Proposition of Analyses in a Vertical Multi-table and Analyses of Links between Two Vertical Multi-tables: Methods (sVMA and sOVMA) and (sCIA3 and sOCIA3)

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Abstract
The co-inertia analysis of two tables leads to study the common structure between these groups of variables. But when the tables are repeated in time or space, to study the stability of the relationship between these two groups of variables, STATICO method seeks to synthesize this relationship. Recently in the same context as STATICO, some methods have been proposed to solve this fundamental question that is the study.
of stability of the relationship between these two groups of variables, such as canonical correlation analysis between two vertical multi-tables and the co-inertia analyses between two vertical multi-tables. In this article, we first propose an approach to determine the link between tables in a vertical multi-tables, and then we propose an extension of this method to the case of the relationship between two vertical multi-tables leading to another co-inertia analysis between two vertical multi-tables. An example data set is analyzed to show the interest of the method and the results are compared to other methods.

Keywords: Co-inertia analysis, Common component and specific weight analysis, STATIS, STATICO

1 Introduction

Simultaneous analysis of M pairs of triplets tables \( \{(X_m, Q_m, D_m), (Y_m, R_m, D_m)\} \) was thoroughly synthesized in STATICO method (Simier et al. (1999)). This method is the combination of the co-inertia analysis of two tables (Chessel and Mercier (1993) and Dolédec and Chessel (1994)) and STATIS method (L’Hermier des Plantes (1976), Lavit (1988) and Lavit et al. (1994)). Other methods concerning the context of STATICO method have recently been proposed, such as canonical correlation analysis between two vertical multi-tables (Kissita et al. (2013)), co-inertia analyses 1 and 2 between two vertical multi-tables respectively of Niéré et al. (2013) and Kissita et al. (2013). The last methods proceed by successive steps to find solutions and are founded on the maximization of a criterion, which is not the case STATICO. A simultaneous approach in this context has also been studied in Kissita et al. (2013).

In this article, we first propose a method to analyze the links of tables in a vertical multi-table. Furthermore, we propose a method for analyzing M pairs of triplets tables that is different from those mentioned above. This is a new co-inertia analysis between two vertical multi-tables based on the maximization of a criterion. The objective of this method is to determine for each vertical multi-table co-inertia axes that are common to all tables which are similar to the common components, as it was the case with Hanafi and Quinnari (2008). It is from these axes we determine the specific weights that are projected inertia on the one hand and the squared coefficients of correlation between synthetic components every pair of tables on the other hand. These quantities enable to study the stability of the relationship between these two vertical multi-tables. This method is called co-inertia analysis between two successive vertical multi-table type 3.

The article consists of four sections. In section 2, before proposing the co-inertia analysis between these two vertical multi-tables that is one of the contributions in this article, we first propose a new approach to synthesize the
links between tables in a vertical multi-table analysis we call successive vertical multi-table analysis acronym sVMA. The co-inertia analysis between two vertical multi-tables developed here is in fact a generalization of this method. In section 3, we make some comments and establish the links with other methods in this context. The last section is established to the application of this method to the data already analyzed by Pegaz-Maucet (1980), by Hanafi (1997), by Blanc et al. (1998) and the new methods that have been mentioned above, this being so purposefully of comparison of methods. These ecological data were measured on Méaudret stream of France.

Data and notations that are used in this article are as follows:

We denote by \( D_m \) the diagonal metric weights of \( n_m \) individuals defined in \( \mathbb{R}^{n_m} \) for \( m = 1, \cdots, M \), \( Q \) and \( R \) are respectively two metrics define in \( \mathbb{R}^p \) and \( \mathbb{R}^q \).

Considering \( M \) pairs of triplets tables \( \{(X_m, Q, D_m), (Y_m, R, D_m)\} \) for \( m = 1, \cdots, M \), \( X_m \). \( X_m \) is a table of dimension \( (n_m, p) \) which consists of \( p \) variables measured on \( n_m \) individuals, \( Y_m \) is a table of dimension \( (n_m, q) \) which consists of \( q \) variables measured on the same \( n_m \) individuals such as \( X_m \).

Let us consider \( X = [X'_1 | X'_2 | \cdots | X'_M]' \) and \( Y = [Y'_1 | Y'_2 | \cdots | Y'_M]' \) two multi-blocks tables respectively associated with group 1 and group 2. Thus, we obtain two studies \( M \)-vertical tables. The first study (group 1) \((X, Q, D)\) is partitioned in \( M \) triplets \((X_m, Q, D_m)\) for \( m = 1, \cdots, M \), the second study (group 2) \((Y, R, D)\) is partitioned in \( M \) triplets \((Y_m, R, D_m)\) for \( m = 1, \cdots, M \), with \( D = diag(D_m, 1 \leq m \leq M) \) diagonal bloc matrix weight in \( \mathbb{R}^n \) with \( n = \sum_{m=1}^{M} n_m \).

We suppose that \( X_m \) and \( Y_m \) for \( m = 1, \cdots, M \) are centred and eventually reduced. We denote by \( V_{X_m Y_m} = X'_m D_m Y_m \) the inter-covariance matrix between tables \( X_m \) and \( Y_m \) at stage \( m \) for \( m = 1, \cdots, M \). We note that \( V_{Y_m X_m} = Y'_m D_m X_m = V'_{X_m Y_m} \).

\( V_{X_m} = X'_m D_m X_m \) denotes the covariance matrix of \( X_m \) for \( m = 1, \cdots, M \). Let us design \( A' \) the transpose of matrix \( A \).

\( s = 1, \cdots, r \) when \( r \) is the rank of \( X_m \) in the case of vertical multi-table and or matrices \( V_{X_m Y_m} = X'_m D_m Y_m \) in the case of two vertical multi-tables.

Methods have been programmed in Scilab language and the graphics are implemented in R language.

## 2 Methods

In this section, we first propose a new method of analysis of a vertical multi-table is similar to Niéré’s (2013) Principal component analysis in a vertical multi-table.
multi-table and also to Quanari and Hanafi’s (2008) Common component and specific weight analysis. Besides, we generalize this method to the case of two multi-tables for obtaining the co-inertia analysis between two vertical multi-tables.

2.1 The successive Vertical multi-tables analysis: sVMA method

When we have a single vertical multi-table, to compare or to analyze the subtables each other of a vertical multi-table \( X \), we determine the vector \( u \) of \( \mathbb{R}^p \) that is common with all the subtables in maximizing the function:

\[
f(u) = \sum_{m=1}^{M} \sum_{i=1}^{p} \text{cov}^2(x_{mi}, X_m Qu) = u' Q \left( \sum_{m=1}^{M} V_{X_m}^2 \right) Qu
\]

subject to the normalization constraint

\[
\|u\|_Q = 1
\]

**Property 1** The axis \( u \) order 1 of a vertical multi-table analysis \( X \) verifies the stationary equation

\[
\left( \sum_{m=1}^{M} V_{X_m}^2 \right) Qu = \alpha u
\]

**Proof:** The solution to this problem is equivalent to maximizing the Lagrangian

\[
L = u' Q \left( \sum_{m=1}^{M} V_{X_m}^2 \right) Qu + \alpha (1 - u' Qu)
\]

where \( \alpha \) is the Lagrange multiplier associated with the constrain (2). Cancelling the derivatives of the Lagrangian function with respect to \( u \) and \( \alpha \) it yields the following stationary equations:

\[
Q \left( \sum_{m=1}^{M} V_{X_m}^2 \right) Qu - \alpha Qu = 0
\]

\[
1 - u' Qu = 0
\]
Using the equation (4) and the normalization constraint (5), the stationary equation becomes

\[
\left( \sum_{m=1}^{M} V_{X_m}^2 \right) Qu = \alpha u
\]

\(u\) is the eigenvector of the matrix \(\left( \sum_{m=1}^{M} V_{X_m}^2 \right)\) related to the largest eigenvalue \(\alpha\). We find the component \(c_{X_m} = X_m Qu\) order 1 for table \(X_m\).

**Property 2** The solutions order \(s\), \(c_s\) \((s = 1, \cdots, r)\) constitute an orthonormal basis and obtain oneself from the stationary equation

\[
Q^{\frac{1}{2}} \left( \sum_{m=1}^{M} V_{X_m}^2 \right) Q^{\frac{1}{2}} c_s = \alpha_s c_s \tag{6}
\]

with \(u_s = Q^{-\frac{1}{2}} c_s\) verifying \(c'_s c_s = u'_s Qu_s = 1\) and \(c'_s c_l = u'_s Qu_l = 0\) for \(s \neq l\).

The synthetic variables \(c_{X_m,s} = X_m Qu_s\) are not \(D_m\)-orthogonal.

In order to find the components order 2 that can be orthogonal at the first synthetic components, \(X_m\) are replaced with their residues of the regression in the criterion on the first components \(c_{X_m} = X_m Qu\):

\[
X_m^{(1)} = \left( I_{n_m} - \frac{c_{X_m} c'_{X_m} D_m}{\|c_{X_m}\|_{D_m}^2} \right) X_m^{(0)}
\]

\(X_m^{(0)} = X_m\) for \(m = 1, \cdots, M\).

This procedure is iterated until we obtain the components \(c_{X_m,s} = X_m^{(s-1)} Qu_s\) for all \(s = 1, \cdots, r\) where \(r\) is the rank of the tables \(X_m\) for \(m = 1, \cdots, M\) and

\[
X_m^{(s-1)} = \left( I_{n_m} - \frac{c_{X_m,s-1} c'_{X_m,s-1} D_m}{\|c_{X_m,s-1}\|_{D_m}^2} \right) X_m^{(s-2)}
\]

Thus, in this way the synthetic components obtained are mutually orthogonal in pairs.

**Property 3** The vectors \(u_s\) \((s = 1, \cdots, r)\) are \(Q\)-orthogonal and verify the stationary equation

\[
\left( \sum_{m=1}^{M} V_{X_m}^2 \right) Qu_s = \alpha_s u_s \tag{7}
\]
Proof: Beginning with the orthogonality of co-inertia axes of the set \{u_s\}, \(s = 1, \cdots, r\). Multiplying to the right the transpose of (7) by \(Qu_l\), for \(l = 1, \cdots, s - 1\), we obtain
\[
\alpha_s u_s'Qu_l = u_s'Q \left( \sum_{m=1}^{M} V_{X(s-1)}^l X_m^{(s-1)'D_mX_m^{(s-1)}} \right) Q u_l
\]

because \(P_{cX_m,t} \perp cX_{m,l} = 0\) and \(X_m^{(s-1)} = \left( \prod_{h=l}^{s-1} P_{cX_m,h} \right) X_m^{(l-1)} = P_{cX_m,s-1} \cdots P_{cX_m,l+1} P_{cX_m,t} X_m^{(l-1)}\) for \(l = 1, \cdots, s - 1\) and \(m = 1, \cdots, M\).

As \(\alpha_s \neq 0\), we have \(u_s'Qu_l = 0\). Hence the orthogonality of co-inertia axes of the set \{u_s\}.

This method is called successive orthogonal vertical multi-table analysis acronym sOVMA.

The principle of this method is each step in retaining only the co-inertia axis \(u_s\) which corresponds to the largest eigenvalue and associated partial synthetic variables. It presents more interest compared with VMA in the sense that sOVMA is interpretable at the individuals level and variables level, but these methods are similar to stage 1.

In the following subsection we will extend a single vertical multi-table analysis into analysis of the relationship between two vertical multi-tables.

### 2.2 Analysis of the link between two vertical multi-tables

To study the common structure between two vertical multi-tables, in subsection, we were going to propose two methods of co-inertia analysis between two vertical multi-tables that are equivalents to the optimum. The first criterion is defined as the following optimization function:

Maximize
\[
g(u, v) = \left[ \sum_{m=1}^{M} \sum_{i=1}^{p} \text{cov}^2(x_{mi}, Y_mRv) \right] \left[ \sum_{m=1}^{M} \sum_{j=1}^{q} \text{cov}^2(X_mQu, y_{mj}) \right]
\] (8)

subject to the normalization constraints

\[
\|u\|_Q = \|v\|_R = 1
\] (9)
The second criterion leads to maximizing the function

\[ h(u, v) = \left[ \sum_{m=1}^{M} \sum_{i=1}^{p} \text{cov}^2(x_{mi}, Y_mRv) \right] + \left[ \sum_{m=1}^{M} \sum_{j=1}^{q} \text{cov}^2(X_mQu, y_{mj}) \right] \quad (10) \]

subject to the same normalization constraints that the first criterion.

### 2.2.1 The co-inertia analysis between two vertical multi-tables type 3

The successive co-inertia analysis between two vertical multi-tables type 3 leads to determine the co-inertia axes \( u \) of \( R_p \) of the group 1 and \( v \) of \( R_q \) of the group 2 at stage 1 achieving the maximum function (8) subject to (9).

**Property 4** \( u \) and \( v \) co-inertia axes order 1 of the co-inertia analysis between two vertical multi-tables type 3 satisfy the relations (11) and (12).

\[ \left( \sum_{m=1}^{M} V_{XmYm}V_{YmXm} \right) Qu = r_u u \quad (11) \]

\[ \left( \sum_{m=1}^{M} V_{YmXm}V_{XmYm} \right) Rv = r_v v \quad (12) \]

with \( \alpha = r_u r_v \) the maximum of the function \( g \), \( r_u = u'Q \left( \sum_{m=1}^{M} V_{XmYm}V_{YmXm} \right) Qu \) and \( r_v = v'R \left( \sum_{m=1}^{M} V_{YmXm}V_{XmYm} \right) Rv \).

**Proof** The function \( g \) can also be written in the following way:

\[ g(u, v) = \left( u'Q \sum_{m=1}^{M} V_{XmYm}V_{YmXm}Qu \right) \left( v'R \sum_{m=1}^{M} V_{YmXm}V_{XmYm}Rv \right) \]

The maximization of this function subject to the normalization constraints over \( u \) and \( v \) is equivalent to the Lagrangian

\[ L = \left( u'Q \sum_{m=1}^{M} V_{XmYm}V_{YmXm}Qu \right) \left( v'R \sum_{m=1}^{M} V_{YmXm}V_{XmYm}Rv \right) + \alpha(1 - u'Qu) + \beta(1 - v'Rv) \]
where \( \alpha \) and \( \beta \) are respectively the Lagrange multipliers associated with constraint (9).

The cancellation of the derivatives of the Lagrangian with respect to \( u, v, \alpha \) and \( \beta \) leads to the following stationary equations:

\[
\frac{1}{2} \frac{\partial L}{\partial u} = \left( \sum_{m=1}^{M} V_{XmYm} V_{YmXm} Qu \right) \left( v' \sum_{m=1}^{M} V_{YmXm} V_{XmYm} Rv \right) - \alpha u = 0 \quad (13)
\]

\[
\frac{1}{2} \frac{\partial L}{\partial u} = \left( \sum_{m=1}^{M} V_{YmXm} V_{XmYm} Rv \right) \left( u' \sum_{m=1}^{M} V_{XmYm} V_{YmXm} Qu \right) - \beta v = 0 \quad (14)
\]

\[
\frac{\partial L}{\partial \alpha} = 1 - u' Qu = 0 \quad (15)
\]

\[
\frac{\partial L}{\partial \beta} = 1 - v' Rv = 0 \quad (16)
\]

Premultiplying (13) and (14) respectively by \( u' Q \) and \( v' R \), and considering the normalization constraints (15) and (16) over the vectors \( u \) and \( v \), the result of this is:

\[
\alpha = \beta = \left( \sum_{m=1}^{M} V_{XmYm} V_{YmXm} Qu \right) \left( v' \sum_{m=1}^{M} V_{YmXm} V_{XmYm} Rv \right) \quad (17)
\]

Consequently, the stationary equations become:

\[
\left( \sum_{m=1}^{M} V_{XmYm} V_{YmXm} Qu \right) \left( v' \sum_{m=1}^{M} V_{YmXm} V_{XmYm} Rv \right) = \alpha u \quad (18)
\]

\[
\left( \sum_{m=1}^{M} V_{YmXm} V_{XmYm} Rv \right) \left( u' \sum_{m=1}^{M} V_{XmYm} V_{YmXm} Qu \right) = \alpha v \quad (19)
\]

By setting \( r_u = u' \sum_{m=1}^{M} V_{XmYm} V_{YmXm} Qu \) and \( r_v = v' \sum_{m=1}^{M} V_{YmXm} V_{XmYm} Rv \), we get \( \alpha = r_u r_v \).

Thus, the stationary equations become again:

\[
\left( \sum_{m=1}^{M} V_{XmYm} V_{YmXm} \right) Qu = r_u u
\]

\[
\left( \sum_{m=1}^{M} V_{YmXm} V_{XmYm} \right) Rv = r_v v
\]

We see that \( u \) and \( v \) are eigenvectors of the matrices \( \left( \sum_{m=1}^{M} V_{XmYm} V_{YmXm} \right) Q \) and \( \left( \sum_{m=1}^{M} V_{YmXm} V_{XmYm} \right) R \) associated with the respective eigenvalues \( r_u \) and \( r_v \), and such as \( \alpha = r_u r_v \).
Property 5 The co-inertia axes between two vertical multi-tables order \( s \) are determined by the stationary equations

\[
Q_s^{\frac{1}{2}} \left( \sum_{m=1}^{M} V_{Xm} V_{Ym} X_m Y_m \right) Q_s^{\frac{1}{2}} c_s = r_{u,s} c_s \quad (20)
\]

\[
R_s^{\frac{1}{2}} \left( \sum_{m=1}^{M} V_{Ym} V_{Xm} X_m Y_m \right) R_s^{\frac{1}{2}} d_s = r_{v,s} d_s \quad (21)
\]

We can construct the orthonormal bases vectors \( u_s \) by setting in (11) \( u_s = Q_s^{-\frac{1}{2}} c_s \) and premultiplying this equation by \( Q_s^{\frac{1}{2}} \), we obtain (20), with \( c_s \) verifying \( c_s^t c_s = u_s^t Q u_s = 1 \) and \( c_s^t c_l = u_s^t Q u_l = 0 \) for \( s \neq l \).

Likewise, we construct the orthonormal bases vectors \( v_s \) by setting in (12) \( v_s = R_s^{-\frac{1}{2}} d_s \) and premultiplying this equation by \( R_s^{\frac{1}{2}} \), we obtain (21). The vectors \( d_s \) are such as \( d_s^t d_s = v_s^t R v_s = 1 \) and \( d_s^t d_l = v_s^t R v_l = 0 \) for \( s \neq l \).

The synthetic components \( c_{Xm,s} = X_m Qu_s \) and \( c_{Ym,s} = Y_m Rv_s \) associated with two groups are not \( D_m \)-orthogonal.

We will define an another way of finding the co-inertia axes in the coupling between two vertical multi-tables so that the explained variation of each table is different at each stage.

2.2.2 The successive orthogonal co-inertia analysis between two vertical multi-tables type 3

We will define in this section a new method for determining the co-inertia axes in the coupling of two vertical multi-tables: the successive orthogonal co-inertia analysis between two vertical multi-tables type 3 soCIA3. The term orthogonality refers to the orthogonality of the partial synthetic components for the two groups of variables obtained at a strictly upper order.

The purpose of this method is identical to the previous one, or to study the stability of the relationship between two vertical multi-tables individually explaining the best tables.

Thus, the first order, the successive orthogonal co-inertia analysis between two vertical multi-tables type 3 is confused to the successive co-inertia analysis between two vertical multi-tables type 3.

Solutions order \( s \) are defined as follows: the co-inertia axes order \( s \), with \( 2 \leq s \leq \min(p,q) \), of the soCIA3 coupling two vertical multi-tables are the axes order 1 of the co-inertia analysis in the coupling of two vertical multi-tables of triplets

\[
\left( X_1^{(s-1)}, Q, D_1 \right), \ldots, \left( X_M^{(s-1)}, Q, D_M \right) \text{ and } \left( Y_1^{(s-1)}, R, D_1 \right), \ldots, \left( Y_M^{(s-1)}, R, D_M \right)
\]
where $X_m^{(s-1)} = P_{c_{X_{m,s-1}}}^\perp X_m^{(s-2)}$, $Y_m^{(s-1)} = P_{c_{Y_{m,s-1}}}^\perp Y_m^{(s-2)}$, $X_m^{(0)} = X_m$ and $Y_m^{(0)} = Y_m$ with

$$P_{c_{X_{m,s-1}}}^\perp = I_{n_m} - \frac{c_{X_{m,s-1}}' c_{X_{m,s-1}} D_m}{\|c_{X_{m,s-1}}\|^2} \text{ and } P_{c_{Y_{m,s-1}}}^\perp = I_{n_m} - \frac{c_{Y_{m,s-1}}' c_{Y_{m,s-1}} D_m}{\|c_{Y_{m,s-1}}\|^2}$$

the $D_m$-orthogonal projectors of the projectors $P_{c_{X_{m,s-1}}}^\perp = \frac{c_{X_{m,s-1}}' c_{X_{m,s-1}} D_m}{\|c_{X_{m,s-1}}\|^2}$ and $P_{c_{Y_{m,s-1}}}^\perp = \frac{c_{Y_{m,s-1}}' c_{Y_{m,s-1}} D_m}{\|c_{Y_{m,s-1}}\|^2}$ respectively.

Therefore, sets of the components $\{c_{X_{m,s}} = X_m^{(s-1)} Qu_s\}_s$ and $\{c_{Y_{m,s}} = Y_m^{(s-1)} Rv_s\}_s$ are $D_m$-orthogonal.

**Property 6** Sets of co-inertia axes $\{u_s\}_s$ and $\{v_s\}_s$ are also respectively $Q$-orthogonal and $R$-orthogonal and the associated axes respectively verify the stationary equations

$$\left(\sum_{m=1}^M V_{X_{m,s}}^{(s-1)} V_{X_{m,l}}^{(s-1)} X_{m,l}^{(s-1)}\right) Qu_s = r_{u,s} u_s$$  \hspace{1cm} (22)

$$\left(\sum_{m=1}^M V_{Y_{m,s}}^{(s-1)} V_{Y_{m,l}}^{(s-1)} X_{m,l}^{(s-1)}\right) Rv_s = r_{v,s} v_s$$  \hspace{1cm} (23)

Proof \hspace{0.5cm} Beginning with the orthogonality of co-inertia axes of the set $\{u_s\}_s$, $s = 1, \cdots, r$. Multiplying to the right the transpose of (22) by $Qu_l$, for $l = 1, \cdots, s - 1$, we obtain

$$r_{u,s} u_s' Qu_l = u_s' \left(\sum_{m=1}^M V_{X_{m,s}}^{(s-1)} V_{X_{m,l}}^{(s-1)} X_{m,l}^{(s-1)} D_m X_{m,l}^{(s-1)}\right) Qu_l = 0$$

because $P_{c_{X_{m,l}}}^\perp c_{X_{m,l}} = 0$ and $X_{m,l}^{(s-1)} = \left(\prod_{h=1}^{s-1} P_{c_{X_{m,h}}}^\perp\right) X_{m,l}^{(l-1)} = P_{c_{X_{m,s-1}}}^\perp P_{c_{X_{m,s-2}}}^\perp \cdots P_{c_{X_{m,l+1}}}^\perp P_{c_{X_{m,l}}}^\perp X_{m,l}^{(l-1)}$ for $l = 1, \cdots, s - 1$ and $m = 1, \cdots, M$. as $r_{u,s} \neq 0$, we get $u_s' Qu_l = 0$. Hence the orthogonality of co-inertia axes of the set $\{u_s\}_s$.

The orthogonality of the co-inertia axes of the set $\{v_s\}_s$ is proved in the same way as the orthogonality of the co-inertia axes of the set $\{u_s\}_s$.

In the same way as previously, this orthogonal approach is more interest than the non-orthogonal approach because of orthogonality at the individuals level and variables level.
Proposition of analyses in a vertical multi-table

2.2.3 The co-inertia analysis between two vertical multi-tables type 4

Finally, we are interested in the second criterion of the successive co-inertia analysis between two vertical multi-tables type 4 to determine respectively the order 1 the co-inertia axes \( u \) of \( \mathbb{R}^p \) of the group 1 and \( v \) of \( \mathbb{R}^q \) of the group 2 achieving the maximum of the function (9) subject to constraints (8).

\[
h(u, v) = \left[ \sum_{m=1}^{M} \sum_{i=1}^{p} \text{cov}^2(x_{mi}, Y_m R v) \right] + \left[ \sum_{m=1}^{M} \sum_{j=1}^{q} \text{cov}^2(X_m Q u, y_{mj}) \right]
\]

subject to the constraint

\[\|u\|_Q = \|v\|_R = 1\]

The function \( h \) can be written

\[
h(u, v) = \left( u' Q \sum_{m=1}^{M} V_{X_m Y_m} V_{Y_m X_m} Q u \right) + \left( v' R \sum_{m=1}^{M} V_{Y_m X_m} V_{X_m Y_m} \right) R v
\]

The maximization of this function subject to the normalization constraints over the vectors \( u \) and \( v \) is equivalent to maximizing the Lagrangian

\[L = \left( u' Q \sum_{m=1}^{M} V_{X_m Y_m} V_{Y_m X_m} Q u \right) + \left( v' R \sum_{m=1}^{M} V_{Y_m X_m} V_{X_m Y_m} \right) R v + \alpha(1 - u' Q u) + \beta(1 - v' R v)\]

where \( \alpha \) and \( \beta \) are respectively the Lagrange multipliers associated with constraints (9).

The cancellation of the derivatives of the Lagrangian function with respect to \( u, v, \alpha \) and \( \beta \) leads to the following stationary equations

\[\frac{1}{2} \frac{\partial L}{\partial u} = \left( \sum_{m=1}^{M} V_{X_m Y_m} V_{Y_m X_m} Q u \right) - \alpha u = 0 \quad (24)\]
\[\frac{1}{2} \frac{\partial L}{\partial v} = \left( \sum_{m=1}^{M} V_{Y_m X_m} V_{X_m Y_m} R v \right) - \beta v = 0 \quad (25)\]
\[\frac{\partial L}{\partial \alpha} = 1 - u' Q u = 0 \quad (26)\]
\[\frac{\partial L}{\partial \beta} = 1 - v' R v = 0 \quad (27)\]
Using the equations (24), (25), (26) and (27), the stationary equations become

\[
\left( \sum_{m=1}^{M} V_{X_m} Y_m V_{Y_m} X_m \right) Qu = \alpha u
\]

\[
\left( \sum_{m=1}^{M} V_{Y_m} X_m V_{X_m} Y_m \right) Rv = \beta v
\]

with \( \rho = \alpha + \beta = h(u, v) \), the maximal value of the function order 1.

In the same way as the co-inertia analysis between two vertical multi-tables on the orthogonal version type 3, an orthogonal approach of the co-inertia analysis between two vertical multi-tables type 4 can be designed.

3 Complements and links between methods

The sCIA3 allows to construct two bases: \( \{u_s\}_s \) is a \( Q \)-orthonormal basis of \( \mathbb{R}^p \) associated with group 1 of variables and \( \{v_s\}_s \) is a \( R \)-orthonormal basis of \( \mathbb{R}^q \) for the group 2 of variables. Therefore, we can represent individuals in group 1 in the two dimensional space of co-inertia axes \( u_s \) and \( u_l \) (\( l \neq s \)) because the coordinates of the relative individuals with tables \( X_m \) \( (m = 1, \ldots, M) \) in the two dimensional space \( (u_s, u_l) \) are the components of vectors \( c_{X_m,s} = X_m Qu_s \) and \( c_{X_m,l} = X_m Qu_l \). Furthermore, individuals in group 2 can be represented in the two dimensional space of co-inertia axes \( v_s \) and \( v_l \) (\( l \neq s \)). The Coordinates of the relative individuals with tables \( Y_m \) \( (m = 1, \ldots, M) \) are the components of vectors \( c_{Y_m,s} = Y_m Rv_s \) and \( c_{Y_m,l} = Y_m Rv_l \). However, like the sets of vectors formed by partial linear combinations \( \{c_{X_m,s} = X_m Qu_s\}_s \) and \( \{c_{Y_m,s} = Y_m Rv_s\}_s \) are not \( D_m \)-orthogonal in \( \mathbb{R}^{n_m} \) for \( m = 1, \ldots, M \) and \( s = 1, \ldots, r \), we can not represent the variables of two groups in the orthogonal spaces of dimension 2. As far as sOCIA3 concerned, the coordinates of the individuals associated with each of tables are contained in the synthetic components \( c_{X_m,s} = X_m^{(s-1)} Qu_s \) and \( c_{Y_m,s} = Y_m^{(s-1)} Rv_s \). There is a bias in these coordinates. In fact, considering the orthogonality of the synthetic components \( c_{X_m,s} = X_m^{(s-1)} Qu_s \), we easily show that

\[
c_{X_m,s} = X_m^{(s-1)} Qu_s = P_{c_{X_{m,s-1}}}^{\perp} P_{c_{X_{m,s-2}}}^{\perp} \cdots P_{c_{X_{m,1}}}^{\perp} X_m Qu_s =
\]

\[
\left( I_{n_m} - \sum_{h=1}^{s-1} P_{c_{X_{m,h}}} \right) X_m Qu_s
\]

or still \( c_{X_m,s} = X_m^{(s-1)} Qu_s = X_m Qu_s - \sum_{h=1}^{s-1} P_{c_{X_{m,h}}} X_m Qu_s \).

In the same way we show that \( c_{Y_m,s} = Y_m^{(s-1)} Rv_s = Y_m Rv_s - \sum_{h=1}^{s-1} P_{c_{Y_{m,h}}} Y_m Rv_s \).
The relative individuals with tables $X_m$ and $Y_m$ can respectively be projected in the two dimensional space of axes $(u_s, u_l)$ and $(v_s, v_l)$ for $s \neq l$. This joint configuration enables to visualize the distances between individuals seen by tables $X_m$ and $Y_m$. The specific weights (projected inertia) make up in the same way as previously.

The orthogonality of components $\{c_{X_{m,s}}\}_s$ and $\{c_{Y_{m,s}}\}_s$ for all $m$, enable to study the inner structures each of tables $X_m$ and $Y_m$. We can obtain the graphics of the variables of each table from the covariances when the variables are centred between variables of the table and two associated components. These covariances correspond to coordinates of the projected variables in the orthonormal space defined by two components. For example, the variables of $X_m$ projected in the orthonormal space $\left(\frac{C_{X_{m,s}}}{\|C_{X_{m,s}}\|_2}, \frac{C_{X_{m,l}}}{\|C_{X_{m,l}}\|_2}\right)$ have on coordinates $\text{cov}(X_m, C_{X_{m,s}})$ and $\text{cov}(X_m, C_{X_{m,l}})$ for all $s \neq l$. In case the variables are centred and reduced, the coordinates of the original variables become the correlations.

The specific weights order $s$ ($s = 1, \ldots, r$) associated with tables $X_m$ and $Y_m$ for $m = 1, \ldots, M$ are respectively defined by $\rho_{X_{m,s}} = \text{var}(X_mQu_s)$ and $\rho_{Y_{m,s}} = \text{var}(Y_mRv_s)$. These weights define projected inertia of the clouds of $n_m$ individuals associated with tables $X_m$ and $Y_m$ over the co-inertia axes $u_s$ and $v_s$ respectively, $r \leq \min(p, q)$ is the rank of the matrices.

Quantities $\rho_{X_{m,s}Y_{m,s}} = \text{cor}^2(C_{X_{m,s}}, C_{Y_{m,s}})$ are squared coefficients of correlation associated with tables $X_m$ and $Y_m$. Weights $\rho_{X_{m,s}}$ and $\rho_{Y_{m,s}}$ for $s = 1, \ldots, r$ and for $m = 1, \ldots, M$ enable to characterize the stability of each group of variables in the space and the time, and on the other hand, weights $\rho_{X_{m,s}Y_{m,s}} = \text{cor}^2(C_{X_{m,s}}, C_{Y_{m,s}})$ enable to characterize equally the stability of the relationship between tables $X_m$ and $Y_m$ in the space and the time after the fashion of the weights that construct the compromise in STATICO.

By setting $X = Y$ in the functions $g$ and $h$, we obtain the criterion (1) subject to (2).

The stationary equation (7) of sOVMA is near to sOPCA’s:

$$\left(\sum_{m=1}^{M} V_{X_{m}^{(s-1)}}\right) Qu_s = \alpha_s u_s$$

is defined in Niéré et al. (2013).

The maximization of the functions $g$ and $h$ subject to normalization constraints over the vectors $u$ and $v$ leads to the same results.

If $M = 1$, the functions $g$ and $h$ are respectively

$$g(u, v) = \langle u' QV_{XY} V_{YX} Qu \rangle \langle v' RV_{YX} V_{XY} Rv \rangle$$

and

$$h(u, v) = \langle u' QV_{XY} V_{YX} Qu \rangle \langle v' RV_{YX} V_{XY} Rv \rangle.$$  

The co-inertia analysis between two tables can be defined in maximizing the function $g(u, v) = \langle u' QV_{XY} V_{YX} Qu \rangle \langle v' RV_{YX} V_{XY} Rv \rangle$ or $h(u, v) = \langle u' QV_{XY} V_{YX} Qu \rangle + \langle v' RV_{YX} V_{XY} Rv \rangle$ subject to normalization constraints (9) (Lafosse and Hanafi (1997)).
4 Application

This section gets organized in two points: the first is devoted to the presentation of the data and the second is reserved to the application and the analysis of the results of this method.

4.1 Data

The example data set was already used by Pegaz-Maucet (1980), by Hanafi (1977) and also by Blanc et al. (1998). These ecological data have been measured along the Méaudret stream in France. 6 stations spread over the Méaudret stream have been visited each once per saison (1-Spring, 2-Summer, 3-Autumn, 4-Winter). In each station, 10 physicochemical parameters of the stream have been measured every time and identified 13 species. We obtain thus in total 24 list (6 stations × 4 seasons).

The species composition table \( X \) of dimension (24,13) is constituted from four tables \( X_m \) of dimension (6,13). 13 present species (ephemeroptera) are: Eda=Ephemera, Bsp=Baetis sp, Brh=Baetis rhodani, Bni=Baetis niger, Bpu=baetis pumilus, Cen=centroptilum, Ecd=Ecdyonurus, Rhi=Rhithrogena, Hla=Habrophlebia lanta, Hab=Habroloctoides modesta, Par=Paraleptophlebia, Cae=Caenis, Eig=Ephemerella ignita.

Likewise the environmental table \( Y \) of dimension (24,10) is formed of four tables \( Y_m \) of dimension (6,10). Two tables \( X_m \) and \( Y_m \) correspond at the same season \( m \) (\( m = 1, 2, 3, 4 \)). 10 variables of the environment are: Temp=Temperature, Flow, pH, Cond=conductivity, Oxyg=Oxygen, BDO5=biological demand for oxygen, Oxyd=Oxydability, Ammo=Ammonium, Nitra=Nitrates, Phosp=Phosphates.

4.2 Application of sOCIA3

Species are centered per saison and environmental variables are centered then globally normalized (Bouroche, 1975). This global normalization allows to take into account the intra-seasonal variance.

Each of these tables corresponds with a season and a triplet \((X_m, Q, D_m)\) for the ephemeroptera species and \((Y_m, R, D_m)\) for the environment.

To have an idea about the internal structures of each vertical multi-tables, we apply each one of them the sVMA method established in subsectin 2.1 in her orthogonal version (sOVMA).

The analysis by this method of the species multi-table (Fig. 2) provides in the first principal map, an image of the stations and descriptors. A size effect of axis 1 is observed: the BPU, Hla and Eda species are present in the S1 stations in Summer. This station is opposite on the axis 1 to station S2. The station
S6 contains species Bsp. However, speaking of sOVMA of the multi-table environmental variables (Fig 3), axis 1 opposes variables Oxygen and pH on the one hand and variables conductivity of the water, phosphates and the oxydability other. The water temperature is highest in station S4 in Summer and Autumn. The Station S2 opposes the stations S1 and S6 in Summer. In the station S4 in Winter, there is a strong flow. After making these two separate sOVMA, it is advisable to make a global analysis in research of the relationship between species and the environment can be made.

Table 1 contains the squared correlations between the partial linear combinations of variables (species of fish) and environmental variables of order 1 and 2 for the sOCIA3 method. These squared correlations enable to describe the evolution of species-environment relationships. Constancy of these squared coefficients of correlation enables to conclude the stability of the link. It emerges from Table 1, a same evolution of species-environment relationships for Summer and Autumn concerning sOCIA3 method (Confer the graph of Figure 1). This observation does not find oneself in Winter and Spring which differ too much from other seasons on this method. The last situation confirms good results from other methods of co-inertia analysis cited above.

Table 2 contains the percentages of projected inertia of each table on the first two axes. On the first two axes according to sOCIA3 method for the first multi-table, Autumn has projected inertia percentages are highest. Regarding the second multi-table corresponding to the environmental variables, it is rather than on the first axis of high percentages of projected inertia for Autumn found. But on the second axis, it is the Summer that has the largest percentage. We find perfectly the same results than previous methods.

In the first two axes of sOCIA3, we show the stations (Fig4X1) and species (Fig4X2). The sOCIA3 method determines simultaneously two sets of orthogonal axes at the individuals level and variables level. It follows from these graphics, any season, a general organization finds again more or less at stations and species. It notes an overall size effect at axis 1 regarding the species. The axis 1 opposes on the one hand station S6 and station S2 on the other hand for all seasons. We can find for all seasons more or less in the station S6 species Baetis sp and Baetis Rhodani. In the Spring, station S6 is characterized by high temperatures and flow (Fig5Y1 and Fig5Y2). Station S2 is characterized by the phosphates and ammonium in Autumn. Near the center marks, we find the rare species that are not taken into account by the sOCIA3. Axis 2 opposes generally one hand station S1 and stations S4 and S6 other hand.

In view of the positions of the environmental variables, we notice some differences in the sOVMA method where Ammo and Nitra were exchanged positions. In contrast to the sOCIA3 method the positions of the environmental variables
are generally those of the previous methods mentioned above.

Table 1: Squared correlations between linear combinations of the variables in fish abundance and environmental variables to order 1 and 2 for sOCIA3

<table>
<thead>
<tr>
<th>Method</th>
<th>Seasons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Spring</td>
</tr>
<tr>
<td>sOCIA3</td>
<td>0.371</td>
</tr>
<tr>
<td></td>
<td>0.436</td>
</tr>
</tbody>
</table>

Table 2: Percentages of projected inertia (specific weights) of each season on the first two axes

<table>
<thead>
<tr>
<th>Methods</th>
<th>Seasons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Spring</td>
</tr>
<tr>
<td>sOCIA3 (X)</td>
<td>11.100</td>
</tr>
<tr>
<td>sOCIA3 (Y)</td>
<td>6.725</td>
</tr>
<tr>
<td>sOCIA3 (Y)</td>
<td>22.657</td>
</tr>
</tbody>
</table>
Fig.1: Position of the seasons on the first two axes of squared correlations between partial synthetic components species-environment for sOCIA3.
Fig. 2: Position of the stations (Fig2X1) and the species (Fig2X2) per season on the first two axes for multi-table $X$ of sOVMA.
Fig. 3: Representation of the stations (Fig3Y1) and the environmental variables (Fig3Y2) per season on the first two axes for multi-table Y of sOVMA.
Fig. 4: Position of the stations (Fig4X1) and the species (Fig4X2) per season on the first two axes for multi-table X of sOCIA3.
Fig.5: Representation of the stations (Fig5Y1) and the environmental variables (Fig5Y2) per season on the first two axes for multi-table $Y$ of sOCIA3.

5 Conclusion

The methods proposed in this paper are firstly a new approach of dual STATIS in research of the link between tables in a vertical multi-table and secondly a new approach in finding the link between two vertical multi-table like the previous methods that are mentioned in this paper. All these methods have the same objective: to study the stability of the relationship between two
vertical multi-tables. To achieve this objective, this method determines the co-inertia axes that are common to all tables for the two groups of variables that determine the squared coefficients of correlation between pairs of tables. The constancy of these coefficients allows to this method to study the stability of the link. None of them is superior to another and can be interchanged by the user. Compared to the same sample application, these methods provide results that are generally consistent with those of Hanafi (1997) in the STATICO method and Thioulouse (2004).

References


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