Refined Stability of an Additive Functional Equation with Several Variables

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Abstract. In this research, we propose the generalized Hyers–Ulam stability of the following Cauchy additive equation

\[ nf \left( \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j), \]

with several variables, where \( n \geq 2 \) and then investigate the refined stability of the equation.

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1. Introduction

In 1940, S.M. Ulam [17] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms: We are given a group \( G_1 \) and a metric group \( G_2 \) with metric \( \varphi(\cdot,\cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if \( f : G_1 \to G_2 \) satisfies
\[ \varphi(f(x + y), f(x) f(y)) < \delta \text{ for all } x, y \in G_1, \text{ then a homomorphism } h : G_1 \to G_2 \text{ exists with } \varphi(f(x), h(x)) < \varepsilon \text{ for all } x \in G_1? \]

Let \( X \) and \( Y \) be Banach spaces with norms \( \| \cdot \| \) and \( \| \cdot \| \), respectively. D.H. Hyers [7] showed that if \( \varepsilon > 0 \) and \( f : X \to Y \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon
\]
for all \( x, y \in X \), then there exists a unique additive mapping \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \varepsilon
\]
for all \( x \in X \).

In 1950 T. Aoki [2] and in 1951 D.G. Bourgin [3] provided a generalized the Hyers theorem for additive mapping and in 1978 Th.M. Rassias [15] generalized the Hyers theorem for linear mapping by allowing the Cauchy difference to be unbounded. Let \( f : X \to Y \) be a mapping such that \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \). Assume that there exist constants \( \varepsilon \geq 0 \) and \( p \in [0, 1) \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon(\| x \|^p + \| y \|^p)
\]
for all \( x, y \in X \). Then Th.M. Rassias proved that there exists a unique \( \mathbb{R} \)-linear mapping \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \frac{2\varepsilon}{2 - 2^p}\| x \|^p
\]
for all \( x \in X \).

In 1982, J.M. Rassias [16] has similarly established the stability for linear mappings involving a product of different powers of norms. Let \( f : X \to Y \) be a mapping from a real normed space \( X \) to a Banach space \( Y \) such that \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \). Assume that there exist constants \( \varepsilon \geq 0 \) and \( p, q \in \mathbb{R} \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon(\| x \|^p \| y \|^q)
\]
for all \( x, y \in X \), where \( x, y \neq 0 \) if \( p, q < 0 \). Then there exists a unique \( \mathbb{R} \)-linear mapping \( L : X \to Y \) such that
\[
\| f(x) - L(x) \| \leq \frac{\varepsilon}{|2 - 2^r|}\| x \|^r
\]
for all \( x \in X \), where \( x \neq 0 \) if \( r < 0 \) and \( r := p + q \neq 1 \). However, the case \( r = 1 \) is singular with an open problem in the above stability theorem, and then a counterexample has been given by P. Găvruța [6].

In general, P. Găvruța [5] established the extended stability result of Th.M. Rassias [15] and J.M. Rassias [16] by allowing the Cauchy difference to be a
generalized control function. Let $G$ be an abelian group, $E$ a Banach space and let $\varphi : G \times G \to [0, \infty)$ be a mapping such that
\[
\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty
\]
for all $x, y \in G$. If a mapping $f : G \to E$ satisfies
\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)
\]
for all $x, y \in G$, then there exists a unique additive mapping $T : G \to E$ satisfying
\[
\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)
\]
for all $x \in G$. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam–Rassias stability to a number of functional equations [1, 4, 8, 9, 10, 11, 12, 14].

Recently, P. Nakmahachalasint [13] has proved that the following $n$-dimensional additive functional equation
\[
nf\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j),
\]
where $n \geq 2$, is equivalent to the Cauchy additive equation $f(x + y) = f(x) + f(y)$ between linear spaces, and then the author has established the stability theorems of which the norm of Cauchy difference operator is bounded by the sum of powers of norms and by the product of different powers of norms, respectively.

In this paper, we improve to establish the generalized Hyers–Ulam stability of the functional equation (1.1) by allowing the functional difference of the equation (1.1) to be bounded by a generalized control function. In addition, we present additional generalized Hyers–Ulam stability results of the functional equation (1.1). Throughout this paper, we now assume that $X$ is a normed linear space with norm $\| \cdot \|$ and $Y$ is a Banach space with norm $\| \cdot \|$, respectively, without specific reference.

2. Refined stability results of functional equation (1.1).

Given a mapping $f : X \to Y$, we set for notational convenience
\[
Df(x_1, \cdots, x_n) := nf\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j),
\]
for all $x_1, \cdots, x_n \in X$, where $n \geq 2$ is fixed. Now, we prove the generalized Hyers–Ulam stability of the functional equation (1.1) that gives a condition
for which an additive mapping exists near an approximate additive mapping \( f : X \to Y \).

**Theorem 2.1.** Let \( f : X \to Y \) be a mapping for which there is a function \( \varphi : X^n \to [0, \infty) \) such that

\[
\Phi(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{2.1}
\]

\[
\|Df(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{2.2}
\]

for all \( x_1, \cdots, x_n \in X \). Then there exists a unique Cauchy additive mapping \( L : X \to Y \), defined as \( L(x) = \lim_{l \to \infty} \frac{f(2^lx)}{2^l}, x \in X \), such that

\[
\|f(x) - L(x)\| \leq \frac{1}{2(n-1)} \Phi(x, x, 0, \cdots, 0) + \frac{n-2}{2} \|f(0)\| \tag{2.3}
\]

for all \( x \in X \), where \( \|f(0)\| \leq \frac{2\varphi(0, \ldots, 0)}{n(n-1)} \).

**Proof.** Putting \( x_1 = \cdots = x_n := 0 \) in (2.2), we get \( \|f(0)\| \leq \frac{2\varphi(0, \ldots, 0)}{n(n-1)} \). Replacing \( x_1 = x_2 := x \) and \( x_3 = \cdots = x_n := 0 \) in (2.2), one has

\[
(n-1)\|f(2x) - 2f(x) - \frac{n-2}{2} f(0)\| \leq \varphi(x, x, 0, \cdots, 0), \tag{2.4}
\]

which yields

\[
\|2f(x) - f(2x)\| \leq \frac{\varphi(x, x, 0, \cdots, 0)}{n-1} + \frac{n-2}{2} \|f(0)\|, \tag{2.5}
\]

\[
\|f(x) - \frac{f(2x)}{2}\| \leq \frac{\varphi(x, x, 0, \cdots, 0)}{2n(n-1)} + \frac{n-2}{4} \|f(0)\|
\]

for all \( x \in X \). Thus, we lead to the crucial inequality due to the last functional inequality

\[
\left\| \frac{f(2^ix)}{2^i} - \frac{f(2^{i+1}x)}{2^{i+1}} \right\| = \frac{1}{2^i} \left\| f(2^ix) - \frac{f(2 \cdot 2^ix)}{2} \right\| \leq \frac{1}{2^{i+1}(n-1)} \varphi(2^ix, 2^ix, 0, \cdots, 0) + \frac{n-2}{2^{i+2}} \|f(0)\| \tag{2.6}
\]

for all \( x \in X \) and all nonnegative integer \( i \). Thus, it follows from (2.6) that

\[
\left\| \frac{f(2^lx)}{2^l} - \frac{f(2^mx)}{2^m} \right\| \leq \sum_{i=l}^{m-1} \frac{\varphi(2^ix, 2^ix, 0, \cdots, 0)}{2^{i+1}(n-1)} + \sum_{i=l}^{m-1} \frac{n-2}{2^{i+2}} \|f(0)\| \tag{2.7}
\]

for all \( x \in X \) and all nonnegative integers \( l \) and \( m \) with \( l < m \), of which the right-hand side approaches 0 as \( l \) tends to infinity. This shows that the sequence \( \{ \frac{f(2^ix)}{2^i} \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete,
the sequence \( \left\{ \frac{f(2^l x)}{2^l} \right\} \) converges in \( Y \) for all \( x \in X \) and so one can define a mapping \( L : X \to Y \) by

\[
L(x) = \lim_{l \to \infty} \frac{f(2^l x)}{2^l}
\]

for all \( x \in X \). Obviously, \( L(0) = 0 \). It follows from (2.2) and the definition of \( L \) that

\[
\|DL(x_1, \ldots, x_n)\| = \lim_{l \to \infty} \frac{1}{2^l} \|Df(2^l x_1, \ldots, 2^l x_n)\|
\]

\[
\leq \lim_{l \to \infty} \frac{1}{2^l} \varphi(2^l x_1, \ldots, 2^l x_n) = 0
\]

for all \( x_1, \ldots, x_n \in X \). Thus, the mapping \( L \) is Cauchy additive by Theorem 2.1 [13]. Putting \( l = 0 \) and letting \( m \to \infty \) in (2.7), we get the desired approximation (2.3).

Now, let \( L' : X \to Y \) be another Cauchy additive mapping satisfying (2.3). Then we have

\[
\|L(x) - L'(x)\| = \frac{1}{2^m} \|L(2^m x) - L'(2^m x)\|
\]

\[
\leq \frac{1}{2^m} \left( \|L(2^m x) - f(2^m x)\| + \|f(2^m x) - L'(2^m x)\| \right)
\]

\[
\leq \frac{1}{2^m(n-1)} \Phi(2^m x, 2^m x, 0, \ldots, 0) + \frac{n-2}{2^m} \|f(0)\|
\]

which tends to zero as \( m \to \infty \) for all \( x \in X \). So we can conclude that \( L(x) = L'(x) \) for all \( x \in X \). This proves the uniqueness of \( L \). \( \square \)

Now, we prove the generalized Hyers–Ulam stability of the functional equation (1.1) which is an alternative stability result of Theorem 2.1.

**Theorem 2.2.** Let \( f : X \to Y \) be a mapping for which there is a function \( \varphi : X^n \to [0, \infty) \) such that

\[
\Psi(x_1, \ldots, x_n) := \sum_{j=1}^{\infty} 2^j \varphi(2^{-j} x_1, \ldots, 2^{-j} x_n) < \infty,
\]

\[
\|Df(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n)
\]

for all \( x_1, \ldots, x_n \in X \). Then there exists a unique Cauchy additive mapping \( L : X \to Y \), defined as \( L(x) = \lim_{l \to \infty} 2^l f(2^{-l} x) \), \( x \in X \), such that

\[
\|f(x) - L(x)\| \leq \frac{1}{2(n-1)} \Psi(x, x, 0, \ldots, 0)
\]

for all \( x \in X \).
Proof. Putting $x_1 = \cdots = x_n := 0$ in (2.2), we get $\|f(0)\| \leq \frac{2\varphi(0, \cdots, 0)}{n(n-1)} = 0$ and so $f(0) = 0$ because of $\varphi(0, \cdots, 0) = 0$ by the convergence of $\Psi(0, \cdots, 0)$. Thus, we lead to the crucial inequality due to the functional inequality (2.5)

$$\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \frac{\varphi\left(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\right)}{n-1},$$

from which one can conclude to get the desired result by applying the same argument as in Theorem 2.1.

\[ \Box \]

**Corollary 2.3.** Let $\delta, \theta$ be nonnegative real numbers and let $p \ (p \neq 1)$ be a positive real number with $\delta = 0$ when $p > 1$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, \cdots, x_n)\| \leq \delta + \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$$

for all $x_1, \cdots, x_n \in X$, then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{(n-2)\|f(0)\|}{2} + \frac{\delta}{n-1} + \frac{2\theta}{(n-1)(2-2p)}\|x\|^p$$

for all $x \in X$, where $\|f(0)\| \leq \frac{2\delta}{n(n-1)}$.

**Proof.** Define $\varphi(x_1, \cdots, x_n) := \delta + \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$ and apply Theorem 2.1 for the case $p < 1$, and apply Theorem 2.2 for the case $p > 1$.

\[ \Box \]

The above corollary is a refined stability result of Theorem 3.1 in the reference [13]. In Corollary 2.3, we observe that the approximation

$$\|f(x) - L(x)\| \leq \frac{(n-2)\|f(0)\|}{2} + \frac{\delta}{n-1} + \frac{2\theta}{(n-1)(2-2p)}\|x\|^p$$

implies

$$\|f(x) - L(x)\| \leq \frac{(n-2)}{2} \frac{2\delta}{n(n-1)} + \frac{\delta}{n-1} + \frac{2\theta}{(n-1)(2-2p)}\|x\|^p$$

$$= \frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2p)}\|x\|^p,$$

which is a refined approximation of Theorem 3.1 [13].

**Corollary 2.4.** Let $\delta, \theta$ be nonnegative real numbers and let $r_i \ (i = 1, \cdots, n)$ be positive real numbers. If a mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, \cdots, x_n)\| \leq \delta + \theta \sum_{1 \leq i < j \leq n} \|x_i\|^{r_i} \|x_j\|^{r_j}$$

implies

$$\|f(x) - L(x)\| \leq \frac{(n-2)}{2} \frac{2\delta}{n(n-1)} + \frac{\delta}{n-1} + \frac{2\theta}{(n-1)(2-2p)}\|x\|^p$$

$$= \frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2p)}\|x\|^p,$$

which is a refined approximation of Theorem 3.1 [13].
for all \( x_1, \ldots, x_n \in X \), where \( r_i + r_j \in (0, 1) \cup (1, \infty) \), and \( \delta = 0 \) when \( r_i + r_j \in (1, \infty) \). then there exists a unique Cauchy additive mapping \( L : X \to Y \) such that

\[
\| f(x) - L(x) \| \leq \frac{(n - 2)\| f(0) \|}{2} + \frac{\delta}{n - 1} + \frac{\theta}{(n - 1)|2 - 2^{r_1 + r_2}|} \| x \|^{r_1 + r_2}
\]

for all \( x \in X \), where \( \| f(0) \| \leq \frac{2\delta}{n(n - 1)} \).

**Proof.** Define \( \varphi(x_1, \ldots, x_n) := \delta + \theta \sum_{1 \leq i < j \leq n} \| x_i \|^{r_i} \| x_j \|^{r_j} \) and apply Theorem 2.1 for the case \( r_i + r_j < 1 \), and apply Theorem 2.2 for the case \( r_i + r_j > 1 \). \( \square \)

The above corollary is a refined stability result of Theorem 4.1 in the reference [13].

3. ALTERNATIVE STABILITY RESULTS OF FUNCTIONAL EQUATION (1.1).

In this section, we now investigate more generalized Hyers–Ulam stability of the functional equation (1.1) that gives a condition for which an additive mapping exists near an approximate additive mapping \( f : X \to Y \). For any \( n \geq 2 \), the following is an alternative stability result of which the approximation may be better than that of Theorem 2.1 in some sense.

**Theorem 3.1.** Let \( f : X \to Y \) be a mapping for which there is a function \( \varphi : X^n \to [0, \infty) \) such that

\[
\tilde{\Phi}(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} \frac{1}{n^j} \varphi(n^j x_1, \ldots, n^j x_n) < \infty,
\]

\[
\| Df(x_1, \ldots, x_n) \| \leq \varphi(x_1, \ldots, x_n)
\]

for all \( x_1, \ldots, x_n \in X \). Then there exists a unique Cauchy additive mapping \( L : X \to Y \), defined as \( L(x) = \lim_{l \to \infty} \frac{f(n^lx)}{n^l} \), \( x \in X \), such that

\[
\| f(x) - L(x) \| \leq \frac{1}{n^2} \tilde{\Phi}(x, x, \ldots, x) + \frac{1}{2n} \tilde{\Phi}(x, x, 0, \ldots, 0) + \frac{(n - 2)}{4} \| f(0) \|
\]

for all \( x \in X \), where \( \| f(0) \| \leq \frac{2\varphi(0, \ldots, 0)}{n(n - 1)} \).

**Proof.** Putting \( x_1 = \cdots = x_n := 0 \) in (3.2), we get \( \| f(0) \| \leq \frac{2\varphi(0, \ldots, 0)}{n(n - 1)} \). Replacing \( x_1 = x_2 = \cdots = x_n := x \) in (3.2), one has

\[
\| nf(nx) - nf(x) - \left( \begin{array}{c} n \\ 2 \end{array} \right) f(2x) \| \leq \varphi(x, x, \ldots, x)
\]

(3.4)
for all $x \in X$. Now, associating (3.4) with (2.4), we lead to the crucial inequality
\[
\left\| nf(nx) - n^2 f(x) - \frac{n(n-1)(n-2)f(0)}{4} \right\| \leq \varphi(x, x, \ldots, x) + \frac{n}{2} \varphi(x, x, 0, \ldots, 0),
\]
\[
\left\| \frac{f(nx)}{n} - f(x) \right\| \leq \frac{\varphi(x, x, \ldots, x)}{n^2} + \frac{\varphi(x, x, 0, \ldots, 0)}{2n} + \frac{(n-1)(n-2)f(0)}{4n}
\]
for all $x \in X$. Thus, one obtains the functional inequality due to the last functional inequality
\[
\left\| \frac{f(n^i x)}{n^i} - \frac{f(n^{i+1} x)}{n^{i+1}} \right\| = \frac{1}{n^i} \left\| f(n^i x) - \frac{f(n \cdot n^i x)}{n} \right\|
\leq \frac{\varphi(n^i x, n^i x, \ldots, n^i x)}{n^{i+2}} + \frac{\varphi(n^i x, n^i x, 0, \ldots, 0)}{2n^{i+1}} + \frac{(n-1)(n-2)f(0)}{4n^{i+1}}
\]
for all $x \in X$ and all nonnegative integer $i$. Therefore, it follows from the last inequality that
\[
\left\| \frac{f(n^l x)}{n^l} - \frac{f(n^m x)}{n^m} \right\| \leq \frac{m-1}{n^l} \sum_{i=l}^{m-1} \frac{\varphi(n^i x, n^i x, \ldots, n^i x)}{n^{i+2}} + \frac{m-1}{n^{l+1}} \sum_{i=l}^{m-1} \frac{\varphi(n^i x, n^i x, 0, \ldots, 0)}{2n^{i+1}} \quad (3.3)
\]
\[
\quad + \frac{m-1}{n^{l+1}} \sum_{i=l}^{m-1} \frac{(n-1)(n-2)f(0)}{4n^{i+1}}
\]
for all $x \in X$ and all nonnegative integers $l$ and $m$ with $l < m$, of which the right-hand side approaches 0 as $l$ tends to infinity. This shows that the sequence $\{ \frac{f(n^l x)}{n^l} \}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{ \frac{f(n^l x)}{n^l} \}$ converges in $Y$ for all $x \in X$, and so we may define a mapping $L : X \to Y$ by
\[
L(x) = \lim_{l \to \infty} \frac{f(n^l x)}{n^l}
\]
for all $x \in X$. Obviously, $L(0) = 0$. It follows from (3.2) and the definition of $L$ that
\[
\| DL(x_1, \ldots, x_n) \| = \lim_{l \to \infty} \frac{1}{n^l} \| Df(n^l x_1, \ldots, n^l x_n) \|
\leq \lim_{l \to \infty} \frac{1}{n^l} \| \varphi(n^l x_1, \ldots, n^l x_n) \| = 0
\]
for all $x_1, \ldots, x_n \in X$. Thus, the mapping $L$ is Cauchy additive by Theorem 2.1 [13]. Putting $l = 0$ and letting $m \to \infty$ in (3.5), we get the desired approximation (3.3).

The uniqueness of the proof is similarly verified by the same argument as in the proof of Theorem 2.1. \qed
Theorem 3.2. Let $f : X \to Y$ be a mapping for which there is a function
\[ \varphi : X^n \to [0, \infty) \] such that
\[ \Psi(x_1, \ldots, x_n) := \sum_{j=1}^{\infty} n^j \varphi\left(\frac{x_1}{n^j}, \ldots, \frac{x_n}{n^j}\right) < \infty, \quad (3.6) \]
\[ \|Df(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n) \quad (3.7) \]
for all $x_1, \ldots, x_n \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$, defined as $L(x) = \lim_{n \to \infty} n^j f\left(\frac{x}{n^j}\right), x \in X$, such that
\[ \|f(x) - L(x)\| \leq \frac{1}{n^2} \Psi(x, x, \ldots, x) + \frac{1}{2n} \Psi(x, x, 0, \ldots, 0) \quad (3.8) \]
for all $x \in X$.

Proof. First, we note that $f(0) = 0 = \varphi(0, \ldots, 0)$ by the convergence of $\Psi(0, \ldots, 0)$. Now, associating (3.4) with (2.4), we lead to the crucial inequality
\[
\|f(x) - nf\left(\frac{x}{n}\right)\| \leq \frac{1}{n^2} \varphi\left(\frac{x}{n}, \frac{x}{n}, \ldots, \frac{x}{n}\right) + \frac{1}{2n} \varphi\left(\frac{x}{n}, \frac{x}{n}, 0, \ldots, 0\right),
\]
\[
\|f(x) - n^j f\left(\frac{x}{n^j}\right)\| \leq \frac{1}{n^2} \sum_{j=1}^{l} n^j \varphi\left(\frac{x}{n^j}, \frac{x}{n^j}, \ldots, \frac{x}{n^j}\right) + \frac{1}{2n} \sum_{j=1}^{l} n^j \varphi\left(\frac{x}{n^j}, \frac{x}{n^j}, 0, \ldots, 0\right),
\]
for all $x \in X$.

Then, by applying the same argument as in the proof of Theorem 3.1, one obtains the desired results. □

Corollary 3.3. Let $\delta, \theta$ be nonnegative real numbers and let $p$ be a positive real number with $p \neq 1$. Assume that $f : X \to Y$ is a mapping such that
\[ \|Df(x_1, \ldots, x_n)\| \leq \delta + \theta(\|x_1\|^p + \cdots + \|x_n\|^p) \]
for all $x_1, \ldots, x_n \in X$, where $\delta = 0$ when $p > 1$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that
\[ \|f(x) - L(x)\| \leq \frac{2\theta}{|n - np|} \|x\|^p + \frac{(n + 2)\delta}{2n(n - 1)} + \frac{(n - 2)\|f(0)\|}{4} \]
for all $x \in X$, where $\|f(0)\| \leq \frac{2\delta}{n(n - 1)}$.

Proof. Letting $\varphi(x_1, \ldots, x_n) := \delta + \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$ and applying Theorem 3.1 and Theorem 3.2, we get the result as claimed. □

We remark that the above upper bound of Corollary 3.3 is finer than that of Corollary 2.3 when $\theta = 0$. 
Corollary 3.4. Let $\delta, \theta$ be nonnegative real numbers and let $r_i (i = 1, \cdots, n)$ be positive real numbers with $r := \sum_{i=1}^{n} r_i \neq 1$. Assume that a mapping $f : X \to Y$ satisfies the functional inequality

$$\| Df(x_1, \cdots, x_n) \| \leq \delta + \theta \prod_{i=1}^{n} \| x_i \|^{r_i}$$

for all $x_1, \cdots, x_n \in X$, where $\delta = 0$ when $r > 1$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \| \leq \frac{\theta}{n|n-n'|} \| x \|^r + \frac{(n+2)\delta}{2n(n-1)} + \frac{(n-2)\| f(0) \|}{4}$$

for all $x \in X$, where $\| f(0) \| \leq \frac{2\delta}{n(n-1)}$.

**Proof.** Letting $\varphi(x_1, \cdots, x_n) := \delta + \theta \prod_{i=1}^{n} \| x_i \|^{r_i}$ and applying Theorem 3.1 and Theorem 3.2, we get the desired result as claimed.

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**References**


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