Reverse Order Law for \( \{1,3\}\)-Inverse of a Two-Operator Product\(^1\)

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Abstract

In this paper, the reverse order law for \( \{1,3\}\)-inverse of a two-operator product is mainly investigated by making full use of block-operator matrix technique. The necessary and sufficient conditions for 
\[ B\{1,3\}A\{1,3\} = (AB)\{1,3\} \] and 
\[ B\{1,4\}A\{1,4\} = (AB)\{1,4\} \]
are presented when all ranges \( R(A), R(B) \) and \( R(AB) \) are closed.

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§1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional Hilbert spaces and $\mathcal{B}(\mathcal{K}, \mathcal{H})$ be the set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ and abbreviate $\mathcal{B}(\mathcal{K}, \mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $N(A)$ and $R(A)$ are the null space and the range of $A$, respectively. And a generalized inverse of $A$ is an operator $G \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which satisfies some of the following four equations, which is said to be the Moore-Penrose conditions:

\begin{align}
(1) \ AGA &= A, \quad (2) \ GAG = G, \quad (3) \ (AG)^* = AG, \quad (4) \ (GA)^* = GA.
\end{align}

Let $A\{i, j, \ldots, l\}$ denote the set of operators $G \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which satisfy equation $(i), (j), \ldots, (l)$ from among the above equations. An operator $G \in A\{i, j, \ldots, l\}$ is called an $\{i, j, \ldots, l\}$-inverse of $A$, and also denoted by $A^{(ij\ldots l)}$. The unique $\{1, 2, 3, 4\}$-inverse of $A$ is denoted by $A^+$, which is called the Moore-Penrose inverse of $A$. As is well known, $A$ is the Moore-Penrose invertible if and only if $R(A)$ is closed.

In recent years, considerable attention has been paid to the reverse order law for the generalized inverses of multiple-matrix and multiple-operator products, and many interesting results have been obtained, see\cite{1-14}. It is a classical result of Greville in \cite{8} that $(AB)^+ = B^+A^+$ if and only if $R(A^*AB) \subset R(B)$ and $R(BB^*A^*) \subset R(A^*)$ for any complex matrices $A$ and $B$. This result was extended for linear bounded operators on Hilbert spaces by Bouldin \cite{1} and Izumino \cite{9}. 2011, Deng considered the reverse order law for the group inverse in \cite{3}. In particular, reverse order laws for $\{1, 3\}$- and $\{1, 4\}$-inverses were considered on matrix algebra by Wei and Guo \cite{13} who obtained the equivalent conditions for $B\{1, 3\}A\{1, 3\} \subset (AB)\{1, 3\}$, $AB\{1, 3\} \subset B\{1, 3\}A\{1, 3\}$ and $B\{1, 3\}A\{1, 3\} = (AB)\{1, 3\}$ by applying product singular value decomposition of Matrices. Djordjević also offered some new necessary and sufficient conditions for the inclusion $AB\{1, 3\} \subset B\{1, 3\}A\{1, 3\}$ \cite{7}. In \cite{2}, the author considered reverse order laws for $\{1, 3\}$, $\{1, 4\}$ and $\{1, 2, 3\}$-inverses on $C^*$-algebra and gave some equivalent conditions for $B\{1, 3\}A\{1, 3\} \subset AB\{1, 3\}$ and $B\{1, 4\}A\{1, 4\} \subset AB\{1, 4\}$ under certain conditions.

In this article, the reverse order law for $\{1, 3\}$-inverse of bounded operators on infinite dimensional Hilbert space is mainly investigated. Using the block-operator matrix technique, we give a necessary and sufficient condition for $B\{1, 3\}A\{1, 3\} = AB\{1, 3\}$ (and $B\{1, 4\}A\{1, 4\} = AB\{1, 4\}$) when $R(A)$, $R(B)$, $R(AB)$ are closed. Moreover, we obtain that $B\{1, 3\}A\{1, 3\} \subset AB\{1, 3\}$ and $B\{1, 3\}A\{1, 3\} = AB\{1, 3\}$ are equivalent.
This paper is organized as follows. Section 2 includes auxiliary results and main result of the paper. In Section 3, the proof of main result is presented. Finally, two corollaries are given by the main result.

§2 Main result

Let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) with closed range. It is well known that \( A \) has the matrix form

\[
A = \begin{pmatrix}
A_1 & 0 \\
0 & 0
\end{pmatrix} : \begin{pmatrix}
R(A^*) \\
N(A)
\end{pmatrix} \rightarrow \begin{pmatrix}
R(A) \\
N(A^*)
\end{pmatrix}
\]  

(2.1)

under the orthogonal decompositions \( \mathcal{H} = R(A^*) \oplus N(A) \) and \( \mathcal{K} = R(A) \oplus N(A^*) \), respectively, where \( A_1 \in \mathcal{B}(R(A^*) \text{ and } R(A)) \) is invertible. The Moore-Penrose inverse \( A^+ \) of \( A \) has the following matrix form:

\[
A^+ = \begin{pmatrix}
A_1^{-1} & 0 \\
0 & 0
\end{pmatrix} : \begin{pmatrix}
R(A) \\
N(A^*)
\end{pmatrix} \rightarrow \begin{pmatrix}
R(A^*) \\
N(A)
\end{pmatrix}
\]

(2.2)

The \( \{1, 3\} \)-inverse has also similarly matrix form. To obtain main result, we will discuss the representation of \( A^{(13)} \) of an operator \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) under certain Hilbert space decompositions firstly.

**Lemma 2.1.** Let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) with closed range. If \( A \) has the matrix form

\[
A = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
0 & A_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3
\end{bmatrix}
\]  

(2.2)

with respect to the orthogonal decompositions \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \) and \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \), respectively, such that \( A_{11} \) is invertible and \( A_{22} \) is surjective, then there are some operators \( G_{ji} \in \mathcal{B}(\mathcal{K}_i, \mathcal{H}_j) \), \( i, j = 1, 2 \), satisfy

\[
\begin{align*}
G_{11} &= A_{11}^{-1} - A_{11}^{-1}A_{12}G_{12}, \\
G_{12} &= -A_{11}^{-1}A_{12}G_{22}, \\
G_{13} &= -A_{11}^{-1}A_{12}G_{23}, \\
R(G_{21}) &\subset N(A_{22}), \\
G_{22} &\in A_{22}\{1\}, \\
R(G_{23}) &\subset N(A_{22}),
\end{align*}
\]  

(2.3)
such that \( A^{(13)} \) has the matrix form

\[
A^{(13)} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{bmatrix}
\] (2.4)

for any \( G_{3i} \in \mathcal{B}(\mathcal{K}_i, \mathcal{H}_3) \), \( i = 1, 2, 3 \).

**Proof.** Since the range of \( A \) is closed, there exists an operator \( G \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( G \in A\{1, 3\} \). Suppose \( G \) has the matrix form

\[
G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{bmatrix}
\] (2.5)

with respect to the orthogonal decompositions \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \) and \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \) respectively. It follows from matrix forms (2.2) and (2.5) that

\[
AG = \begin{pmatrix} A_{11}G_{11} + A_{12}G_{21} & A_{11}G_{12} + A_{12}G_{22} & A_{11}G_{13} + A_{12}G_{23} \\ A_{21}G_{21} & A_{22}G_{22} & A_{22}G_{23} \\ 0 & 0 & 0 \end{pmatrix}
\] (2.6)

and

\[
AGA = \begin{pmatrix} (A_{11}G_{11} + A_{12}G_{21})A_{11} & (A_{11}G_{11} + A_{12}G_{21})A_{12} + (A_{11}G_{12} + A_{12}G_{22})A_{22} & 0 \\ A_{21}G_{21}A_{11} & A_{22}G_{21}A_{12} + A_{22}G_{22}A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (2.7)

By the formula (2.7) and the Moore-Penrose condition (1) \( AGA = A \), we obtain that

\[
\begin{cases} 
(A_{11}G_{11} + A_{12}G_{21})A_{11} = A_{11}, \\
(A_{11}G_{11} + A_{12}G_{21})A_{12} + (A_{11}G_{12} + A_{12}G_{22})A_{22} = A_{12}, \\
A_{22}G_{21}A_{11} = 0, \\
A_{22}G_{21}A_{12} + A_{22}G_{22}A_{22} = A_{22}. 
\end{cases}
\]

Therefore,

\[
\begin{cases} 
G_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}G_{21}, \\
G_{12} = -A_{11}^{-1}A_{12}G_{22}, \\
R(G_{21}) \subset N(A_{22}), \\
G_{22} \in A_{22}\{1\},
\end{cases}
\]
since $A_{11}$ is invertible. Combining the formula (2.6) with Moore-Penrose condition (3), we have $A_{11}G_{13} + A_{12}G_{23} = 0$ and $A_{22}G_{23} = 0$. Hence $G_{13} = -A_{11}^{-1}A_{12}G_{23}$ and $R(G_{23}) \subset N(A_{22})$. It is easy to derive from the above computation that $G_{31}, G_{32}, G_{33}$ can be arbitrary linear bounded operators.

This ended the proof of Lemma 2.1. □

**Corollary 2.2.** if $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ with closed range and has the matrix form (2.1), then

$$A^{(13)} = \begin{bmatrix} A_{11}^{-1} & 0 \\ G_{21} & G_{22} \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix}$$  \hspace{1cm} (2.8)

for any $G_{21} \in \mathcal{B}(R(A), N(A))$ and $G_{22} \in \mathcal{B}(N(A^*), N(A))$.

In [2], the author gave an equivalent condition of $B \{1,3\}A \{1,3\} \subset AB \{1,3\}$ under certain conditions of operators $A, B, AB$ and $A - ABB^+$ are regular, which equivalent to all ranges of $A, B, AB$ and $A - ABB^+$ are closed since $A$ is regular if and only if $A^+$ exists. Here, the equivalent condition for $B \{1,3\}A \{1,3\} = AB \{1,3\}$ will be presented when $R(A), R(B)$ and $R(AB)$ are closed. Thus, the main result is stated as follows.

**Theorem 2.3.** Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that all ranges $R(A), R(B)$ and $R(AB)$ are closed. Then $B \{1,3\}A \{1,3\} = AB \{1,3\}$ if and only if $R(A^*AB) = R(B) \oplus (R(B) \cap N(A))$.

§3 The proof of main result

Now, we give the proof of Theorem 2.3. Firstly, some notations are listed. Set

$$\begin{cases} \mathcal{H}_1 = R(B) \cap N(A), \\ \mathcal{H}_2 = R(B) \oplus (R(B) \cap N(A)), \\ \mathcal{H}_3 = N(B^*) \cap N(A), \\ \mathcal{H}_4 = N(B^*) \oplus (N(B^*) \cap N(A)), \end{cases}$$

$$\begin{cases} \mathcal{K}_1 = R(AB), \\ \mathcal{K}_2 = R(A) \ominus R(AB), \text{ and} \end{cases} \begin{cases} \mathcal{J}_1 = B^+\mathcal{H}_1, \\ \mathcal{J}_2 = R(B^*) \ominus B^+\mathcal{H}_1, \\ \mathcal{J}_3 = N(B), \end{cases}$$

respectively, where $B^+$ is the Moore-Penrose inverse of $B$. Then it is known that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$. In particular, it is
elementary that $A$ and $B$ have the following matrix forms

$$A = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \\ \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \end{bmatrix} ) , \quad (3.1)$$

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \end{bmatrix} : \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \end{bmatrix} , \quad (3.2)$$

This implies that

$$AB = \begin{bmatrix} 0 & A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \end{bmatrix} : \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \end{bmatrix} ) . \quad (3.3)$$

$$A^*AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{12}^*A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & A_{14}^*A_{14}B_{22} & 0 \\ \end{bmatrix} : \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \end{bmatrix} . \quad (3.4)$$

Next, we divide the proof into three cases. 

**Case 1.** $\mathcal{H}_2 \neq \{0\}$ and $\mathcal{H}_1 \neq \{0\}$. In this case, operators $A_{12}$, $B_{11}$, $B_{22}$ which in matrix forms (3.1) and (3.2) are invertible, and $A_{24}$ is surjective. By Lemma 2.1 and Corollary 2.2, $\{1, 3\}$-inverses $B^{(13)}$ of $B$ and $A^{(13)}$ of $A$ have the matrix forms as follows,

$$B^{(13)} = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \end{bmatrix} \to \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \end{bmatrix}$$

and

$$A^{(13)} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \\ G_{41} & G_{42} & G_{43} \\ \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \end{bmatrix} \quad (3.5)$$
Reverse order law for \( \{1, 3\} \)-inverse of a two-operator product respectively, where \( F_{3k}, k \in \{1, 2, 3, 4\} \) and \( G_{ij}, i \in \{1, 3\}, j \in \{1, 2, 3\} \) are arbitrary, and

\[
\begin{align*}
G_{22} &= -A_{12}^{-1} A_{14} G_{42}, \\
G_{23} &= -A_{12}^{-1} A_{14} G_{43}, \\
R(G_{43}) &\subseteq N(A_{24}), \\
G_{42} &\in A_{24}^{(1)}, \\
G_{21} &= A_{12}^{-1} - A_{12}^{-1} A_{14} G_{41}.
\end{align*}
\] (3.6)

Denote

\[
\begin{align*}
P_{11} &= B_{11}^{-1} G_{11} - B_{11}^{-1} B_{12} B_{22}^{-1} G_{21}, \\
P_{12} &= B_{11}^{-1} G_{12} - B_{11}^{-1} B_{12} B_{22}^{-1} G_{22}, \\
P_{13} &= B_{11}^{-1} G_{13} + B_{11}^{-1} B_{12} B_{22}^{-1} G_{23}
\end{align*}
\]

and

\[
P_{31} = \sum_{i=1}^{4} F_{3i} G_{i1}, \quad P_{32} = \sum_{i=1}^{4} F_{3i} G_{i2}, \quad P_{33} = \sum_{i=1}^{4} F_{3i} G_{i3}.
\]

Then it follows from matrix forms of \( A^{(13)} \) and \( B^{(13)} \) that

\[
B^{(13)} A^{(13)} = \begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
B_{22}^{-1} G_{21} & B_{22}^{-1} G_{22} & B_{22}^{-1} G_{23} \\
P_{31} & P_{32} & P_{33}
\end{bmatrix}.
\] (3.7)

On the other hand, using Corollary 2.2 again, we get that

\[
(AB)^{(13)} = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
B_{22}^{-1} A_{12}^{-1} & 0 & 0 \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\] (3.8)

for any \( M_{ij} \in B(K_j, J_i), i \in \{1, 3\}, j \in \{1, 2, 3\} \).

Assume that \( B\{1, 3\} A\{1, 3\} = AB\{1, 3\} \), it follows from (3.7) and (3.8) that

\[
G_{21} = A_{12}^{-1}, G_{22} = 0, G_{23} = 0.
\]

Combining \( G_{23} = 0 \) with the equality \( G_{23} = -A_{12}^{-1} A_{14} G_{43} \) in (3.6), we can obtain that \( A_{14} G_{43} = 0 \), and consequently \( R(G_{43}) \subseteq N(A_{14}) \). In deed, \( A_{14} = 0 \).

To see it, we consider two cases \( K_2 \neq \{0\} \) and \( K_2 = \{0\} \), respectively. If \( K_2 \neq \{0\} \), then \( A_{24} \neq 0 \). From the relation \( R(G_{43}) \subseteq N(A_{24}) \) in (3.6), we can infer that there exists an non-zero operator \( G_{43} \) if \( N(A_{24}) \neq \{0\} \). This shows that \( \{0\} \neq R(G_{43}) \subseteq N(A_{14}) \cap N(A_{24}) \). It is a contradiction with definitions.
of \( \mathcal{H}_3 \) and \( \mathcal{H}_4 \). So \( \mathcal{N}_{24} = \{0\} \) and then \( A_{24} \) is invertible. From relations in (3.6), it follows that \( G_{42} = A_{24}^{-1} \) and \( G_{22} = -A_{12}^{-1}A_{14}A_{24}^{-1} \). Therefore \( A_{14} = 0 \) since \( G_{22} = 0 \). If \( \mathcal{K}_2 = \{0\} \), then \( A_{24} = 0 \) and \( \mathcal{N}(A_{24}) = \mathcal{H}_4 \). Combining with \( R(G_{43}) \subset \mathcal{N}(A_{24}) \) in (3.6) and \( R(G_{43}) \subset \mathcal{N}(A_{14}) \), we get that \( A_{14} = 0 \). So \( R(A^*AB) = \mathcal{H}_2 = R(B) \oplus (R(B) \cap \mathcal{N}(A)) \) since \( A_{12} \) and \( B_{22} \) are invertible.

On the contrary, suppose that \( R(A^*AB) = R(B) \oplus (R(B) \cap \mathcal{N}(A)) \). It is easy to prove that \( A_{14}^*A_{12}B_{22} = 0 \) from the formula (3.4). This shows \( A_{14} = 0 \) since \( A_{12} \) and \( B_{22} \) are invertible. Then \( G_{21} = A_{12}^{-1}, G_{22} = 0, G_{23} = 0 \) in \( A^{(13)} \). Combining formulae (3.7) with (3.8), we infer that \( B\{1, 3\}A\{1, 3\} = AB\{1, 3\} \) by the arbitrary of \( G_{1i}, M_{1i}, M_{3i}, i \in \{1, 2, 3\} \).

**Case 2.** \( \mathcal{H}_2 \neq \{0\} \) and \( \mathcal{H}_1 = \{0\} \). Obviously, \( \mathcal{J}_1 = \{0\} \), consequently,

\[
A = \begin{bmatrix} A_{12} & 0 & A_{14} \\ 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{bmatrix},
\]

\[
B = \begin{bmatrix} B_{22} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{J}_2 \\ \mathcal{J}_3 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}.
\]

Similar to Case 1, it is elementary that \( B\{1, 3\}A\{1, 3\} = AB\{1, 3\} \) if and only if \( R(A^*AB) = R(B) \oplus (R(B) \cap \mathcal{N}(A)) \).

**Case 3.** \( \mathcal{H}_2 = \{0\} \). In such a case, \( AB = 0, \mathcal{K}_1 = \{0\} \) and \( \mathcal{J}_2 = \{0\} \). It is not difficult to see that formulae (3.1) and (3.2) become

\[
A = \begin{bmatrix} 0 & 0 & A_{14} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{K}_3 \end{bmatrix},
\]

\[
B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_3 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
\]

where \( A_{14} \) and \( B_{11} \) are invertible. From Corollary 2.2, we have

\[
A^{(13)} = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ A_{14}^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{K}_3 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}.
\]
Reverse order law for \(\{1,3\}\)-inverse of a two-operator product

\[
B^{(13)} = \begin{bmatrix}
B_{11}^{-1} & 0 & 0 \\
F_{21} & F_{22} & F_{23}
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
J_1 \\
J_3
\end{bmatrix},
\]

where \(G_{ij}, i, j \in \{1,2\}\) and \(F_{2k}, k \in \{1,2,3\}\) are arbitrarily. By direct computation, it is clearly that

\[
B^{(13)}A^{(13)} = \begin{bmatrix}
B_{11}^{-1}G_{11} & B_{11}^{-1}G_{12} \\
F_{21}G_{11} + F_{22}G_{21} + F_{23}A_{14}^{-1} & F_{21}G_{12} + F_{22}G_{22}
\end{bmatrix}.
\]

This shows that \(B\{1,3\}A\{1,3\} = B(\mathcal{H}) = AB\{1,3\}\). Moreover, \(R(A^*AB) = R(B) \ominus (R(B) \cap N(A)) = \{0\}\) always holds in this case.

The proof is completed. \(\square\)

§4 Two corollaries

By the proof of Theorem 2.3, we have

**Corollary 4.1** Let \(A \in B(\mathcal{H},\mathcal{K})\) and \(B \in B(\mathcal{K},\mathcal{H})\) such that all ranges \(R(A), R(B)\) and \(R(AB)\) are closed. Then

1. \(AB\{1,3\} \subset B\{1,3\}A\{1,3\}\) always holds.
2. \(B\{1,3\}A\{1,3\} \subset AB\{1,3\}\) and \(B\{1,3\}A\{1,3\} = AB\{1,3\}\) are equivalent.

From the relationship of \(\{1,3\}\)-inverse and \(\{1,4\}\)-inverse, we can obtain the following result without proof.

**Corollary 4.2** Let \(A \in B(\mathcal{H},\mathcal{K})\) and \(B \in B(\mathcal{K},\mathcal{H})\) such that all ranges \(R(A), R(B)\) and \(R(AB)\) are closed. Then

1. \(B\{1,4\}A\{1,4\} = AB\{1,4\}\) if and only if \(R(BB^*A^*) = R(A^*) \ominus (R(A^*) \cap N(B^*))\).
2. \(AB\{1,4\} \subset B\{1,4\}A\{1,4\}\) always holds.
3. \(B\{1,4\}A\{1,4\} \subset AB\{1,4\}\) and \(B\{1,4\}A\{1,4\} = AB\{1,4\}\) are equivalent.

References


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