((X, d)\) be a complete metric space, \( \alpha \in \Lambda \), and let \( \psi \in \Psi \).

Suppose that a mapping \( T : X \to X \) satisfies

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(m(x, y))
\]

for all \( x, y \in X \). Assume that there exists \( x_0 \in X \) such that

\[
\alpha(T^i x_0, T^j x_0) \geq 1
\]

for all \( i, j \in \mathbb{N} \cup \{0\} \) with \( i \neq j \). Suppose that either \( T \) is continuous or

\[
\limsup \alpha(T^n x_0, x) \geq 1
\]

for any cluster point \( x \) of \( \{T^n x_0\} \).

Then \( T \) has a fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) such that \( \alpha(T^i x_0, T^j x_0) \geq 1 \) for all \( i, j \in \mathbb{N} \cup \{0\} \) with \( i \neq j \).

Define a sequence \( \{x_n\} \subset X \) by \( x_{n+1} = Tx_n \) for all \( n \in \mathbb{N} \cup \{0\} \).

If \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \cup \{0\} \), then \( T \) has a fixed point.

Assume that \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N} \cup \{0\} \). By assumption, \( \alpha(x_i, x_j) \geq 1 \) for all \( i, j \in \mathbb{N} \) with \( i \neq j \).

From (2.1) with \( x = x_{n-1} \) and \( y = x_n \) we obtain

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\
\leq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \\
\leq \psi(m(x_{n-1}, x_n)),
\]

where \( m(x_{n-1}, x_n) = \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), \frac{1}{2}[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)]\} \).

We have

\[
m(x_{n-1}, x_n) = \max\{d((x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\} \\
\leq \max\{d((x_n, x_{n-1}), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} \\
= \max\{d((x_{n-1}, x_n), d(x_n, x_{n+1})\}.
\]
Hence, from (2.5) we have $d(x_n, x_{n+1}) \leq \psi(\max\{d((x_{n-1}, x_n), d(x_n, x_{n+1}))\})$, because $\psi$ is nondecreasing.

If $\max\{d((x_{n-1}, x_n), d(x_n, x_{n+1}))\} = d(x_n, x_{n+1})$, then $d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$, which is a contradiction.

Thus, $\max\{d((x_{n-1}, x_n), d(x_n, x_{n+1}))\} = d(x_{n-1}, x_n)$, and so

$$d(x_{n-1}, x_n) \leq \psi(d(x_{n-1}, x_n))$$

(2.6)

for all $n \in \mathbb{N}$. Hence we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1))$$

for all $n \in \mathbb{N}$. Thus we have

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

We now show that $\{x_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \epsilon - \psi(\epsilon)$$

(2.7)

for all $n > N$.

We claim

$$d(x_{n+1}, x_{n+k}) \leq \psi(\epsilon)$$

for all $n > N$ and $k = 2, 3, \ldots$

From (2.6) and (2.7) we obtain

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_{n+1}, x_n)) < \psi(\epsilon - \psi(\epsilon)) < \psi(\epsilon)$$

for all $n > N$.

Note that $d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) < \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon$

for all $n > N$, and $d(x_{n+2}, x_{n+3}) \leq \psi(d(x_{n+1}, x_{n+2})) < \psi^2(\epsilon)$ for all $n > N$.

From (2.1) we obtain

$$d(x_{n+1}, x_{n+3}) = d(Tx_n, Tx_{n+2}) \leq \alpha(x_n, x_{n+2})d(Tx_n, Tx_{n+2}) \leq \psi(m(x_n, x_{n+2}))$$

for all $n > N$, where $m(x_n, x_{n+2}) = \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3}), \frac{1}{2}[d(x_n, x_{n+3}) + d(x_{n+2}, x_{n+1})]\}$.

Hence, $d(x_{n+1}, x_{n+3}) \leq \psi(\max\{\epsilon, \epsilon - \psi(\epsilon), \psi^2(\epsilon), \frac{1}{2}[\psi(\epsilon) + d(x_{n+1}, x_{n+3})]\})$. 
Suppose that \( \max\{\epsilon, \epsilon - \psi'(\epsilon), \psi^2(\epsilon), \frac{1}{2}[\epsilon + d(x_{n+1}, x_{n+3})]\} = \frac{1}{2}[\epsilon + d(x_{n+1}, x_{n+3})] \), then \( \frac{1}{2}[\epsilon + d(x_{n+1}, x_{n+3})] > 0 \) and \( d(x_{n+1}, x_{n+3}) < \frac{1}{2}[\epsilon + d(x_{n+1}, x_{n+3})] \). Thus, we have \( d(x_{n+1}, x_{n+3}) < \epsilon \), and so \( \max\{\epsilon, \epsilon - \psi'(\epsilon), \psi^2(\epsilon), \frac{1}{2}[\epsilon + d(x_{n+1}, x_{n+3})]\} = \frac{1}{2}[\epsilon + d(x_{n+1}, x_{n+3})] < \frac{1}{2}[\epsilon + \epsilon] = \epsilon \), which is a contradiction.

Thus, \( \max\{\epsilon, \epsilon - \psi'(\epsilon), \psi^2(\epsilon), \frac{1}{2}[\psi(\epsilon) + d(x_{n+1}, x_{n+3})]\} = \epsilon \), and hence we obtain
\[
d(x_{n+1}, x_{n+3}) \leq \psi(\epsilon)
\]
for all \( n > N \).

Note that \( d(x_n, x_{n+3}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+3}) < \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon \).

With the way as in the above, we obtain
\[
d(x_{n+1}, x_{n+4}) \leq \psi(\epsilon)
\]
for all \( n > N \).

By induction,
\[
d(x_{n+1}, x_{n+k}) \leq \psi(\epsilon)
\]
for all \( n > N \) and \( k = 2, 3, \ldots \), and above claim is proved.

From (2.1) we have
\[
d(x_n, x_m) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \\
= d(x_n, x_{n+1}) + \alpha(x_n, x_{m-1})d(Tx_n, Tx_{m-1}) \\
\leq \epsilon - \psi(\epsilon) + \psi(\max\{d(x_n, x_{m-1}), d(x_n, x_{n+1}), d(x_{m-1}, x_m), \frac{1}{2}[d(x_{m-1}, x_{n+1}) + d(x_m, x_n)]\}) \\
\leq \epsilon - \psi(\epsilon) + \psi(\max\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{m-1}), d(x_n, x_{n+1}), d(x_{m-1}, x_m), \frac{1}{2}[d(x_{m-1}, x_{n+1}) + d(x_m, x_n)]\}) \\
\leq \epsilon - \psi(\epsilon) + \psi(\max\{\epsilon, \epsilon - \psi(\epsilon), \frac{1}{2}[\psi(\epsilon) + d(x_m, x_n)]\})
\]
for all \( m, n > N \).

Suppose that \( \max\{\epsilon, \epsilon - \psi'(\epsilon), \frac{1}{2}[\psi(\epsilon) + d(x_m, x_n)]\} = \frac{1}{2}[\psi(\epsilon) + d(x_m, x_n)] \).

Then, \( d(x_m, x_n) < \frac{1}{2}[\psi(\epsilon) + d(x_m, x_n)] \), and so \( d(x_m, x_n) < \psi(\epsilon) \). Thus, \( \max\{\epsilon, \epsilon - \psi'(\epsilon), \frac{1}{2}[\psi(\epsilon) + d(x_m, x_n)]\} < \psi(\epsilon) < \epsilon \), which is a contradiction.

Hence, \( \max\{\epsilon, \epsilon - \psi'(\epsilon), \frac{1}{2}[\psi(\epsilon) + d(x_m, x_n)]\} = \epsilon \), and hence
\[
d(x_n, x_m) \leq \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon
\]
for all \( m, n > N \).

Thus, \( \{x_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, there exists \( x_* \in X \) such that \( x_* = \lim_{n \to \infty} x_n \).
If \( T \) is continuous, then \( \lim_{n \to \infty} x_n = Tx_\ast \). So \( x_\ast \) is a fixed point.

Assume that \( \limsup \alpha(T^n x_0, x) \geq 1 \) for any cluster point \( x \) of \( \{T^n x_0\} \).

Then, \( \limsup \alpha(x_n, x_\ast) \geq 1 \). Hence, there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} \alpha(x_{n(k)}, x_\ast) \geq 1 \). Thus, we have

\[
d(x_\ast, Tx_\ast) = \lim_{k \to \infty} d(x_{n(k)+1}, Tx_\ast)
\leq \lim_{k \to \infty} \alpha(x_{n(k)}, x_\ast)d(Tx_{n(k)}, Tx_\ast)
\leq \lim_{k \to \infty} \psi(m(x_{n(k)}, x_\ast)) \tag{2.8}
\]

where \( m(x_{n(k)}, x_\ast) = \max\{d(x_{n(k)}, x_\ast), d(x_{n(k)}, x_{n(k)+1}), d(x_\ast, Tx_\ast), \frac{1}{2}\{d(x_{n(k)}, Tx_\ast) + d(x_\ast, x_{n(k)+1})\}\} \).

Suppose that \( d(x_\ast, Tx_\ast) = a > 0 \).

Since \( \lim_{n \to \infty} x_{n(k)} = x_\ast \), there exists \( N_1 \in \mathbb{N} \) such that

\[
d(x_\ast, x_{n(k)}) < \frac{a}{2}
\]

for all \( n \geq N_1 \).

Then we have

\[
d(x_{n(k)}, x_{n(k)+1}) \leq d(x_\ast, x_{n(k)}) + d(x_\ast, x_{n(k)+1}) = a
\]

and

\[
\frac{1}{2}\{d(x_{n(k)}, Tx_\ast) + d(x_\ast, x_{n(k)+1})\} \\
\leq \frac{1}{2}\left(\frac{a}{2} + d(x_{n(k)}, x_\ast) + d(x_\ast, Tx_\ast)\right) \\
< \frac{1}{2}\left(\frac{a}{2} + \frac{a}{2} + a\right) = a.
\]

Thus, we obtain

\[
m(x_{n(k)}, x_\ast) = d(x_\ast, Tx_\ast)
\]

for all \( n \geq N_1 \), and so

\[
\lim \psi(m(x_{n(k)}, x_\ast)) = \psi(d(x_\ast, Tx_\ast))
\]

Hence, from (2.8) we have

\[
d(x_\ast, Tx_\ast) \leq \lim \psi(m(x_{n(k)}, x_\ast)) = \psi(d(x_\ast, Tx_\ast)) < d(x_\ast, Tx_\ast)
\]

which is a contradiction. Thus, \( d(x_\ast, Tx_\ast) = 0 \), and so \( x_\ast \) is a fixed point of \( T \). \( \Box \)
Example 2.1. Let \( X = [0, \infty) \) and \( d(x, y) = |x - y| \) for all \( x, y \in X \), and let
\[
\psi(t) = \begin{cases} 
\frac{1}{4}t, & \text{if } t \in [0, 1], \\
\frac{1}{1+t}, & \text{if } t > 1.
\end{cases}
\]

Then, \( \psi \in \Psi \).

Note that \( \psi \) is not continuous at \( t = 1 \), and \( \sum_{n=1}^{\infty} \psi^n(1) = \sum_{n=1}^{\infty} \frac{1}{1+n} = \infty \).

Define a mapping \( T : X \to X \) by
\[
Tx = \begin{cases} 
\frac{1}{4}x, & \text{if } x \in [0, 1], \\
2x - \frac{7}{4}, & \text{if } x > 1.
\end{cases}
\]

Obviously, \( T \) is continuous.

We define \( \alpha : X \times X \to [0, \infty) \) by
\[
\alpha(x, y) = \begin{cases} 
1, & \text{if } x, y \in [0, 1], \\
0, & \text{if } x \not\in [0, 1] \text{ or } y \not\in [0, 1].
\end{cases}
\]

Clearly, (2.1) of Theorem 2.1 is satisfied. Condition (2.2) of Theorem 2.1 holds with \( x_0 = 1 \).

By applying Theorem 2.1, \( T \) has a fixed point.

Consider the general contractive condition:
\[
(G) \quad d(fx, fy) \leq \psi(\max\{d(x, y), d(x, fx), d(fx, fy), \frac{1}{2}(d(x, fy) + d(y, fx))\})
\]
for all \( x, y \in X \).

It is well known that if \( f \) is a self mapping of a complete metric space and \( f \) satisfies the general contractive condition \((G)\), then \( f \) has a fixed point.

Note that \( T \) do not satisfy condition \((G)\). In fact, \( \psi(m(1, 2)) = \psi(\frac{7}{4}) < d(T1, T2) \).

Example 2.2. In Example 2.1, let
\[
Tx = \begin{cases} 
\frac{1}{4}x, & \text{if } x \in [0, 1], \\
2x - \frac{7}{4}, & \text{if } x > 1.
\end{cases}
\]

Then it is easy to see that condition (2.1) of Theorem 2.1 holds. Obviously, condition (2.2) of Theorem 2.1 is satisfied with \( x_0 = 1 \).

For all \( n \in \mathbb{N} \cup \{0\} \), \( T^n x_0 = T^n 1 = \frac{1}{5^n} \in [0, 1] \). Hence, (2.4) of Theorem 2.1 is satisfied.

By applying Theorem 2.1, \( T \) has a fixed point.
Corollary 2.3. Let $(X, d)$ be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies

$$\alpha(x, y)d(Tx, Ty) \leq \psi(\max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\})$$

for all $x, y \in X$. Assume that condition (2.2) is satisfied. If either (2.3) or (2.4) hold, then $T$ has a fixed point in $X$.

Corollary 2.4. Let $(X, d)$ be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$. Assume that condition (2.2) is satisfied. If either (2.3) or (2.4) hold, then $T$ has a fixed point in $X$.

Remark 2.2. If we replace condition (2.2) of Theorem 2.1 with $\alpha(T^ix_0, T^jx_0) \geq 1$ for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$, then the conclusion of Theorem 2.1 is still satisfied.

Remark 2.3. Let $(X, d)$ be a metric space, and let $\alpha \in \Lambda$.
Consider the following condition:

(1) for each $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$;

(2) for each $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;

(3) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(4) if $\{x_n\}$ is a sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n \to \infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;

(5) there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$;

(6) $\limsup \alpha(T^n x_0, x) \geq 1$ for all cluster point $x$ of $\{T^n x_0\}$.

Then conditions (1), (2) and (3) imply (5), and condition (4) implies (6).

Remark 2.4. If we replace condition (2.2) of Theorem 2.1 (resp. Corollary 2.2, Corollary 2.3) with above conditions (1), (2) and (3) and replace condition (2.4) of Theorem 2.1 (resp. Corollary 2.2, Corollary 2.3) with above condition (4), then $T$ has a fixed point.
Corollary 2.5. Let $(X,d)$ be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$$

for all $x,y \in X$. Suppose that conditions (1) – (3) of Remark 2.2 are satisfied. Assume that either $T$ is continuous or if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$ then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.

Then $T$ has a fixed point in $X$. Further if, for all $x,y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$, then $T$ has a unique fixed point.

Proof. By Remark 2.4, $T$ has fixed point.

Let $y_* \in X$ be another fixed point of $T$.

Then by assumption, there exists $z \in X$ such that $\alpha(x_*,z) \geq 1$ and $\alpha(y_*,z) \geq 1$. From (2) we obtain $\alpha(x_*,T^nz) \geq 1$ and $\alpha(y_*,T^nz) \geq 1$ for all $n \in \mathbb{N}$.

Hence we have

$$d(x_*,T^nz)$$
$$\leq \alpha(x_*,T^{n-1}z)d(Tx_*,TT^{n-1}z)$$
$$\leq \psi(d(x_*,T^{n-1}z)).$$

for all $n \in \mathbb{N}$.

This implies that $d(x_*,T^nz) \leq \psi^n(d(x_*,z))$ for all $n \in \mathbb{N}$. Thus, we have $\lim_{n \to \infty} T^nz = x_*$. Similarly, we can show that $\lim_{n \to \infty} T^nz = y_*$. Thus, we have $x_* = y_*$. \(\square\)

Remark 2.5. By Remark 2.1, Corollary 2.4 is a generalization of Theorem 2.1, Theorem 2.2 and Theorem 2.3 in [5].

By Theorem 2.1, we have the following corollary.

Corollary 2.6. Let $(X, \preceq, d)$ be a complete ordered metric space, and let $\psi \in \Psi$.

Suppose that a mapping $T : X \to X$ satisfies

$$d(Tx,Ty) \leq \psi(m(x,y))$$

for all comparable elements $x,y \in X$. Assume that there exists $x_0 \in X$ such that $T^ix_0$ and $T^jx_0$ are comparable for all $i,j \in \mathbb{N} \cup \{0\}$ with $i \neq j$.

If either $[T$ is continuous$]$ or $[T^nx_0$ and $x$ are comparable for all $n \in \mathbb{N}$ whenever $\lim_{n \to \infty} T^nx_0 = x$], then $T$ has a fixed point in $X$.\)
Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
1, (x \text{ and } y \text{ are comparable}), \\
0, \text{ (otherwise)}. 
\end{cases}$$

By using Theorem 2.1, $T$ has a fixed point in $X$. \hfill $\Box$

**Corollary 2.7.** [1] Let $(X, \preceq, d)$ be a complete ordered metric space, and let $\psi \in \Psi$. Suppose that a nondecreasing mapping $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \psi(m(x, y))$$

for all $x, y \in X$ with $x \preceq y$. Assume that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Suppose that either $[T$ is continuous$]$ or $\{x_n\}$ is a sequence in $X$ such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then $T$ has a fixed point in $X$.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
1, (x \preceq y), \\
0, \text{ (otherwise)}. 
\end{cases}$$

By using Remark 2.3, $T$ has a fixed point in $X$. \hfill $\Box$

**Remark 2.6.** By remark 2.1, Corollary 2.6 is a generalization of Corollary 2.4 in [3].

**Theorem 2.8.** Let $(X, d)$ be a complete metric space, $\alpha \in \Lambda$, and let $\psi : [0, \infty) \to [0, \infty)$ be upper semicontinuous with $\psi(t) < t$ for all $t > 0$. Suppose that a mapping $T : X \to X$ satisfies

$$\alpha(x, y)d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty)\})$$

for all $x, y \in X$. Assume that (2.2) is satisfied. If either $T$ is continuous or (2.4) holds, then $T$ has a fixed point in $X$.

Proof. By (2.2), there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \in \mathbb{N} \cup \{0\}$ with $i \neq j$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $T$ has a fixed point.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

By assumption, $\alpha(x_i, x_j) \geq 1$ for all $i, j \in \mathbb{N}$ with $i \neq j$. 

Thus, we have
\[
\begin{align*}
    d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
    &\leq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \\
    &\leq \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\
    &= \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})
    \end{align*}
\]
for all \( n \in \mathbb{N} \).

Suppose that \( d(x_{n-1}, x_n) \leq d(x_n, x_{n+1}) \).
Then we have \( d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n+1})) \), which is a contradiction. Thus we obtain
\[
    d(x_n, x_{n+1}) \leq \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) = \psi(d(x_{n-1}, x_n))
\]
for all \( n \in \mathbb{N} \).

By induction, we obtain \( d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \) for all \( n \in \mathbb{N} \). Thus we have
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.9}
\]

We now show that \( \{x_n\} \) is a Cauchy sequence.

Suppose that \( \{x_n\} \) is not a Cauchy sequence.
Then there exists \( \epsilon > 0 \) such that, for all \( k > 0 \), there exist \( m(k) > n(k) > k \) with
\[
    d(x_{m(k)}, x_{n(k)}) \geq \epsilon \quad \text{and} \quad d(x_{m(k)-1}, x_{n(k)}) < \epsilon. \tag{2.10}
\]
Then we have
\[
\begin{align*}
    \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
    &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\
    &< d(x_{m(k)}, x_{m(k)-1}) + \epsilon.
    \end{align*}
\]
Letting \( k \to \infty \) in above inequality, we have
\[
\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.11}
\]

By using (2.9), (2.10) and (2.11), we obtain \( \lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon \) and \( \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon \).

Note that \( m(k)-1 > n(k) \). In fact, if \( m(k)-1 = n(k) \) then \( d(x_{m(k)}, x_{n(k)}) = d(x_{n(k)+1}, x_{n(k)}) < \epsilon \), which is a contradiction.
Hence, \( \alpha(x_{n(k)}, x_{m(k)}) \geq 1 \); and so we have
\[
d(x_{m(k)}, x_{n(k)+1}) = d(Tx_{m(k)-1}, Tx_{n(k)}) \\
\leq \alpha(x_{n(k)}, x_{m(k)-1})d(Tx_{m(k)-1}, Tx_{n(k)}) \\
\leq \psi(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1})\}).
\]

Letting \( k \to \infty \) in above inequality and using upper semicontinuity of \( \psi \), we have \( \epsilon \leq \psi(\epsilon) \), which is a contradiction. Thus \( \{x_n\} \) is a Cauchy sequence.

It follows from the completeness of \( X \) that there exists
\[
x_* = \lim_{n \to \infty} x_n \in X.
\]

If \( T \) is continuous, then \( \lim_{n \to \infty} x_n = Tx_* \), and so \( x_* = Tx_* \).

Assume that (2.4) is satisfied.

Then, \( \lim \sup \alpha(x_n, x_n) \geq 1 \). Hence, there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} \alpha(x_{n(k)}, x_*) = 1 \).

We have \( \lim_{k \to \infty} \max\{d(x_{n(k)}, x_*), d(x_{n(k)}, x_{n(k)+1}), d(x_*, Tx_*)\} = d(x_*, Tx_*) \).

By using upper semicontinuity of \( \psi \), we obtain
\[
d(x_*, Tx_*) = \lim_{k \to \infty} d(x_{n(k)+1}, Tx_*) \\
\leq \lim_{k \to \infty} \alpha(x_{n(k)}, x_*)d(Tx_{n(k)}, Tx_*) \\
\leq \lim \sup \psi(\max\{d(x_{n(k)}, x_*), d(x_{n(k)}, x_{n(k)+1}), d(x_*, Tx_*)\}) \\
\leq \psi(d(x_*, Tx_*)).
\]

If \( d(x_*, Tx_*) > 0 \), then \( d(x_*, Tx_*) \leq \psi(d(x_*, Tx_*)) < d(x_*, Tx_*) \), which is a contradiction. Hence \( d(x_*, Tx_*) = 0 \), and hence \( x_* \) is a fixed point of \( T \).

\[ \square \]

**Remark 2.7.** If we replace condition (2.2) of Theorem 2.7 with conditions (1), (2) and (3) of Remark 2.2 and replace condition (2.4) of Theorem 2.7 with (4) of Remark 2.2, then \( T \) has a fixed point.

**Corollary 2.9.** Let \( (X, \preceq, d) \) be a complete ordered metric space, and let \( \psi : [0, \infty) \to [0, \infty) \) be upper semicontinuous with \( \psi(t) < t \) for all \( t > 0 \).

Suppose that a mapping \( T : X \to X \) satisfies
\[
d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty)\})
\]
for all comparable elements \( x, y \in X \). Assume that there exists \( x_0 \in X \) such that \( T^ix_0 \) and \( T^jx_0 \) are comparable for all \( i, j \in \mathbb{N} \cup \{0\} \) with \( i \neq j \).

If either \( [T \text{ is continuous}] \) or \( [T^nx_0 \text{ and } x \text{ are comparable for all } n \in \mathbb{N} \text{ whenever } \lim_{n \to \infty} T^nx_0 = x] \), then \( T \) has a fixed point in \( X \).
Corollary 2.10. [1] Let \((X, \preceq, d)\) be a complete ordered metric space, and let \(\psi : [0, \infty) \to [0, \infty)\) be upper semicontinuous with \(\psi(t) < t\) for all \(t > 0\). Suppose that a mapping \(T : X \to X\) is nondecreasing such that \(d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty)\})\) for all \(x, y \in X\) with \(x \preceq y\).

Assume that there exists \(x_0 \in X\) such that \(x_0 \preceq Tx_0\). Suppose that either \([T\text{ is continuous}]\) or \([\text{if }\{x_n\}\text{ is a sequence in }X\text{ such that }x_n \preceq x_{n+1}\text{ for all }n \in \mathbb{N}\text{ and }\lim_{n \to \infty} x_n = x\text{ then }x_n \preceq x\text{ for all }n \in \mathbb{N}]\).

Then \(T\) has a fixed point in \(X\).

Theorem 2.11. Let \((X, d)\) be a complete metric space, \(\alpha \in \Lambda\), and let \(\psi : [0, \infty) \to [0, \infty)\) be upper semicontinuous with \(\psi(t) < t\) for all \(t > 0\). Suppose that a mapping \(T : X \to X\) satisfies
\[
\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))
\]
for all \(x, y \in X\). Assume that (2.2) is satisfied.

Suppose that either \(T\) is continuous or
\[
\limsup \alpha(T^n x_0, x) > 0
\]
for any cluster point \(x\) of \(\{T^n x_0\}\).

Then \(T\) has a fixed point in \(X\).

Proof. By (2.2), there exists \(x_0 \in X\) such that \(\alpha(T^i x_0, T^j x_0) \geq 1\) for all \(i, j \in \mathbb{N} \cup \{0\}\) with \(i \neq j\). Define a sequence \(\{x_n\} \subset X\) by \(x_{n+1} = Tx_n\) for all \(n \in \mathbb{N} \cup \{0\}\).

As in proof of Theorem 2.7, \(x_* = \lim_{n \to \infty} x_n \in X\) exists.

Obviously, \(T\) has a fixed point whenever \(T\) is continuous.

Assume that \(\limsup \alpha(T^n x_0, x) > 0\) for any cluster point \(x\) of \(\{T^n x_0\}\).

Then, there exists a subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) such that \(p := \lim_{k \to \infty} \alpha(x_{n(k)}, x_*) > 0\).

By using upper semicontinuity of \(\psi\), we obtain
\[
d(x_*, Tx_*) \leq \lim_{k \to \infty} \psi(d(x_{n(k)}, x_*)) \leq \frac{1}{p} \lim_{k \to \infty} \alpha(x_{n(k)}, x_*) \lim_{k \to \infty} \psi(d(x_{n(k)}, x_*)) \leq \frac{1}{p} \lim_{n \to \infty} \psi(d(x_{n(k)}, x_*)) = 0.
\]

Hence, \(x_*\) is a fixed point of \(T\). \(\square\)
References


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