Gradient System with Dependence on the Gradient at Nonlinearity

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Abstract

In this paper we establish existence of solutions for quasilinear elliptic system of gradient form with dependence on the gradient at nonlinearity.

1 Introduction

Let us consider the problem

\[
\begin{align*}
-\Delta u &= F_u(x, u, v, \nabla u, \nabla v) \quad \text{in} \quad \Omega \\
-\Delta v &= F_v(x, u, v, \nabla u, \nabla v) \quad \text{in} \quad \Omega \\
u = v &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial \Omega$ and $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. This type of system has been studied when there is not the presence of gradient at nonlinearity we can cite [2], [3], [4] and [5]. in this case the problem can be treated variationally. On the other hand, the system (1) is not variational. In this paper we will adapt the technique developed by D.G. De Figueiredo, M. Girardi, M. Matzeu, in [6]. This technique consists of associating with problem (1) a family of gradient
elliptic systems with no dependence on the gradient of the solution. Namely, for each \( w, z \in H^1_0(\Omega) \), we consider the problem

\[
\begin{cases}
-\Delta u = F_u(x, u, v, \nabla w, \nabla z) & \text{in } \Omega \\
-\Delta v = F_v(x, u, v, \nabla w, \nabla z) & \text{in } \Omega \\
u = v = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(2)

Now problem (2) is variational and we can treat it by Variational Methods. So we assign hypotheses on \( F \) in such a way that problem (2) can be treated by the mountain-pass theorem by Ambrosetti and Rabinowitz (see [1]).

On the set of assumptions on the nonlinearity \( F \) is the following

\((F_0)\) \( F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) is \( C^1 \) and \( F_t(x, t, s, \xi, \eta), F_s(x, t, s, \xi, \eta) \) are locally Lipschitz continuous;

\((F_1)\) There exists \( \epsilon > 0 \) such that \( |F(x, t, s, \xi, \eta)| \leq C_1(|t|^r + |s|^{q}) \), for \( |t|, |s| \leq \epsilon \), for all \( x \in \Omega, \xi, \eta \in \mathbb{R}^N \) and \( 2 < r, n < \frac{2N}{N-2} \);

\((F_2)\) \( \lim_{(t,s) \to 0} \frac{F_t(x, t, s, \xi, \eta)}{|t| + |s|} = 0 \) and \( \lim_{(t,s) \to 0} \frac{F_s(x, t, s, \xi, \eta)}{|t| + |s|} = 0 \) uniformly for \( x \in \overline{\Omega}, \xi, \eta \in \mathbb{R}^N \);\n
\((F_3)\) \( |F_t(x, t, s, \xi, \eta)| \leq C_2(1+|t|^p+|s|^q) \) and \( |F_s(x, t, s, \xi, \eta)| \leq C_3(1+|t|^p+|s|^q) \), for all \( x \in \overline{\Omega}, t, s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N \) and \( 1 < p, q < \frac{N+2}{N-2} \);

\((F_4)\) \( 0 < F(x, t, s, \xi, \eta) \leq \theta_t F_t(x, t, s, \xi, \eta) + \theta_s F_s(x, t, s, \xi, \eta) \), for all \( x \in \overline{\Omega}, \xi, \eta \in \mathbb{R}^N \) and \( |t|, |s| \geq R \), where \( R \) is a positive number and

\[
0 < \theta_t, \theta_s < \frac{1}{2};
\]

\((F_5)\) There exists positive constants \( a_1, a_2, a_3 \) such that

\[
F(x, t, s, \xi, \eta) \geq a_1|t|^\alpha + a_2|s|^\beta - a_3,
\]

for all \( x \in \overline{\Omega}, t, s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N, 2 < \alpha \leq p+1 \) and \( 2 < \beta \leq q+1 \).

Remark 1.1 Under conditions \((F_1)\) and \((F_3)\) there exist a positive constant \( C \) such that \( |F(x, t, s, \xi, \eta)| \leq C_1(|t|^r + |s|^n + |t|^{p+1} + |s|^{q+1} + |t|^r + |s|^r), \) for all \( x \in \Omega, t, s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N, r, n, p, q \) as previously.

Our first result concerns the solvability of (2) in \( E = H^1_0(\Omega) \times H^1_0(\Omega) \), and obtaining bounds on their solutions.
Theorem 1.1 Suppose that \((F_0)\) – \((F_5)\) holds. Then there exists positive constants \(M_1, M_2\) such that, for each \(w, z \in H^1_0(\Omega)\) the problem (2) has solution \((u_{wz}, v_{wz}) \in H^1_0(\Omega) \times H^1_0(\Omega)\) such that

\[ M_1 \leq \|u_{wz}\|^2_{H_0^1(\Omega)} + \|v_{wz}\|^2_{H_0^1(\Omega)} \leq M_2. \]

Our main result concern the solvability of system (1). For that matter we need a further assumption:

\((F_6)\) The functions \(F_t\) and \(F_s\) satisfies the following local Lipschitz conditions, that is, there exists \(L^t_i, L^s_i, i = 1, 2, 3, 4\) such that

\[
\begin{align*}
(i) & \quad |F_t(x, t', s, \xi, \eta) - F_t(x, t'', s, \xi, \eta)| \leq L^t_i |t' - t''| \\
(ii) & \quad |F_t(x, t, s', \xi, \eta) - F_t(x, t, s'', \xi, \eta)| \leq L^t_j |s' - s''| \\
(iii) & \quad |F_t(x, t, s, \xi', \eta) - F_t(x, t, s, \xi'', \eta)| \leq L^t_k |\xi' - \xi''| \\
(iv) & \quad |F_t(x, t, s, \xi', \eta') - F_t(x, t, s, \xi'', \eta'')| \leq L^t_l |\eta' - \eta''| \\
v & \quad |F_s(x, t', s, \xi, \eta) - F_s(x, t'', s, \xi, \eta)| \leq L^s_i |t' - t''| \\
v & \quad |F_s(x, t, s', \xi, \eta) - F_s(x, t, s'', \xi, \eta)| \leq L^s_j |s' - s''| \\
vii & \quad |F_s(x, t, s, \xi', \eta) - F_s(x, t, s, \xi'', \eta)| \leq L^s_k |\xi' - \xi''| \\
vii & \quad |F_s(x, t, s, \xi', \eta') - F_s(x, t, s, \xi'', \eta'')| \leq L^s_l |\eta' - \eta''|
\end{align*}
\]

all \(x \in \overline{\Omega}\), \(|t'|, |t''| \leq \rho, |s| \leq \rho, |\xi| \leq \rho_3\) and \(|\eta| \leq \rho_4\).

Thus we obtain the following result:

Theorem 1.2 Suppose that \((F_0)\) – \((F_6)\) Holds. Then the system (1) has solution, provided

\[
\lambda_1^{-\frac{1}{2}} \left( L^t_3 + L^t_4 + L^s_3 + L^s_4 \right) < 1 - \lambda_1^{-1} \left( L^t_1 + L^t_2 + L^s_1 + L^s_2 \right),
\]

where \(\lambda_1\) is the first eigenvalue of \(-\Delta\).
2 Preliminaries

In this section we will establish notation to be used in resting work. Let $E := H^1_0(\Omega) \times H^1_0(\Omega)$ be Hilbert space with inner product

$$(u_1, v_1), (u_2, v_2) >_E := <u_1, u_2 >_{H^1_0(\Omega)} + <v_1, v_2 >_{H^1_0(\Omega)}$$

and norm

$$\|(u, v)\|_E = \left(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\right)^{\frac{1}{2}}.$$

For $w, z \in H^1_0(\Omega)$ fixed, the associating functional from (2) is $I_{wz} : E \to \mathbb{R}$, defined by

$$I_{wz}(u, v) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} F(x, u, v, \nabla w, \nabla z) \, dx. \quad (4)$$

We will consider $I := I_{wz}$, $|u|^p = \left(\int_{\Omega} |u|^p \right)^{\frac{1}{p}}$, $|\Omega| := med\Omega$, $2^* := \frac{2N}{N-2}$ and $\phi_1$ the first eigenfunction of $(-\Delta, H^1_0(\Omega))$ with $|\phi_1|_2 = 1$.

3 Proof of Theorem 1.1

As usual, a weak solution of a problem (2), which is variational, is obtained as a critical point of an associated functional (4). For each $w, z \in H^1_0(\Omega)$ fixed we prove that the functional (4) has the geometry of the Mountain-Pass Theorem, that is, it satisfies the Palais- Smale condition (PS-condition) and finally that the obtained solutions have the uniform bounds stated in the theorem.

**Lemma 3.1** Suppose that $(F_1)$ hold and let $w, z \in H^1_0(\Omega)$. Then there exists positive numbers $\rho$ and $\gamma$, which are independent of $w$ and $z$, such that

$$I(u, v) \geq \gamma, \quad \forall u, v \in H^1_0(\Omega) : \|(u, v)\|_E = \rho. \quad (5)$$

**Proof:** It follows from $(F_1)$ and $(F_3)$ that, given $\epsilon > 0$, there exist a positive constant $C_\epsilon$, independent of $w$ and $z$, such that

$$|F(x, u, v, \nabla w, \nabla z)| \leq C_\epsilon (|u|^r + |v|^n + |u|^{p+1} + |u|^{q+1} + |v|^{p+1} + |v|^{q+1}). \quad (6)$$

From (6) and (4) we obtain

$$I(u, v) \geq \frac{1}{2} \|u\|^2_{H^1_0(\Omega)} + \frac{1}{2} \|v\|^2_{H^1_0(\Omega)} - C_\epsilon \int_{\Omega} (|u|^r + |v|^n + |u|^{p+1} + |u|^{q+1} + |v|^{p+1} + |v|^{q+1}).$$
Since \(r, n, p + 1, q + 1 < \frac{2N}{N-2}\), using Poincaré Inequality and Sobolev Embedding Theorem, we estimate

\[
I(u, v) \geq \|u\|_{H^1_0(\Omega)}^2 \left( \frac{1}{2} - C_\epsilon \|u\|_{H^1_0(\Omega)}^{r-1} - C_\epsilon \|u\|_{H^1_0(\Omega)}^{q-1} \right)
\]

as \(r, n, p + 1, q + 1 > 2\) and (7), the result follows.

\[\square\]

**Lemma 3.2** Suppose \((F_5)\). Let \(w, z \in H^1_0(\Omega)\) and \(\phi_1\) as previously. Then there is a \(T > 0\), independent of \(w\) and \(z\), such that

\[
I(t\phi_1, t\phi_1) \leq 0 \quad \text{for all} \quad t \geq T.
\]

**Proof:** It follows from (4) and \((F_5)\) that

\[
I(t\phi_1, t\phi_1) = t^2 \lambda_1 - \int_\Omega F(x, t\phi_1, t\phi_1, \nabla w, \nabla z)
\]

\[
\leq t^2 \lambda_1 - a_1 |t|^{\alpha} \int_\Omega |\phi_1|^\alpha - a_2 |t|^{\beta} \int_\Omega |\phi_1|^\beta + a_3 |\Omega|.
\]

Since \(\alpha, \beta < \frac{2N}{N-2}\) again by Sobolev Embedding Theorem we obtain

\[
I(t\phi_1, t\phi_1) \leq t^2 \left( \lambda_1 - a_1 S_\alpha^\alpha |t|^{\alpha-\frac{\alpha}{2}} \lambda_1^{\frac{\alpha}{2}} - a_2 S_\beta |t|^{\beta-\frac{\beta}{2}} \lambda_1^{\frac{\beta}{2}} + a_3 |\Omega| \right),
\]

where \(S_\alpha\) is the constant of the embedding of \(H^1_0(\Omega)\) into \(L^\alpha(\Omega)\) and \(S_\beta\) is the constant of the embedding of \(H^1_0(\Omega)\) into \(L^\beta(\Omega)\). For this (8) hold.

\[\square\]

**Lemma 3.3** \(I\) satisfies the \(PS -\) Condition.

**Proof:** Let \((u_n, v_n) \in E\) be a sequence \((PS)\), that is, \((u_n, v_n)\) is such that

\[
I(u_n, v_n) \to A \quad \text{as} \quad n \to +\infty
\]

and

\[
I'(u_n, v_n) \to 0 \quad \text{as} \quad n \to +\infty.
\]

Since \(F_u\) and \(F_v\) are subcritical we have show that \((u_n, v_n)\) is bounded on \(E\). By contradiction, suppose that \(\|(u_n, v_n)\|_E \to +\infty\) as \(n \to +\infty\).
From (4) and \((F_4)\) we have
\[
I(u_n, v_n) \geq \frac{1}{2} \|(u_n, v_n)\|_E^2 - \theta_t \int_\Omega u_n F_u(x, u_n, v_n, \nabla w, \nabla z) - \theta_s \int_\Omega v_n F_v(x, u_n, v_n, \nabla w, \nabla z).
\] (11)

On the other hand, from (10) up to a subsequence we can assume that there exist \((\epsilon_n)_{n \in \mathbb{N}}, \epsilon_n > 0\) such that
\[
|\langle I'(u_n, v_n), (u, v) \rangle| < \epsilon_n, \forall u, v \in H^1_0(\Omega).
\] (12)

Making \(u = u_n\) and \(v = v_n\) we obtain
\[
\|u_n\|_{H^1_0(\Omega)}^2 - \int_\Omega u_n F_u(x, u_n, v_n, \nabla w, \nabla z) \geq -\epsilon_n
\] (13)
and
\[
\|v_n\|_{H^1_0(\Omega)}^2 - \int_\Omega v_n F_v(x, u_n, v_n, \nabla w, \nabla z) \geq -\epsilon_n.
\] (14)

From (11), (13) and (14) it follows
\[
I(u_n, v_n) \geq \left(\frac{1}{2} - \theta_t\right) \|u_n\|_{H^1_0(\Omega)}^2 + \left(\frac{1}{2} - \theta_s\right) \|v_n\|_{H^1_0(\Omega)}^2 - 2\epsilon_n.
\]

Since \(\left(\frac{1}{2} - \theta_t\right) > 0\) and \(\left(\frac{1}{2} - \theta_s\right) > 0\) passing limit as \(n \to +\infty\) we have a contradiction. The lemma is proved.

It follows from Lemmas 3.1, 3.2 and 3.3 and Mountain-Pass Theorem that for each \(w, z \in H^1_0(\Omega)\) the system (2) has a solution \((u_{wz}, v_{wz})\). This solution satisfies
\[
I'_{wz}(u_{wz}, v_{wz}) = 0, \quad I_{wz}(u_{wz}, v_{wz}) = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_{wz}(\gamma(t))
\]
where \(\Gamma = \{\gamma \in C^0([0, 1]; E) : \gamma(0) = 0, \gamma(1) = (T\phi_1, T\phi_1)\}\) for \(T\) as defined in Lemma 3.2. From now on we fix this \(T\).

**Lemma 3.4** Let \(w, z \in H^1_0(\Omega)\). There exist a positive constant \(M_1\), independent of \(w\) and \(z\), such that
\[
\|(u_{wz}, v_{wz})\|_E \geq M_1
\] (15)
for all solutions \((u_{wz}, v_{wz})\) obtained previously.
Proof: Writing \((u, v) = (u_w, v_w)\), where \((u_w, v_w)\) is a solution of (2) we have

\[
\int_{\Omega} |\nabla u|^2 = \int_{\Omega} F_u(x, u, v, \nabla w, \nabla z) u.
\]  

(16)

Therefore, from \((F_2)\) and \((F_3)\) there exist \(\varepsilon > 0\) such that

\[
|F_u(x, u, v, \nabla w, \nabla z)| \leq \varepsilon (|u| + |v|) + C(|u|^p + |v|^q).
\]  

(17)

From (16) and (17) we obtain

\[
\int_{\Omega} |\nabla u|^2 \leq \varepsilon \int_{\Omega} |u|^2 + \varepsilon \int_{\Omega} |u||v| + C \int_{\Omega} |u|^{p+1} + C \int_{\Omega} |v|^q |u|.
\]  

(18)

Analogously,

\[
\int_{\Omega} |\nabla v|^2 \leq \varepsilon \int_{\Omega} |v|^2 + \varepsilon \int_{\Omega} |u||v| + C \int_{\Omega} |u|^p |v| + C \int_{\Omega} |v|^{q+1}.
\]  

(19)

From (18), (19) and Young Inequality, results

\[
\|(u, v)\|^2_E = \|u\|_{H^1_0(\Omega)} + \|v\|_{H^1_0(\Omega)}
\leq 2 \frac{\varepsilon}{\lambda_1} \left(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\right)
\]

\[
+ C \int_{\Omega} |u|^{p+1} + C \int_{\Omega} |v|^q |u| + C \int_{\Omega} |u|^p |v| + C \int_{\Omega} |v|^{q+1}.
\]  

(20)

Since \(u, v \in L^{q+1}(\Omega) \cap L^{p+1}(\Omega)\) and Hölder Inequality we obtain

\[
\int_{\Omega} |v|^q |u| \leq |v|_{q+1}^{q+1} |u|_{q+1}^{q+1}, \quad \int_{\Omega} |u|^p |v| \leq |u|_{p+1}^{p+1} |v|_{p+1}^{p+1}.
\]

Therefore

\[
\left(1 - 2 \frac{\varepsilon}{\lambda_1}\right) \|(u, v)\|^2_E \leq C \left(|u|_{p+1}^{p+1} + |v|_{q+1}^{q+1} + |u|_{p+1}^{p+1} |v|_{p+1} + |v|_{q+1}^{q+1} |u|_{q+1}\right)
\]  

(21)

By Young inequality and (21) we have that

\[
\left(1 - 2 \frac{\varepsilon}{\lambda_1}\right) \left(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\right) \leq C \left(\|u\|^p_{H^1_0(\Omega)} + \|v\|^p_{H^1_0(\Omega)} + \|u\|^q_{H^1_0(\Omega)} + \|v\|^q_{H^1_0(\Omega)}\right).
\]

which implies (15).
Lemma 3.5 Let \( w, z \in H^1_0(\Omega) \). There exists a positive constant \( M_2 \), independent of \( w \) and \( z \), such that

\[
\|(u_{wz}, v_{wz})\|_E \leq M_2
\]  

(22)

for all solutions \((u_{wz}, v_{wz})\) of (2).

Proof: From the inf max characterization of \((u_{wz}, v_{wz})\) obtained previously we obtain

\[
I_{wz}(u_{wz}, v_{wz}) \leq \max_{t \geq 0} I_{w,z}(t\phi_1, t\phi_1).
\]

We estimate \( I_{w,z}(t\phi_1, t\phi_1) \) using \((F_5)\):

\[
I_{w,z}(t\phi_1, t\phi_1) = t^2\|\phi_1\|_{H^1_0(\Omega)}^2 - \int_\Omega F(x, u, v, \nabla w, \nabla z)
\]

\[
\leq t^2\lambda_1 - a_1|t|^\alpha\lambda_1^\frac{\alpha}{2} - a_2|t|^\beta\lambda_1^\frac{\beta}{2} + a_3|\Omega|
\]

\[
= h(t)
\]

whose maximum is achieved at some \( 0 < t_0 \leq T \) and the value \( h(t_0) \) can be taken as \( M_2 \). Clearly it is independent of \( w, z \).

From \((F_4)\) and writing \((u, v) = (u_{wz}, v_{wz})\) we obtain

\[
\frac{1}{2}\left(\|u\|_{H^1_0(\Omega)}^2 + \|v\|_{H^1_0(\Omega)}^2\right) \leq K + \theta_u \int_\Omega F_u(x, u, v, \nabla w, \nabla z)
\]

\[
+ \theta_v \int_\Omega F_v(x, u, v, \nabla w, \nabla z)
\]

\[
= K + \theta_u \|u\|_{H^1_0(\Omega)}^2 + \theta_v \|v\|_{H^1_0(\Omega)}^2,
\]

since \( \theta_u, \theta_v < \frac{1}{2} \) we have that (22).

\[\square\]

Remark 3.1 (On the regularity of the solution of (2)) For each \( w, z \in H^1_0(\Omega) \) we have obtained a weak solution \((u_{wz}, v_{wz})\) of (2). Since \( p < \frac{N+2}{N-2} \), a standard bootstrap argument, using the \( L^p \)-regularity theory, show that \((u_{wz}, v_{wz})\) is, in fact, in \( C^{0,\alpha}(\overline{\Omega}) \). So, if \( w \) and \( z \) are \( C^1 \) using the Schauder regularity theory we obtain \( u_{wz}, v_{wz} \in C^{2,\alpha}(\overline{\Omega}) \) (see [7]).

As a consequence of the Sobolev embedding theorems and Lemma 3.5 we conclude the following

Lemma 3.6 Let \( w, z \in H^1_0(\Omega) \cap C^1(\overline{\Omega}) \). Then there exists positive constants \( \rho_1, \rho_2, \rho_3 \) and \( \rho_4 \), independent of \( w, z \), such that the solution \((u_{wz}, v_{wz})\) of (2) satisfies

\[
\|u_{wz}\|_{C^0} \leq \rho_1, \|v_{wz}\|_{C^0} \leq \rho_2, \|\nabla u_{wz}\|_{C^0} \leq \rho_3 \text{ and } \|\nabla v_{wz}\|_{C^0} \leq \rho_4.
\]
4 Proof of Theorem 1.2

The idea of the proof consists of using Theorem 1.1 in an interactive way, as following. We construct a sequence \( \{ (u_n, v_n) \} \in H_0^1(\Omega) \times H_0^1(\Omega) \) as solutions of

\[
\begin{align*}
- \Delta u_n &= F_u(x, u_n, v_n, \nabla u_{n-1}, \nabla v_{n-1}) \quad \text{em } \Omega \\
- \Delta v_n &= F_v(x, u_n, v_n, \nabla u_{n-1}, \nabla v_{n-1}) \quad \text{em } \Omega \\
u_n = v_n = 0 \quad \text{sobre } \partial \Omega,
\end{align*}
\]

obtained by the mountain pass theorem in Theorem 1.1, starting with an arbitrary \((u_0, v_0) \in (H_0^1(\Omega) \cap C^1(\Omega)) \times (H_0^1(\Omega) \cap C^1(\Omega))\).

By Remark 3.1, we see that

\[
\| u_n \|_{C^0} \leq \rho_1, \quad \| v_n \|_{C^0} \leq \rho_2, \quad \| \nabla u_n \|_{C^0} \leq \rho_3 \quad \text{e} \quad \| \nabla v_n \|_{C^0} \leq \rho_4.
\]

On the other hand, using \((P_n)\) and \((P_{n+1})\), we obtain

\[
\begin{align*}
\int_{\Omega} \nabla u_{n+1} (\nabla u_{n+1} - \nabla u_n) &= \int_{\Omega} F_u(x, u_{n+1}, v_{n+1}, \nabla u_n, \nabla v_n) (u_{n+1} - u_n), \\
\int_{\Omega} \nabla u_n (\nabla u_{n+1} - \nabla u_n) &= \int_{\Omega} F_u(x, u_n, v_n, \nabla u_{n-1}, \nabla v_{n-1}) (u_{n+1} - u_n), \\
\int_{\Omega} \nabla v_{n+1} (\nabla v_{n+1} - \nabla v_n) &= \int_{\Omega} F_v(x, u_{n+1}, v_{n+1}, \nabla u_n, \nabla v_n) (v_{n+1} - v_n)
\end{align*}
\]

and

\[
\int_{\Omega} \nabla v_n (\nabla v_{n+1} - \nabla v_n) = \int_{\Omega} F_v(x, u_n, v_n, \nabla u_{n-1}, \nabla v_{n-1}) (v_{n+1} - v_n)
\]

which gives

\[
\begin{align*}
\| (u_{n+1}, v_{n+1}) - (u_n, v_n) \|^2_E &= \int_{\Omega} [F_u(x, u_{n+1}, v_{n+1}, \nabla u_n, \nabla v_n) - F_u(x, u_n, v_n, \nabla u_n, \nabla v_n)] (u_{n+1} - u_n) \\
&+ \int_{\Omega} [F_u(x, u_n, v_n, \nabla u_{n-1}, \nabla v_n) - F_u(x, u_n, v_n, \nabla u_n, \nabla v_n)] (u_{n+1} - u_n) \\
&+ \int_{\Omega} [F_u(x, u_n, v_n, \nabla u_{n-1}, \nabla v_n) - F_u(x, u_n, v_n, \nabla u_{n-1}, \nabla v_n)] (u_{n+1} - u_n) \\
&+ \int_{\Omega} [F_u(x, u_{n+1}, v_{n+1}, \nabla u_{n-1}, \nabla v_n) - F_u(x, u_{n+1}, v_{n+1}, \nabla u_{n-1}, \nabla v_n)] (u_{n+1} - u_n) \\
&+ \int_{\Omega} [F_v(x, u_{n+1}, v_{n+1}, \nabla u_n, \nabla v_n) - F_v(x, u_{n+1}, v_{n+1}, \nabla u_n, \nabla v_n)] (v_{n+1} - v_n) \\
&+ \int_{\Omega} [F_v(x, u_n, v_{n+1}, \nabla u_n, \nabla v_n) - F_v(x, u_n, v_{n+1}, \nabla u_n, \nabla v_n)] (v_{n+1} - v_n)
\end{align*}
\]
+ \int_{\Omega} [F_v(x, u_n, v_n, \nabla u_n, \nabla v_n) - F_v(x, u_n, v_n, \nabla u_{n-1}, \nabla v_n)] (v_{n+1} - v_n) \\
+ \int_{\Omega} [F_v(x, u_n, v_n, \nabla u_{n-1}, \nabla v_n) - F_v(x, u_n, v_n, \nabla u_{n-1}, \nabla v_{n-1})] (v_{n+1} - v_n) .

We can then estimate the integrals above using hypothesis (F_6):

\begin{align*}
\|(u_{n+1}, v_{n+1}) - (u_n, v_n)\|_E^2 & = L_1^1 |u_{n+1} - u_n|^2 + L_2^1 |u_{n+1} - u_n| |v_{n+1} - v_n|^2 \\
& + L_3^1 |u_{n+1} - u_n| |\nabla u_n - \nabla u_{n-1}| |v_{n+1} - v_n|^2 + L_4^1 |u_{n+1} - u_n| |\nabla v_n - \nabla v_{n-1}| |v_{n+1} - v_n|^2 \\
& + L_5^1 |v_{n+1} - v_n| |\nabla u_n - \nabla u_{n-1}| |v_{n+1} - v_n|^2 (23)
\end{align*}

Next, using Sobolev embedding and Young inequality, we estimate further (23):

\begin{align*}
(1 - \lambda_1^{-1}(L_1^t + L_2^s)) \|(u_{n+1}, v_{n+1}) - (u_n, v_n)\|_E^2 & \leq \frac{\lambda_1^{-1}}{2} (L_2^t + L_3^s) \left( \|u_{n+1} - u_n\|_{H^3_0(\Omega)}^2 + \|v_{n+1} - v_n\|_{H^3_0(\Omega)}^2 \right) \\
& + \lambda_1^{-\frac{t}{2}} L_3^t \|u_n - u_{n-1}\|_{H^3_0(\Omega)}^2 \|u_{n+1} - u_n\|_{H^3_0(\Omega)}^2 \\
& + \lambda_1^{-\frac{t}{2}} L_4^t \|v_n - v_{n-1}\|_{H^3_0(\Omega)}^2 \|u_{n+1} - u_n\|_{H^3_0(\Omega)}^2 \\
& + \lambda_1^{-\frac{t}{2}} L_3^s \|u_n - u_{n-1}\|_{H^3_0(\Omega)}^2 \|v_{n+1} - v_n\|_{H^3_0(\Omega)}^2 \\
& + \lambda_1^{-\frac{t}{2}} L_4^s \|v_n - v_{n-1}\|_{H^3_0(\Omega)}^2 \|v_{n+1} - v_n\|_{H^3_0(\Omega)}^2
\end{align*}

which results

\[ \|(u_{n+1}, v_{n+1}) - (u_n, v_n)\|_E \leq \frac{\lambda_1^{-\frac{t}{2}} (L_3^t + L_4^t + L_3^s + L_4^s)}{1 - \lambda_1^{-1}(L_1^t + L_2^s + L_3^t + L_4^s)} \|(u_n, v_n) - (u_{n-1}, v_{n-1})\|_E .\]

Since \( \lambda_1^{-\frac{t}{2}} \frac{(L_3^t + L_4^t + L_3^s + L_4^s)}{L_2^s} < 1 \) strongly converges in \( E = H^3_0(\Omega) \times H^3_0(\Omega) \), it easily proving that \((u, v)\) is a Cauchy sequence in \( E = H^3_0(\Omega) \times H^3_0(\Omega) \).

Since \( \|(u_n, v_n)\|_E \geq M_1 \) for all \( n \) (see Lemma 3.4), it follows that \((u, v) \neq 0\). In this way we obtain a nontrivial solution of 1.

**Example 4.1** We consider \( F : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) defined by

\[ F(x, t, s, \xi, \eta) = b_1 |t|^{\alpha} |s|^\beta g(\xi) h(\eta) \]

where \( b_1 \) is a positive constant, \( 2 < \alpha, \beta < \frac{N+1}{2} \), \( g, h \in C^1 \) are bounded, \( g, h \geq b_2 > 0 \) and \( N = 3 \).
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We have that

\[ F_t(x, t, s, \xi, \eta) = \alpha b_1 |t|^{\alpha-2}t|s|^\beta g(\xi)h(\eta) \] (24)

and

\[ F_s(x, t, s, \xi, \eta) = \beta b_1 |t|^\alpha |s|^\beta-2 s g(\xi)h(\eta) \] (25)

for \(2 < \alpha, \beta\). We conclude \(F\) is \(C^1\). By (24) and (25) we have \(F_t, F_s\) are locally Lipschitz continuous for any \(b_1 > 0\) small. This verifies \((F_0)\).

The \((F_1)\)-condition \((i)\) and \((ii)\) are obtained from Young Inequality and using the fact that \(g, h\) are bounded, so

\[ |F(x, t, s, \xi, \eta)| \leq C(|t|^{2\alpha} + |s|^{2\beta}). \]

Clearly \((F_2)\) is hold. \((F_3)\) is analogous to \((F_1)\) and \((F_4)\) is hold for small values \(b_1 > 0\).

For \((F_5)\) we break in two parts

(I) For \(|t|, |s| \geq 1\) we have

\[ F(x, t, s, \xi, \eta) \geq \frac{1}{2}b_1 |t|^\alpha b_2^2 + \frac{1}{2}b_1 |s|^\beta b_2^2. \] (26)

(II) For \(|t|, |s| \leq 1\) there exist \(m > 0\) such that

\[ |F(x, t, s, \xi, \eta)| \leq m. \] (27)

From (26) and (27) results \((F_5)\).

The \((F_6)\)-conditions are obtained from (24) and (25) for small values \(b_1 > 0\).

5 Multiplicity of Nontrivial Solutions

In this section we treated multiplicity of solutions for the problem (1). For this, we suppose that

\((F'_2)\)

\((i)\) \( \lim_{t \to 0} \frac{F_t(x, t, s, \xi, \eta)}{t} = 0, \lim_{t \to 0} \frac{F_s(x, t, s, \xi, \eta)}{t} = 0 \) uniformly in \(x \in \Omega, \xi, \eta \in \mathbb{R}^N\) and locally uniformly in \(s\); also this limit is hold for all \(s \in \mathbb{R}\), and

\((ii)\) \( \lim_{s \to 0} \frac{F_t(x, t, s, \xi, \eta)}{s} = 0, \lim_{s \to 0} \frac{F_s(x, t, s, \xi, \eta)}{s} = 0 \) uniformly in \(x \in \Omega, \xi, \eta \in \mathbb{R}^N\) and locally uniformly in \(s\); also this limit is hold for all \(t \in \mathbb{R}\).
From \((F'_2)\)-condition we obtain
\[
\frac{\partial F}{\partial u}(x, 0, v) = \frac{\partial F}{\partial v}(x, 0, v) = 0, \quad \forall x \in \Omega, \ v \in \mathbb{R}; \quad (28)
\]
and
\[
\frac{\partial F}{\partial u}(x, u, 0) = \frac{\partial F}{\partial u}(x, u, 0) = 0, \quad \forall x \in \Omega, \ v \in \mathbb{R}. \quad (29)
\]

Define the following functions
\[
F^v_+ (-) = \begin{cases} 
F(x, u, v, \nabla u, \nabla v); & u > 0 (u < 0), \ v > 0 (v < 0) \\
0; & \text{other case}
\end{cases}
\]

We have that the four functions: \(F^v_+, F^v_- \) are \(C^1\) by \((F'_2)\)-condition.

**Theorem 5.1** Suppose that \((F_0), (F_1), (F'_2), (F_3) - (F'_6)\) are holds. Then the problem \((1)\) has at least four nontrivial solutions, since
\[
\lambda_1^{-\frac{1}{2}} \left( L_3^t + L_4^t + L_3^s + L_4^s \right) < 1 - \lambda_1^{-1} \left( L_1^t + L_2^t + L_1^s + L_2^s \right).
\]

**Proof:** We consider the function \(F^v_+\). It is clearly that \(F^v_+\) satisfies \((F_4)\) and \((F'_5)\) for all \(t \geq 0\). The Palais-Smale Condition is hold the same form as Lemma 3.3. Thus, the functional \(F^v_+\) satisfies the mountain pass geometry. By Theorem (1.2) we have that

\[
\begin{cases}
-\Delta u = (F^v_+(x, u, v, \nabla u, \nabla v)) \quad \text{in} \quad \Omega \\
-\Delta v = (F^v_+(x, u, v, \nabla u, \nabla v)) \quad \text{in} \quad \Omega \\
u = v = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\]

has a solution \((u, v) \in E\).

We multiplies the equations of (30) for \(u^-\) and \(v^-\), respectively, we obtain
\[
\int_{\Omega} |\nabla u^-|^2 = \int_{\Omega} (F^v_+(x, u, v, \nabla u, \nabla v)) u^- = 0
\]
and
\[
\int_{\Omega} |\nabla v^-|^2 = \int_{\Omega} (F^v_+(x, u, v, \nabla u, \nabla v)) v^- = 0.
\]
Thus \(u \geq 0\) and \(v \geq 0\) and \((u, v)\) is a nontrivial solution of (1).

Now, if \(u \neq 0\) then \(v \neq 0\), conversely.

In fact, if \(v \equiv 0\) then by \(\frac{\partial F}{\partial u}(x, u, 0, \nabla u, 0) = 0\) we have that
\[
\begin{cases}
-\Delta u = \frac{\partial F}{\partial u}(x, u, 0, \nabla u, 0) = 0 \quad \text{in} \quad \Omega \\
u = 0, \quad \text{on} \quad \partial \Omega.
\end{cases}
\]
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has only trivial solution.

Therefore, \( u > 0 \) and \( v > 0 \).

Analogously, we obtain solutions for \( F_{u-}^v, F_{v-}^u \) and \( F_{u-}^v \). We observe that the four solutions are different.

Example 5.1 Let

\[
F(x, t, s, \xi, \eta) = b_1 g(\xi) h(\eta) \left( |t|^\alpha s^4 + |s|^\beta t^4 + 1 \right).
\]

where \( b_1 \) is a positive constant, \( 2 < \alpha, \beta < \frac{2N}{N-2} - 1 \), \( g, h \in C^1 \) are bounded, \( g, h \geq b_2 > 0 \) and \( N = 3, 4, 5 \).

We observe that

\[
F_t(x, t, s, \xi, \eta) = b_1 g(\xi) h(\eta) \left( \alpha |t|^\alpha |s^4 + 1| + 4|s|^\beta \frac{t^4}{(t^4 + 1)^2} \right)
\]

and

\[
F_s(x, t, s, \xi, \eta) = b_1 g(\xi) h(\eta) \left( 4|t|^\alpha \frac{s^3}{(s^4 + 1)^2} + \beta |s|^\beta s^2 \frac{t^4}{(t^4 + 1)} \right).
\]

Thus \( F \) satisfies \((F_0), (F_1), (F_2), (F_3) - (F_6)\).

References


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