Generalized Mannheim Quaternionic Curves in Euclidean 4-Space

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Abstract

In this paper, we define a generalized Mannheim quaternionic curve in the four-dimensional Euclidean space $\mathbb{R}^4$ and investigate the properties of it.

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1 Introduction

In the theory of space curves in differential geometry, the associated curve, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve have an important role for the characterizations of space curves. The well-known examples of such curves are Bertrand curves and Mannheim curves.
Space curves of which principal normals are the binormals of another curve are called Mannheim curves in the three dimensional Euclidean space. The notion of Mannheim curves was discovered by A. Mannheim in 1878. The articles concerning the Mannheim curves are rather few. O. Tigano([7]) obtained the locus of Mannheim curves in the three dimensional Euclidean space. Mannheim partner curves in the three dimensional Euclidean space and the three dimensional Minkowski space are studied and the necessary and sufficient conditions for the Mannheim partner curves are obtained in [4] and [6]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves in the four dimensional Euclidean space are introduced by [5].

For the study of a quaternionic curve, M. A. Gungor and M. Tosun([3]) investigated quaternionic rectifying curves in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \) and D. W. Yoon ([9]) investigated the quaternionic general helices in \( \mathbb{R}^4 \). Also, A. C. Coken and A. Tuna([2]) studies quaternionic inclined curves as harmonic curvature functions in the 4-dimensional semi-Euclidean space with \( \mathbb{R}^4_2 \).

In this paper, we study a generalized Mannheim quaternionic curve in the four-dimensional Euclidean space \( \mathbb{R}^4 \) and give some characterizations of generalized Mannheim quaternionic curves.

2 Preliminary Notes

Let \( Q_H \) be the four-dimensional vector space over a field \( H \) whose characteristic greater than 2. Let \( e_i (1 \leq i \leq 4) \) be a basis for the vector space. Let the rule of multiplication on \( Q_H \) be defined on \( e_i \) and extended to the whole of the vector space distributivity as follows:

A real quaternionic is defined with \( q = ae_1 + be_2 + ce_3 + de_4 \) such that

(i) \( e_i \times e_i = -e_4 = -1, \quad (1 \leq i \leq 3) \)

(ii) \( e_i \times e_j = -e_j \times e_i = e_k \quad (1 \leq i, j \leq 3) \),

where \( a, b, c, d \) are real numbers and \( (ijk) \) is even permutation of \( (123) \) in the four dimensional Euclidean space \( \mathbb{R}^4 \).

We put \( S_q = d \) and \( V_q = ae_1 + be_2 + ce_3 \). Then a quaternion \( q \) can rewrite as \( q = S_q + V_q \), where \( S_q \) and \( V_q \) are the scalar part and vectorial part of \( q \), respectively.

Let \( p \) and \( q \) be any two elements of \( Q_H \). Then the product of \( p \) and \( q \) is defined by

\[ p \times q = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \land V_q, \]

where we have used the inner product and the cross product in \( \mathbb{R}^3 \).

On the other hand, the conjugate of \( q = ae_1 + be_2 + ce_3 + de_4 = S_q + V_q \in Q_H \) is denoted by \( aq \) and given by

\[ aq = -ae_1 - be_2 - ce_3 + de_4 = S_q - V_q. \]
From this, we define the symmetric non-degenerate valued bilinear form $h$ as follows:

$$ h : Q_H \times Q_H \rightarrow R $$

$$(p, q) \mapsto h(p, q) = \frac{1}{2}(p \times \alpha q + q \times \alpha q), $$

it is called the quaternionic inner product. And the norm of $q$ defined by

$$ ||q||^2 = h(q, q) = q \times \alpha q = a^2 + b^2 + c^2 + d^2. $$

A number $q$ is said to be a spatial quaternion if $q + \alpha q = 0$. It is called a temporal quaternion if $q - \alpha q = 0$. Therefore, any quaternion $q$ can be written as

$$ q = \frac{1}{2}(q + \alpha q) + \frac{1}{2}(q - \alpha q). $$

The spatial part of $q$ is $\frac{1}{2}(q - \alpha q)$ and is a spatial quaternion, while $\frac{1}{2}(q + \alpha q)$ the temporal part of $q$ and is temporal quaternion.

In 1985, K. Bharathi and M. Nagaraj([1]) studied the Frenet formulas for quaternionic curves in the three-dimensional Euclidean space $R^3$ and the four-dimensional Euclidean space $R^4$ and they proved the following theorems:

**Theorem 2.1** The three-dimensional Euclidean space $R^3$ is identified with the space of spatial quaternion $\{q \in Q_H | q + \alpha q = 0\}$. Let

$$ \beta : I = [0, 1] \subset R \rightarrow Q_H $$

$$ s \rightarrow \beta(s) = \sum_{i=1}^{3} \beta_i(s)e_i, \quad (1 \leq i \leq 3) $$

be an arc-lengthed smooth curve with non-zero principal curve $k$ and torsion $r$. Then Frenet formula of $\beta$ is given by

$$ t' = kn_1, $$

$$ n_1' = -kt + rn_2, $$

$$ n_2' = -rn_1, $$

(1)

where $t$, $n_1$, $n_2$ are the unit tangent, the unit principal normal and the unit binormal vector of $\beta$, respectively.

**Theorem 2.2** The four-dimensional Euclidean space $R^4$ is identified with the space of unit quaternion $Q_H$. Let

$$ \gamma : I = [0, 1] \subset R \rightarrow Q_H $$

$$ s \rightarrow \gamma(s) = \sum_{i=1}^{4} \gamma_i(s)e_i, \quad e_4 = 1 $$

be an arc-lengthed smooth curve with non-zero principal curve $K$, torsion $k$ and bitorsion $r - K$. Then Frenet formula of $\gamma$ is given by

$$ T'(s) = KN_1(s), $$

$$ N_1'(s) = -KT(s) + kN_2(s), $$

$$ N_2'(s) = -kN_1(s) + (r - K)N_3(s), $$

$$ N_3'(s) = -(r - K)N_2(s), $$

(2)

where $T$, $N_1$, $N_2$, $N_3$ are the unit tangent, the first normal, the second normal and the third normal vectors of $\gamma$, respectively.
At each point of \( \gamma \), a line \( l_i(i = 1, 2, 3) \) in the direction of \( N_i \) is called the \( i \)-th normal line.

This Frenet formula of the curve \( \gamma \) is obtained by making use the Frenet formula for a curve \( \beta \) in \( \mathbb{R}^3 \). Moreover, there are relationships between curvatures of the curve \( \gamma \) and \( \beta \). These relations can be explained that the torsion of \( \gamma \) is the principal curvature of the curve \( \beta \). Also, the bitorsion \( r - K \) of \( \gamma \) is determined by the torsion \( r \) and the principal curvature \( K \) of \( \beta \).

3 Generalized Mannheim Quaternionic Curves in \( Q_H \)

In this section, we define generalized Mannheim quaternionic curves in the four-dimensional Euclidean space \( \mathbb{R}^4 \) and investigate the properties of it.

**Definition 3.1** A quaternionic curve \( \gamma \) in the four-dimensional Euclidean space \( \mathbb{R}^4 \) is a generalized Mannheim curve if there is a quaternionic curve \( \bar{\gamma} \) in \( \mathbb{R}^4 \) such that the first normal line at each point of \( \gamma \) is included in the plane generated by the second normal line and the third normal line of \( \bar{\gamma} \) at the corresponding point under a bijective smooth function \( \phi: \gamma \to \bar{\gamma} \). In this case, \( \bar{\gamma} \) is called the generalized Mannheim partner quaternionic curve of \( \gamma \).

**Theorem 3.2** The distance between corresponding points of a generalized Mannheim quaternionic curve and of its generalized Mannheim partner quaternionic curve in \( Q_H \) is a constant.

**Proof.** Let \( \gamma \) be a generalized Mannheim quaternionic curve in \( Q_H \) and \( \bar{\gamma} \) a generalized Mannheim partner quaternionic curve of \( \gamma \). \( \bar{\gamma} \) is distinct from \( \gamma \). Let the pair of \( \gamma(s) \) and \( \bar{\gamma}(s) = \bar{\gamma}(\bar{s}(s)) \) be of corresponding points of \( \gamma \) and \( \bar{\gamma} \). Then the curve \( \bar{\gamma} \) is given by

\[
\bar{\gamma}(\bar{s}) = \bar{\gamma}(\bar{s}(s)) = \gamma(s) + \lambda(s)N_1(s)
\]

for some smooth function \( \lambda \). Let \( \{T, N_1, N_2, N_3\} \) and \( \{\bar{k}_1, \bar{k}_2, \bar{k}_3\} \) be the Frenet frame and the curvature functions of \( \bar{\gamma} \), respectively.

By differentiating of equation (3) with respect to \( s \) as a vector valued function in \( \mathbb{R}^4 \) and using the chain rule and Frenet formula of \( \gamma \), we obtain

\[
\varphi(s)T(\bar{s}(s)) = (1 - \lambda(s)K(s))T(s) + \lambda'(s)N_1(s) + \lambda(s)k(s)N_2(s),
\]

where \( \varphi(s) = \frac{ds}{d\bar{s}} \). By definition 3.1, \( N_1(s) \) can be represented as the following form:

\[
N_1(s) = f_1(s)N_2(\bar{s}(s)) + f_2(s)N_3(\bar{s}(s))
\]
for some smooth functions \( f_1 \) and \( f_2 \). If we consider
\[
\langle \mathbf{T}(\bar{s}(s)), f_1(s)\mathbf{N}_2(\bar{s}(s)) + f_2(s)\mathbf{N}_3(\bar{s}(s)) \rangle = 0
\]
and equation (4), then we have \( \lambda'(s) = 0 \). This means that \( \lambda \) is a nonzero constant. On the other hand, from the distance function between two points, we have
\[
d(\bar{\gamma}(\bar{s}), \gamma(s)) = |\lambda|.
\]
Namely, \( d(\bar{\gamma}(\bar{s}), \gamma(s)) \) is a constant. This completes the proof.

**Theorem 3.3** If a quaternionic curve \( \gamma \) in \( Q_H \) is a generalized Mannheim curve, then the principal curvature \( K \) and the torsion \( k \) of \( \gamma \) satisfy the equation
\[
K(1 - \lambda K) - \lambda k^2 = 0 \tag{6}
\]
for nonzero constant \( \lambda \).

**Proof.** Since \( \lambda \) is a nonzero constant, equation (4) is reduced to
\[
\bar{K}(\bar{s}(s))\varphi(s)\mathbf{N}_1(\bar{s}(s)) = \left( \frac{1 - \lambda K(s)}{\varphi(s)} \right)' T(s) + \frac{\lambda k(s)}{\varphi(s)} \mathbf{N}_2(s) + \frac{\lambda k(s)(r(s) - K(s))}{\varphi(s)} \mathbf{N}_3(s). \tag{7}
\]

By taking differentiation both sides of equation (7) with respect to \( s \) as a vector valued function in \( R^4 \),
\[
\bar{K}(\bar{s}(s))\varphi(s)\mathbf{N}_1(\bar{s}(s)) = \left( \frac{1 - \lambda K(s)}{\varphi(s)} \right)' T(s) + \frac{\lambda k(s)(r(s) - K(s))}{\varphi(s)} \mathbf{N}_3(s). \tag{8}
\]
Since \( \mathbf{N}_1(\bar{s}(s)) \) is orthogonal to \( f_1(s)\mathbf{N}_2(\bar{s}(s)) + f_2(s)\mathbf{N}_3(\bar{s}(s)) \) in \( R^4 \), the coefficient of \( \mathbf{N}_1(s) \) in (8) vanishes, that is,
\[
K(s)(1 - \lambda K(s)) - \lambda k^2(s) = 0.
\]
This completes the proof.

**Theorem 3.4** For non-constant principal curvature \( K(s) \) and the torsion \( k(s) \) of a quaternionic curve \( \gamma \) in \( Q_H \), if \( \gamma \) satisfies the equation (6), then a curve
\[
\bar{\gamma}(s) = \gamma(s) + \lambda \mathbf{N}_1(s)
\]
is a generalized Mannheim partner curve \( \gamma \).
\textbf{Proof.} Let $\bar{s}$ be the arc-length of a curve $\bar{\gamma}$. Then by using (6) we have

$$\bar{s} = \int_0^s ||\frac{d\bar{\gamma}(s)}{ds}||ds = \int_0^s \sqrt{1 - \lambda K}ds$$

By differentiating both sides of the equation $\bar{\gamma}(s) = \gamma(s) + \lambda N_1(s)$ with respect to $s$ we have

$$\varphi(s)\bar{T}(\bar{s}(s)) = (1 - \lambda K(s))\bar{T}(s) + \lambda k(s)N_2(s),$$

where $\varphi(s) = \frac{ds}{d\gamma}$. It follows

$$\bar{T}(\bar{s}(s)) = \frac{1 - \lambda K(s)}{\sqrt{1 - \lambda K}}T(s) + \frac{\lambda k(s)}{\sqrt{1 - \lambda K}}N_2(s). \quad (9)$$

The differentiation of the above equation with respect to $s$ is

$$\varphi(s)\bar{K}(\bar{s}(s))\bar{N}_1(\bar{s}(s)) = \left(\sqrt{1 - \lambda K}\right) T(s) + \left(\frac{(1 - \lambda K(s))K(s) - \lambda k^2(s)}{\sqrt{1 - \lambda K}}\right) N_1(s)$$

$$+ \left(\frac{\lambda k(s)}{\sqrt{1 - \lambda K}}\right)^2 N_2(s) + \left(\frac{\lambda k(s)(r(s) - K(s))}{\sqrt{1 - \lambda K}}\right) N_3(s). \quad (10)$$

From (6), the coefficient of $N_1$ in (10) is identically zero. On the other hand, from (9) $\bar{T}(\bar{s}(s))$ is a linear combination of $T(s)$ and $N_2(s)$. And from (10) $\bar{N}_1$ is also linear combination of $T(s)$, $N_1(s)$ and $N_3(s)$. Therefore, $N_1(s)$ is expressed by $T(\bar{s}(s))$ and $\bar{N}_1(\bar{s}(s))$. Thus, the first normal line of $\gamma$ lies in the plane generated by the second normal line and the third normal line of $\bar{\gamma}$ at the corresponding points under a bijection $\varphi$ which is defined by $\varphi(\gamma(s)) = \bar{\gamma}(\bar{s}(s))$. This completes the proof.

\textbf{Remark.} In the three-dimensional Euclidean space $\mathbb{R}^3$, a smooth curve $\xi$ in $\mathbb{R}^3$ is a Mannheim curve if and only if the curvature $k$ and the torsion $r$ of $\xi$ satisfy the relation $k = \lambda(k^2 + r^2)$ for some constant $\lambda$(cf.[6]). Also, for a spatial quaternionic curve $\beta$ in $Q_H$ defined in Theorem 2.1 the above statement holds because of having close analogies to the Euclidean type. But, for a quaternionic curve $\gamma$ in $Q_H$ defined in Theorem 2.2 satisfying the relation (6) it is not clear that a smooth curve $\bar{\gamma}$ given by $\bar{\gamma}(s) = \gamma(s) + \lambda N_1(s)$ is a quaternionic curve. Thus, it is unknown whether the reverse of Theorem 3.3 is true or false.

Next, we consider a special generalized Mannheim quaternionic curve $\gamma$ in $Q_H$ and give necessary and sufficient condition for the curve $\gamma$.

\textbf{Theorem 3.5} Let $\gamma(s)$ be a quaternionic curve in $Q_H$ with non-zero bitor-sion $r - K$. Then there is a curve $\bar{\gamma}$ in $Q_H$ such that the first normal line of $\gamma$ is linearly dependent on the third normal line of $\bar{\gamma}$ at the corresponding points $\gamma(s)$ and $\bar{\gamma}(\bar{s})$ if and only if the principal curvature $K$ and the torsion $k$ of $\gamma$ are constants.
Proof. Let the first normal line $N_1$ of $\gamma$ be linearly dependent on the third normal line $N_3$ of $\bar{\gamma}$ at the corresponding points $\gamma(s)$ and $\bar{\gamma}(\bar{s})$. Then the parametrization of $\bar{\gamma}$ is

$$\bar{\gamma}(\bar{s}) = \gamma(s) + \lambda(s)N_1(s).$$

Differentiating the above equation with respect to $s$ and using Frenet formula, we obtain

$$\varphi(s)\bar{T}(\bar{s}(s)) = (1 - \lambda(s)K(s))T(s) + \lambda'(s)N_1(s) + \lambda(s)k(s)N_2(s), \quad (11)$$

where $\varphi(s) = \frac{ds}{d\bar{s}}$. Since $N_1 = \pm\bar{N}_3$, $\langle \bar{T}(\bar{s}(s)), N_1(s) \rangle = 0$ which implies from (11) we have $\lambda'(s) = 0$. Therefore, equation (11) becomes

$$\bar{T}(\bar{s}(s)) = \frac{1 - \lambda K(s)}{\varphi(s)}T(s) + \frac{\lambda k(s)}{\varphi(s)}N_2(s). \quad (12)$$

By taking differentiation both sides of equation (12) with respect to $s$, we have

$$\varphi(s)\bar{K}(\bar{s}(s))\bar{N}_1(\bar{s}(s)) = \left(\frac{1 - \lambda K(s)}{\varphi(s)}\right)'T(s) + \left(\frac{(1 - \lambda K(s)K(s) - \lambda k^2(s))}{\varphi(s)}\right)N_1(s)$$

$$+ \left(\frac{\lambda k(s)}{\varphi(s)}\right)'N_2(s) + \left(\frac{\lambda k(s)(\bar{r}(s) - K(s))}{\varphi(s)}\right)N_3(s). \quad (13)$$

Since $\langle \bar{N}_1(\bar{s}(s)), \bar{N}_3(\bar{s}(s)) \rangle = 0$ and $N_1 = \pm\bar{N}_3$, we have from (13)

$$\lambda = \frac{K}{K^2 + k^2}. \quad (14)$$

On the other hand, the differentiation of equation (13) with respect to $s$ is

$$\varphi(s)(-\bar{K}(\bar{s}(s))\bar{T}(\bar{s}(s)) + \bar{k}(\bar{s}(s))\bar{N}_2(\bar{s}(s)))$$

$$= \left(\frac{1}{\bar{K}(\bar{s}(s))\varphi(s)}\right)\left(\frac{1 - \lambda K(s)}{\varphi(s)}\right)'T(s)$$

$$+ \left(\frac{1}{\bar{K}(\bar{s}(s))\varphi(s)}\right)'\left(\frac{k(s)}{\varphi(s)}\right)N_1(s)$$

$$+ \left(\frac{1}{\bar{K}(\bar{s}(s))\varphi(s)}\right)'\left(\frac{\lambda k(s)}{\varphi(s)}\right)N_2(s)$$

$$+ \left(\frac{1}{\bar{K}(\bar{s}(s))\varphi(s)}\right)'\left(\frac{\lambda k(s)((\bar{r}(s) - K(s))^2)}{\varphi(s)}\right)N_3(s) \quad (15)$$

Since $\langle N_1(s), T(\bar{s}(s)) \rangle = 0$ and $\langle N_1(s), N_2(\bar{s}(s)) \rangle = 0$, we obtain from (15)

$$(-\lambda K(s)K'(s) - \lambda k(s)k'(s))\varphi(s) + (-K(s) + \lambda K^2(s) + \lambda k^2(s))\varphi'(s) = 0. \quad (16)$$

By differentiating equation (13) with respect to $s$, we get

$$K'(s) - 2\lambda(K(s)K'(s) + k(s)k'(s)) = 0, \quad (17)$$
which implies from (13) and (16) we easily show that $K'(s) = 0$, that is, $K'(s)$ is a constant. Also, from (17) $k(s)$ is a constant.

Conversely, let us suppose that the principal curvature $K$ and the torsion $k$ of a quaternionic curve $\gamma$ in $Q_H$ are constants. Then $\frac{K}{k^2}$ is a constant, say, $\lambda$.

The representation of a curve $\bar{\gamma}$ is

$$\bar{\gamma}(s) = \gamma(s) + \lambda N_1(s).$$

If the are-length parameter of $\bar{\gamma}$ is given by $\bar{s}$, then we have

$$\bar{s} = \int_0^s ||d\bar{\gamma}(s)||ds = \int_0^s \sqrt{|1 - \lambda K|}ds.$$

Therefore, by differentiating both sides of the equation $\bar{\gamma}(\bar{s}(s)) = \gamma(s) + \lambda N_1(s)$ with respect to $s$ we have

$$\varphi(s)\dot{T}(\bar{s}(s)) = (1 - \lambda K)T(s) + \lambda kN_2(s),$$

where $\varphi(s) = \frac{ds}{d\bar{s}}$. It follows

$$T(\bar{s}(s)) = \sqrt{|1 - \lambda K|}T(s) + \frac{\lambda k}{\sqrt{|1 - \lambda K|}}N_2(s). \quad (18)$$

By taking differentiation both sides of the above equation with respect to $s$, we have

$$\varphi(s) \frac{dT(\bar{s}(s))}{d\bar{s}} = \left(\frac{\lambda k(r - K)(s)}{\sqrt{|1 - \lambda K|}}\right)N_3(s), \quad (19)$$

because $K, k$ are constants and $\lambda = \frac{K}{k^2 + k^2}$. Since $r - K$ is non-zero function, we find

$$K(\bar{s}(s)) = ||\frac{d\bar{T}(\bar{s}(s))}{d\bar{s}}|| = \varepsilon \frac{\lambda k(r - K)(s)}{1 - \lambda K}, \quad (20)$$

where $\varepsilon(=\pm 1) = \text{sign } (r - K)$ is the sign of the function $r - K$ so that sign $(r - K)(r - K)$ is positive. We have put

$$\bar{N}_1(\bar{s}(s)) = \frac{1}{K(\bar{s}(s))} \frac{dT(\bar{s}(s))}{d\bar{s}}.$$

It follows that from (19) and (20)

$$\bar{N}_1(\bar{s}(s)) = \mp N_3(s). \quad (21)$$

Differentiating of the above equation with respect to $s$, we find

$$\frac{d\bar{N}_1(\bar{s}(s))}{d\bar{s}} = -\varepsilon \frac{(r - K)(s)}{\sqrt{|1 - \lambda K|}}N_2(s).$$
which implies we have from (18) and (20)
\[
\frac{d\bar{N}_1(\bar{s}(s))}{d\bar{s}} + \bar{K}(\bar{s}(s))\mathbf{T}(\bar{s}(s)) = \varepsilon \frac{\lambda k(r - K)(s)}{\sqrt{1 - \lambda K}} \mathbf{T}(s) - \varepsilon (r - K)(s)\sqrt{1 - \lambda K}\mathbf{N}_2(s).
\]

From this, we have
\[
\bar{k}(\bar{s}(s)) = \frac{dN(\bar{s}(s))}{d\bar{s}} + \bar{K}(\bar{s}(s))\mathbf{T}(\bar{s}(s))
\]
\[= \frac{\lambda k^2((r - K)(s))^2}{1 - \lambda K} + (1 - \lambda K)((r - K)(s))^2
\]
\[= \varepsilon (r - K)(s).
\]

Thus, we can put
\[
\bar{N}_2(\bar{s}(s)) = \frac{1}{k(\bar{s}(s))}\left(\frac{dN(\bar{s}(s))}{d\bar{s}} + \bar{K}(\bar{s}(s))\mathbf{T}(\bar{s}(s))\right)
\]
\[= \frac{\lambda k}{\sqrt{1 - \lambda K}} \mathbf{T}(s) - \sqrt{1 - \lambda K}\mathbf{N}_2(s). \tag{22}
\]

Differentiating of the last equation with respect to \(s\), we reach
\[
\varphi(s)\frac{d\bar{N}_2(\bar{s}(s))}{d\bar{s}} = \frac{k}{\sqrt{1 - \lambda K}} \mathbf{N}_1(s) - (r - K)(s)\sqrt{1 - \lambda K}\mathbf{N}_3(s).
\]

Since \(\bar{k}(\bar{s}(s))\mathbf{N}_1(\bar{s}(s)) = (r - K)(s)\mathbf{N}_3(s)\), we get
\[
\frac{d\bar{N}_2(\bar{s}(s))}{d\bar{s}} + \bar{k}(\bar{s}(s))\mathbf{N}_1(\bar{s}(s)) = \frac{k}{1 - \lambda K} \mathbf{N}_1(s),
\]
from which we get
\[
\bar{N}_3(\bar{s}(s)) = \delta \mathbf{N}_1(s), \tag{23}
\]
where \(\delta = \pm 1\). From (18), (21), (22) and (23) we have thus
\[
\det[\mathbf{T}(\bar{s}(s)) \quad \bar{N}_1(\bar{s}(s)) \quad \bar{N}_2(\bar{s}(s)) \quad \bar{N}_3(\bar{s}(s))] = \varepsilon \delta,
\]
because of \(\det[\mathbf{T}(s) \quad \mathbf{N}_1(s) \quad \mathbf{N}_2(s) \quad \mathbf{N}_3(s)] = 1\). It follows that \(\varepsilon = \delta\). Thus, we obtain
\[
\bar{N}_3(\bar{s}(s)) = \varepsilon \mathbf{N}_1(s)
\]
and
\[
(r - \bar{K})(\bar{s}(s)) = \frac{\left(\frac{d\bar{N}_2(\bar{s}(s))}{d\bar{s}}, \bar{N}_3(\bar{s}(s))\right)}{\varepsilon \frac{k}{1 - \lambda K}}.
\]

By the above facts, \(\bar{\gamma}\) is a Frenet curve in \(Q_H\) and the first normal line at each point of \(\gamma\) is linearly dependent on the third normal line of \(\bar{\gamma}\) at corresponding each point under a bijection \(\phi\) which is defined by \(\phi(\gamma(s)) = \bar{\gamma}(\bar{s}(s))\). This completes the proof.
References


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