Symmetry Group Approach to the Solution of Generalized Burgers Equation: $U_t + UU_x = \lambda U_{xx}$

M.E. Oduor Okoya*, N. Omolo Ongati, T. J. O. Aminer and J. Nyakinda Otula

School of Mathematics and Actuarial Science
Bondo University College, P.O. Box 210-40601, Bondo, Kenya
* e-mail: oduor.okoya@yahoo.com

Abstract

Symmetry of a system of differential equations is a transformation that maps any solution to another solution of the system. In Lie’s framework such transformations are groups that depend on continuous parameters and consist of point transformations (point symmetries), acting on the system’s space of independent and dependent variables, or, more generally, contact transformations (contact symmetries), acting on independent and dependent variables as well as on all first derivatives of the dependent variables. Lie groups, and hence their infinitesimal generators, can be naturally prolonged to act on the space of independent variables, dependent variables, and derivatives of the dependent variables. We present a Lie symmetry approach in solving Burgers Equation: $U_t + UU_x = \lambda U_{xx}$ which is a nonlinear partial differential equation, which arises in model studies of turbulence and shock wave theory. In physical application of shock waves in fluids, coefficient $\lambda$, has the meaning of viscosity. So far in both analytic and numerical approaches the solutions have only been established for $0 \leq \lambda \leq 1$. In this paper, we give a global solution to the Burgers equation with no restriction on $\lambda$ i.e. for $\lambda \in (-\infty, \infty)$.

Keywords: Symmetries, Lie Group, Burgers Equation, Invariant, Global, Solutions

1 The Burgers Equation

We consider the Burgers equation:

$$U_t + UU_x = \lambda U_{xx}$$  \hspace{1cm} (1)
1.1 Lie Symmetry Analysis

First we consider a $K^{th}$ order partial differential equation of the form

$$F(x, u, u_1, u_2, u_3, ..., u_k) = 0 \quad (2)$$

where $x = (x_1, x_2, x_3, ..., x_n)$ denotes $n$ independent variables, $u_j$ denotes the set of coordinates corresponding to all the $j$–th order partial derivatives with respect to $x$.

1.1.1 Definition

The one-parameter Lie group of transformations

$$x^* = X(x, u; \varepsilon) \quad (3)$$

$$u^* = U(x, u; \varepsilon) \quad (4)$$

leaves the partial differential equation (2) invariant if and only if its $K$–th extension, $x^*, u^*, u^*_1, u^*_2, ..., u^*_k$ leaves the surface $F(x, u, u_1, u_2, u_3, ..., u_k) = 0$ invariant.

1.1.2 Theorem

Let $F_r(x, u^{(k)}) = 0$ be a non degenerate system of partial differential equations and

$$V = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u} \quad (5)$$

be the infinitesimal generator of the one parameter Lie group of transformations (3, 4)

Let

$$V^{(k)} = \xi(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u} + \eta^1_i + ... + \eta^k_{i_1i_2...i_k}(x, u, u_1, u_2, ..., u_k) \frac{\partial}{\partial u_{i_1i_2...i_k}} \quad (6)$$

be the corresponding $k^{th}$ extended infinitesimal generator of (5) where

$$\eta^1_i = D_i \eta - (D_i \xi_j) u_j, i = 1, 2, 3, ..., n; \quad (7)$$

and $\eta^{(k)}_{i_1i_2i_3i_4i_5...i_k}$ is given by

$$\eta^{(k)}_{i_1i_2i_3i_4i_5...i_k} = D_{i_k} \eta^{(k-1)}_{i_1i_2i_3...i_k-1} - (D_{i_k} \xi_j) u_{i_1i_2...i_kj} \quad (8)$$
$i_j = 1, 2, 3, ..., n$ for $j = 1, 2, 3, ..., k$ with $k = 1, 2, ...$ in terms of $(\xi(x, u), \eta(x, u))$. Then a connected local group of transformations $G$ of the form; $(3, 4)$ is a symmetrical group of the system of the partial differential equations if and only if

$$ V^{(k)}[F_r(x, u^{(k)})] = 0, r = 1, 2, 3, ... l, \text{ whenever } F(x, u^{(k)}) = 0 \quad (9) $$

[6, 3]

### 1.2 Lie Group Solution of the Burgers Equation

From (1) the symmetry groups of transformations are of the form

$$ t^* = T(t, x, u; \varepsilon), \quad x^* = X(t, x, u; \varepsilon), \quad u^* = U(t, x, u; \varepsilon) $$

with corresponding infinitesimals

$$ \xi(t, x, u) = \frac{\partial X(t, x, u; \varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}, \tau(t, x, u) = \frac{\partial T(t, x, u; \varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}, \phi(t, x, u) = \frac{\partial U(t, x, u; \varepsilon)}{\partial \varepsilon}|_{\varepsilon=0} $$

We let the generator $V$, of (1), be of the form

$$ V = \xi(t, x, u) \frac{\partial}{\partial x} + \tau(t, x, u) \frac{\partial}{\partial t} + \phi(t, x, u) \frac{\partial}{\partial u} $$

From equation (1), $F = u_t + uu_x - \lambda u_{xx} = 0$. By theorem (1.1.2) it follows that

$$ V^{(2)}F = V^{(2)}(u_t + uu_x - \lambda u_{xx}) = 0 \quad (10) $$

when $F = 0$, and so we obtain

$$ [\xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u} + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \eta^{tx} \frac{\partial}{\partial u_{tx}} + \eta^{xx} \frac{\partial}{\partial u_{xx}}] [u_t + uu_x - \lambda u_{xx}] = 0 $$

The infinitesimal condition above reduces to

$$ \phi u_x + \eta^t + \eta^x u - \lambda \eta^{xx} = 0 \quad (11) $$

with $\eta^t$, $\eta^x$, $\eta^{xx}$ explicitly defined [3]

Substituting $\eta^t$, $\eta^x$, $\eta^{xx}$ into equation (11), we obtain the equation

$$ [\phi u_x + \{ \phi u - \xi_t - \xi_x u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_x^2 \} + u \{ \phi_x - \tau_x u_t + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_u u_x \} - \lambda [\phi_{xx} + (2\phi_{xx} - \xi_{xx})u_x - \tau_{xx} u_t + (\phi_{uu} - 2\xi_{xx})u_x^2 - 2\tau_{uu} u_x u_t - \tau_{uu} u_x^3] - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_x u_x - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xx} = 0 \quad (12) $$

Equate to zero the coefficients of monomials in the first and second partial derivatives of $u$ and on substituting $u_t + uu_x = \lambda u_{xx}$, $u_t = \lambda uu_{xx} - u_{xx}$, $uu_x = \lambda u_{xx} - u_t$; wherever it occurs in equation (11) we arrive at the
solutions of the infinitesimal functions $\tau, \xi, \phi$ as

\[
\xi = c_2 + c_4x + c_5xt \\
\tau = c_1 + (c_3 + 2c_4)t + c_5t^2 \\
\phi = c_3 - c_4u + c_5x - c_5tu + \alpha(x, t) : \quad \alpha_t = \lambda \alpha_{xx}
\]  

(13) \quad (14) \quad (15)

$\alpha$, is an arbitrary solution to the generalized heat equation

Representation of the infinitesimal functions $\xi, \tau, \phi$ in the standard basis

\[
\begin{array}{cccccc}
 v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \xi & 0.c_1 + 1.c_2 + 0.c_3 + c_4x + c_5tx + 0.c_\alpha \\
 \tau & 1.c_1 + 0.c_2 + 1.c_3t + 2.c_4t + 1.c_5t^2 + 0.c_\alpha \\
 \phi & 0.c_1 + 0.c_2 + 1.c_3 - 1.c_4u + 1.c_5(x - tu) + 1.c_\alpha \alpha
\end{array}
\]  

(16)

enables one to determine infinitesimal generators.

We derive the corresponding basis infinitesimal generators $v_i$'s of the form

\[
v_i = \hat{\xi}_i \frac{\partial}{\partial x} + \hat{\tau}_i \frac{\partial}{\partial t} + \hat{\phi}_i \frac{\partial}{\partial u},
\]

where $\hat{\xi}_i, \hat{\tau}_i, \hat{\phi}_i$ are the coefficients $c_i$ in the standard solutions of $\xi, \tau, \phi$. Hence $v_4, v_5$ are listed below as the infinitesimal generators of Burgers equation.

\[
v_4 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u},
\]

\[
v_5 = tx \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - [ut - x] \frac{\partial}{\partial u}
\]  

(17)

The heat solutions $\omega$ referred to here, are the invariant solutions of the generalized heat equation [3]

1.3 Lie Groups Admitted by Equation (1)

The one parameter groups $G_i$ admitted by the Burgers equation, are determined by solving the corresponding Lie equations The one-parameter groups $G_i$ admitted by the Burgers equation, are determined by solving the corresponding Lie equations

\[
v_4 : \frac{d\tau}{de} = 2\tau, \frac{d\tau}{de} = \tau, \frac{d\tau}{de} = -\tau; \quad v_5 : \frac{d\tau}{de} = \tau, \frac{d\tau}{de} = \tau^2, \frac{d\tau}{de} = \tau - \tau\tau
\]

with initial conditions: $\tau_{e=0} = t, \tau|_{e=0} = x, \tau|_{e=0} = u$ which lead to;

\[
v_4; \quad G_4 : X(x, t, u; \varepsilon) \rightarrow X_4(e^{\varepsilon}x, e^{2\varepsilon}t, e^{-\varepsilon}u)
\]

\[
v_5; \quad G_5 : X(x, t, u; \varepsilon) \rightarrow X_5[\frac{x}{1-\varepsilon t}, \frac{t}{1-\varepsilon t}, u(1 - \varepsilon t) + \varepsilon x]
\]

(18) \quad (19)
1.4 Group Transformations of Solutions

The method is based on the fact that a symmetry group transforms any solutions of the equation in question into solutions of the same equation. If $u$, $x$, $t$ are group transformations of equation (1) with $u$, of the form $u = \Psi(u, x, t, \varepsilon)$, for some explicit function $\Psi$, then applying the inverse mapping, the new symmetry solution $\hat{u}$ is defined by

$$\hat{u} = \Psi(\Phi(g_\varepsilon^{-1}(\overline{\varepsilon}), g_\varepsilon^{-1}(\overline{\tau}), g_\varepsilon^{-1}(\overline{\tau}), \varepsilon^{-1})$$

where $u = \Phi(x, t)$ is any known solution of equation (2). Finally we obtain a one parameter family (with a parameter $\varepsilon$) of new solutions to equation (2) as

$$\hat{u} = \Psi(\Phi(g_\varepsilon^{-1}(\overline{\varepsilon}), g_\varepsilon^{-1}(\overline{\tau}), g_\varepsilon^{-1}(\overline{\tau}), \varepsilon^{-1})$$

1.5 Symmetry Solutions of the Burgers Equation

By symmetry group inversion theory above, if each $G_i$ is a symmetry group and $u = \Phi(x, t)$ is a solution of the Burgers equation (1), then transformation groups of the Burgers equation (1), solve the equation (1). The above solution can also be written in the new variables: $u = \Phi(\overline{\varepsilon}, \overline{\tau})$. If $u$, $x$, $t$ are group transformations of the Burgers equation (1) with $u$, of the form $u = \Psi(u, x, t, \varepsilon)$, for some explicit function $\Psi$, then applying the inverse mapping, the new symmetry solution $\hat{u}$ satisfies relation

$$\hat{u} = \Psi(\Phi(g_\varepsilon^{-1}(\overline{\varepsilon}), g_\varepsilon^{-1}(\overline{\tau}), g_\varepsilon^{-1}(\overline{\tau}), \varepsilon^{-1})$$

where $u = \Phi(x, t)$ is any known solution of equation (1). $v_4; G_4 : X(x, t, u; \varepsilon) \rightarrow X_4(e^\varepsilon x, e^{2\varepsilon} t, e^\varepsilon u)$

Then the new symmetry solution $\hat{u}_4, \hat{u}_5$ are defined by

$$\hat{u}_4 = e^{-\varepsilon}\Phi(e^{-\varepsilon}x, e^{-2\varepsilon}t)$$

$$\hat{u}_5 = \frac{\varepsilon}{1+\varepsilon t} + \frac{1}{1+\varepsilon t} \Phi(\frac{x}{1+\varepsilon t}, \frac{t}{1+\varepsilon t})$$

2 Invariant Solutions of the Burgers Equation

If a group of transformations maps a solutions into itself, we arrive at what is called a self-similar or group invariant solution [1, 4, 5]. Given infinitesimal symmetry of equation (1) the invariant solution under the one-parameter group generated by a generator $V$ are obtained as follows: We calculate two independent invariants $J_1 = k(x, t)$ and $J_2 = \mu(x, t, u)$ by solving the equation

$$V(J) = \tau(t, x, u) \frac{\partial J}{\partial t} + \zeta(t, x, u) \frac{\partial J}{\partial x} + \eta(t, x, u) \frac{\partial J}{\partial u} = 0$$

(23)
or its system of characteristics

\[
\frac{dt}{\tau(t,x,u)} = \frac{dx}{\varsigma(t,x,u)} = \frac{du}{\eta(t,x,u)} \quad (24)
\]

Then we designate one of the invariants as a function of the other e.g.

\[
\mu = \phi(k) \quad (25)
\]

Finally we substitute expression for \(\mu\) in equation (25) and obtain ordinary differential equation for the unknown function \(\phi(k)\) of one variable. This procedure reduces the number of independent variables by one [3, 4]. The invariant solution of (1) under the transformation group generated by equation (7) are obtained as \(u_i, u_{\omega j}\) in the table below. That is, for each of the infinitesimal generator \(V_i\) above we form the corresponding system of characteristics

\[
\frac{dt}{\tau(t,x,u_i)} = \frac{dx}{\varsigma(t,x,u_i)} = \frac{du}{\eta(t,x,u_i)}
\]

Hence we solve for \(u_i\), the invariant solution of the Burgers equation corresponding to each of the infinitesimal generator [2, 4, 7].

2.1 Invariant Solutions of the Burgers Equation

Below is a list of Generator \((v_i)\) and their corresponding Invariant Solutions \((u_m)\):

\(v_4\)

\[
u = [\sqrt{t}C_3 - (2\sqrt{2\lambda} - 1) \sqrt{t\pi e^{\frac{x^2}{2}}} \text{erf}(\alpha)]^{-1} : C = 0
\]

\(v_5\)

\[
u = \frac{x}{t} - \frac{2\lambda}{x + tc_1}
\]

\(v_5\)

\[
u = \frac{x}{t} - \frac{k}{t} \sqrt{2} \tanh[\frac{\sqrt{2}kx}{2\lambda t} + C_2] : C = k^2, \ C > 0
\]

\(v_5\)

\[
u = \frac{x}{t} - \frac{r}{t} \sqrt{2} \tan[C_3 - \frac{\sqrt{2}r x}{2\lambda t}] : C = -r^2, \ C < 0
\]

\(v_w4\)

\[
u = \frac{K\sqrt{x} \text{erf}(x\sqrt{\frac{x}{2n}}) + L}{C_3 + F_1(x, t)}
\]
The most general solution \( u_g \), i.e. **global** solution of equation (1) is obtained by

\[
u_{w5}
\]

\[
u = \frac{1}{\sqrt{t}} e^{\frac{x^2}{2t}} \left[ e^{-\left(\frac{4\lambda x^2}{t+4\lambda x^2}\right)^\frac{3}{2}} + K \right]
\]

(31)

\[
F_1(x, t) = \int \left[ \sqrt{\frac{\pi}{t}} \text{erf}\left[ \frac{x}{\sqrt{4t}} \right] \right] x dt,
\]

3 Global Symmetry Solutions of the Burgers Equation

Consider the Lie groups \( G_1, G_2, G_3, G_4, G_5, G_\alpha \) admitted by the burgers equation in section (1.2). By symmetry group inversion theory of section (1.3) and (1.4), if each \( G_i \) is a symmetry group and \( u = \Phi(x,t) \) is a known solution of the Burgers equation (1), then the functions \( \hat{u}_j \) expressed new solutions of the Burgers equation (1) [6]

The most general one-parameter group of symmetries is obtained by considering a general linear combination \( v_g = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + c_5 v_5 + c_\alpha v_\alpha \) of the given field vectors. We may represent an arbitrary group transformation \( g \) as the composition of transformations in the various one-parameter subgroups \( G_1, G_2, G_3, G_4, G_5, G_\alpha \). In particular if \( g \) is near the identity, it can be represented uniquely in the exponential form

\[
g = \exp(\varepsilon_\alpha v_\alpha)^* \exp(\varepsilon_5 v_5)^* \exp(\varepsilon_4 v_4)^* \exp(\varepsilon_3 v_3)^* \exp(\varepsilon_2 v_2)^* \exp(\varepsilon_1 v_1)^*
\]

Thus the most general solution \( u_g \), i.e. **global** solution of equation (1) is obtained by

\[
u_g = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 - \varepsilon_3}{1 + \varepsilon_5 t} + \left[ \frac{e^{-\varepsilon_4}}{1 + \varepsilon_5 t} \right] \times \Phi\left[ \frac{xe^{-\varepsilon_4} - (\varepsilon_2 \varepsilon_5 + \varepsilon_3)t - \varepsilon_3 \varepsilon_5 t^2 - \varepsilon_2}{1 + \varepsilon_5 t} \right], \frac{te^{-2\varepsilon_4} - \varepsilon_1 \varepsilon_5 t - \varepsilon_1}{1 + \varepsilon_5 t}
\]

(32)

where \( u = \Phi(x,t) \) is a known solution of the Burgers equation (1).

Substituting \( u_i(x, t) = \Phi(x, t) \), the invariant solution of (1) corresponding to the infinitesimal generator \( V_i \) into equation (26), we obtain \( u_{gi} \) the global symmetry solutions of the burgers equation as shown below.[3]

3.1 Global Symmetry Solutions of the Burgers Equation

Below is a list of Generator \( (v_i) \) and their corresponding Global Symmetry Solutions \( (u_m) \) \( v_1 \) of the Burgers Equation
\[ u_{gw4}(x, t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1 + \varepsilon_5 t} + \left[ \frac{e^{-\varepsilon_4}}{1 + \varepsilon_5 t} \right] * \left[ \sqrt[\varepsilon_1 \varepsilon_5 t - \varepsilon_1}{C_3} \right] \]

\[ = (2\sqrt{2\lambda})^{-1} \sqrt{\left(\frac{te^{-2\varepsilon_4} - \varepsilon_1 \varepsilon_5 t - \varepsilon_1}{1 + \varepsilon_5 t}\right)\pi e^{\frac{a^2}{2}} erf(\bar{a})} \] (33)

\[ u_{g5}(x, t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1 + \varepsilon_5 t} + \left[ \frac{e^{-\varepsilon_4}}{1 + \varepsilon_5 t} \right] * \left[ \frac{x e^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2 \varepsilon_5)t - \varepsilon_3 \varepsilon_5 t^2 - \varepsilon_2}{te^{-2\varepsilon_4} - \varepsilon_1 \varepsilon_5 t - \varepsilon_1} \right] \]

\[ \* \sqrt{2\lambda} \text{tanh}\left[\frac{\sqrt{2}k(x e^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2 \varepsilon_5)t - \varepsilon_3 \varepsilon_5 t^2 - \varepsilon_2)}{-2\lambda(te^{-2\varepsilon_4} - \varepsilon_1 \varepsilon_5 t - \varepsilon_1)} + C_2 \right] \] (34)

\[ u_{g5}(x, t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1 + \varepsilon_5 t} + \left[ \frac{e^{-\varepsilon_4}}{1 + \varepsilon_5 t} \right] * \left[ \frac{x e^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2 \varepsilon_5)t - \varepsilon_3 \varepsilon_5 t^2 - \varepsilon_2 - r(1 + \varepsilon_5 t)}{te^{-2\varepsilon_4} - \varepsilon_1 \varepsilon_5 t - \varepsilon_1} \right] \]

\[ \* \sqrt{2\lambda} \text{tanh}[C_3 - \frac{\sqrt{2}k(x e^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2 \varepsilon_5)t - \varepsilon_3 \varepsilon_5 t^2 - \varepsilon_2)}{-2\lambda(te^{-2\varepsilon_4} - \varepsilon_1 \varepsilon_5 t - \varepsilon_1)}] \] (35)

\[ u_{g5}(x, t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1 + \varepsilon_5 t} + \left[ \frac{e^{-\varepsilon_4}}{1 + \varepsilon_5 t} \right] * \left[ \frac{\dot{x}}{t} - \frac{2\lambda(1 + \varepsilon_5 t)}{\dot{x} + tc_1} \right] \] (36)

\[ \bar{a}(x, t) = \alpha(\dot{x}, \dot{t}), \quad \dot{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2 \varepsilon_5)t - \varepsilon_3 \varepsilon_5 t^2 - \varepsilon_2, \quad \dot{t} = te^{-2\varepsilon_4 - \varepsilon_1 \varepsilon_5 t - \varepsilon_1} \]

4 Conclusions

In this paper, we have managed to find global solutions of the Burgers Equation using the Lie symmetry approach. The solutions (33 - 36) are appearing here in literature for the first time. Other previous attempts to solve this very important equation only managed to find solutions when the coefficient \( \lambda \) is restricted to \( \lambda \in [0, 1] \).

References


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