On the Representability Number of Lexicographic Products in a Dedekind-Complete Chain

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Abstract

For any two linear orderings $L$ and $M$, the representability number $\text{repr}_M(L)$ of $L$ in $M$ is the least ordinal $\alpha$ such that $L$ can be order-embedded into the lexicographic power $M^\alpha_{\text{lex}}$. We study $\text{repr}_M(L)$ in the particular case that $L$ is a lexicographic product and $M$ is a Dedekind-complete chain.

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1 Preliminaries

Let $L$ and $M$ be two chains (i.e., linear orderings). The representability number of $L$ in $M$ is the least ordinal $\alpha$ with the property that $L$ can be order-embedded into the lexicographic power $M^\alpha_{\text{lex}}$; this ordinal is denoted by $\text{repr}_M(L)$. In [6] we computed $\text{repr}_R(L)$ for some special chains $L$ (e.g., cardinal numbers, lexicographic powers of $\mathbb{R}$, and Aronszajn lines). In this short note we determine an upper bound for $\text{repr}_M(L)$ in the case that the base chain $M$ is Dedekind-complete and $L$ is a suitable lexicographic product of chains. This fact is relevant in view of a possible classification of (short) chains according to their representability number in $\mathbb{R}$ (cf. [7, 8]). Furthermore, the results obtained
here can be useful to provide some insight into the topic of lexicographic preferences and their representability by (extended forms of) utility functions: see, e.g., [1, 2, 4] for lexicographic preferences, and [3] for an overview of the topic of representability.

To begin we establish some basic terminology. A chain is denoted either by \((L, \prec)\) or simply by \(L\). If \(\alpha\) is a nonzero ordinal and \((L_\xi)_{\xi<\alpha}\) is a family of chains, the lexicographic product of this family is denoted by \(\left(\prod_{\xi<\alpha}^{} L_\xi, \prec_{\text{lex}}\right)\) or simply by \(\prod_{\xi<\alpha}^{} L_\xi\). In particular, \(L_\alpha^\text{lex}\) denotes the lexicographic product \(\prod_{\xi<\alpha}^{} L_\xi\) in which every factor is the chain \(L\).

An order-homomorphism (henceforth, homomorphism) is a map \(f : L \rightarrow M\) between two chains such that for all \(x, y \in L\), \(x \prec y\) implies \(f(x) \preceq f(y)\). In particular, an embedding (resp., isomorphism) is an injective (resp., bijective) homomorphism; the notation \(L \hookrightarrow M\) stands for embeddability of the chain \(L\) into the chain \(M\), whereas \(L \cong M\) denotes that \(L\) and \(M\) are isomorphic chains.

The density \(d(L)\) of a chain \((L, \prec)\) is the density of the topological space \((L, \tau_{\text{ord}})\), where \(\tau_{\text{ord}}\) is the order topology on \(L\) induced by \(\prec\). The perfect density \(d'(L)\) of \((L, \prec)\) is the least infinite cardinal \(\kappa\) such that there exists a set \(D \subseteq L\) which has cardinality \(\leq \kappa\) and intersects every closed interval containing more than one point. Observe that \(\text{repr}_\mathbb{R}(L) \leq 1\) if and only if \(d'(L) = \aleph_0\) if and only if \((L, \tau_{\text{ord}})\) is second-countable. A chain \((L, \prec)\) is dense-in-itself if for any \(x, y \in L\), \(x \prec y\) implies that the open interval \((x, y)\) is nonempty; in this case, \(d(L) = d'(L)\). The cellularity \(c(L)\) of \((L, \prec)\) is the least infinite cardinal \(\kappa\) such that every family of pairwise disjoint nonempty open intervals of \(L\) has cardinality \(\leq \kappa\); in particular, \(L\) has the c.c.c. (countable chain condition) if \(c(L) = \aleph_0\). Note that \(c(L) \leq d(L) \leq d'(L)\) for any chain \((L, \prec)\).

A chain is short if it embeds neither \(\omega_1\) nor its reverse ordering \(\omega_1^*\). An Aronszajn line is an uncountable short chain which embeds no uncountable subset of \(\mathbb{R}\). A Souslin line is a chain which has the c.c.c. but is not separable. The Souslin Hypothesis (SH) says that there exists no Souslin line; it is well-known that SH is independent from the usual axioms of set theory (see [9]).

The next result collects some facts.

**Lemma 1.1** For any chain \(L\), we have:

(i) \(\text{repr}_2^{}(L) \leq \min\{d'(L), d(L) + 1\}\);

(ii) if \(L\) has the c.c.c., then \(d(L) \leq \aleph_1\);

(iii) if \(L\) is a Souslin line, then it contains an Aronszajn line (which is dense in \(L\));

(iv) if \(L\) is an Aronszajn line, then \(\text{repr}_2^{}(L) = \text{repr}_\mathbb{R}(L) = \omega_1\).
Proof. For (i) and (iv), see [6]; for (ii) and (iii), see [10]. □

An immediate consequence of Lemma 1.1 is the following.

Corollary 1.2 For any Souslin line $S$, $\omega_1 \leq \text{repr}_R(S) \leq \text{repr}_2(S) \leq \omega_1 + 1$.

If there exists a Souslin line, then there is one whose representability number in 2 is $\omega_1$.

Example 1.3 Let $S$ be a Souslin line. Define on $S$ an equivalence relation $\sim$ as follows: for each $x, y \in S$, $x \sim y$ if the interval with endpoints $x$ and $y$ is separable. Let $S' := S/\sim$ be the quotient chain (whose linear order is defined from the order on $S$ in the obvious way). Then $S'$ is a dense-in-itself Souslin line (see [9]) such that $\text{repr}_2(S') = \omega_1$.

This paper contains a brief analysis of the representability numbers of a class of chains which includes Souslin lines. In particular, we resolve the ambiguity in Corollary 1.2 by showing that $\text{repr}_2(S) = \omega_1$ for any Souslin line $S$.

2 Thin chains

The well-ordering number of a chain $L$, denoted by $\text{wo}(L)$, is the supremum of the set of all cardinals $\kappa$ such that either $\kappa$ or $\kappa^*$ embeds into $L$. Thus, $L$ is short if and only if $\text{wo}(L) \leq \aleph_0$. The next result relates the size and the well-ordering number of a chain (see [10], Theorem 3.4).

Lemma 2.1 For any chain $L$, $|L| \leq 2^{\text{wo}(L)}$.

Definition 2.2 A chain $L$ is said to be thin if $d(L) \leq \text{wo}(L)^+$; it is thick otherwise. In particular, a thin chain is ultrathin if $d(L) \leq \text{wo}(L)$.

Example 2.3 Finite lexicographic sums of ordinals and reverse ordinals are ultrathin chains. $\mathbb{Q}$ and $\mathbb{R}$ are short ultrathin chains. Aronszajn lines and Souslin lines are short thin chains. Under $\neg\text{CH}$, $\mathbb{R}_{\text{lex}}^\alpha$ is a short thick chain for any ordinal $2 \leq \alpha < \omega_1$.

The last example shows that, under $\neg\text{CH}$, thinness of a chain is not preserved by lexicographic products, even finite. On the other hand, Lemma 2.1 implies that under CH there are no short thick chains, and under GCH there are no thick chains at all.

In the reminder of this section we give a subordering characterization of thin chains. To begin we recall some basic facts about cardinal invariants for topological spaces (see [5]). The tightness $t(X)$ of a topological space $(X, \tau)$ is
the supremum of the set \( \{ t(x, X) : x \in X \} \), where \( t(x, X) \) is the least infinite cardinal \( \kappa \) with the property that if \( x \in A \), then there exists \( A_0 \subseteq A \) such that \( |A_0| \leq \kappa \) and \( x \in A_0 \). If \( (L, \prec) \) is a chain, then \( L \) will be implicitly considered as a topological space with the order topology \( \tau_{\text{ord}} \); in this case, \( t(L) \leq \text{wo}(L) \).

If \( A \) is a subset of a chain \( L \), then the subspace topology \( \tau_{\text{sub}} \) on \( A \) is finer than the order topology \( \tau_{\text{ord}} \) on \( A \); thus, in particular, \( d(A, \tau_{\text{ord}}) \leq d(A, \tau_{\text{sub}}) \).

**Lemma 2.4** Let \( X \) be a topological space and \( \kappa \) a regular cardinal \( \geq \max \{ d(X), t(X)^+ \} \). There exists an increasing sequence \( (X_\xi)_{\xi<\kappa} \) of subspaces of \( X \) with the property that \( X = \bigcup_{\xi<\kappa} X_\xi \) and for all \( \xi < \kappa \), \( d(X_\xi) < \kappa \).

**Proof.** Let \( D \) be a dense subset of \( X \) such that \( |D| = d(X) \). Since \( \kappa \geq d(X) \), we can write \( D = \bigcup_{\xi<\kappa} D_\xi \), where \( (D_\xi)_{\xi<\kappa} \) is an increasing sequence of subsets of \( X \) such that for each \( \xi < \kappa \), we have \( |D_\xi| < \kappa \). (If \( \kappa > d(X) \), define \( D_\xi := D \) for \( d(X) \leq \xi < \kappa \).) Note that \( \bigcup_{\xi<\kappa} D_\xi = \bigcup_{\xi<\kappa} D_\xi^\prec \), because \( \kappa \) is regular and is greater than \( t(X) \). Set \( X_\xi := D_\xi^\prec \) for all \( \xi < \kappa \). The sequence \( (X_\xi)_{\xi<\kappa} \) satisfies the claim. \( \square \)

**Proposition 2.5** The following statements are equivalent for a chain \( (L, \prec) \):

(i) \( L \) is thin;

(ii) for any regular cardinal \( \kappa \geq \text{wo}(L)^+ \), there exists an increasing sequence of chains \( (L_\xi)_{\xi<\kappa} \) such that \( L = \bigcup_{\xi<\kappa} L_\xi \) and for each \( \xi < \kappa \), \( d(L_\xi, \tau_{\text{sub}}) < \kappa \).

In particular, a short chain is thin if and only if it is the union of an increasing \( \omega_1 \)-sequence of subchains which are separable in the subspace topology (equivalently, in the order topology).

**Proof.** Assume that \( L \) is thin and let \( \kappa \) be a regular cardinal \( \geq \text{wo}(L)^+ \). It follows that \( \max \{ d(L), t(L)^+ \} \leq \kappa \). Thus (i) implies (ii), using Lemma 2.4.

Next assume that (ii) holds. Then \( L = \bigcup_{\xi<\text{wo}(L)^+} L_\xi \), where \( (L_\xi)_{\xi<\text{wo}(L)^+} \) is an increasing sequence of chains such that for each \( \xi < \text{wo}(L)^+ \), \( d(L_\xi, \tau_{\text{ord}}) \leq \text{wo}(L)^+ \). Note that if \( D_\xi \subseteq L_\xi \) is dense in \( (L_\xi, \tau_{\text{ord}}) \) for each \( \xi < \text{wo}(L)^+ \), then \( D := \bigcup_{\xi<\text{wo}(L)^+} D_\xi \) is dense in \( L \), because \( (L_\xi)_{\xi<\text{wo}(L)^+} \) is an increasing sequence indexed on a limit ordinal. It follows that \( d(L) \leq \sum_{\xi<\text{wo}(L)^+} d(L_\xi, \tau_{\text{ord}}) \) (where \( \sum \) denotes cardinal sum), and so \( L \) is thin. \( \square \)

### 3 Main result

If \( L \) is a thin chain, then Lemma 1.1 implies that \( \text{repr}_2(L) \leq \text{wo}(L)^+ + 1 \).

In this section we prove the main result of this paper (Theorem 3.2), which yields as a consequence an improvement of the above upper bound for suitable lexicographic products of thin chains (Corollaries 3.3 and 3.4).
Definition 3.1 Given two subsets $A$ and $B$ of a chain $(Z, \prec)$, the notation $A \prec B$ means that $a \prec b$ for each $a \in A$ and $b \in B$. We call a chain $(Z, \prec)$ Dedekind-complete if for every pair $(A, B)$ of (not necessarily empty) subchains of $Z$ such that $A \prec B$, there exists $z \in Z$ satisfying $A \preceq \{z\} \preceq B$.

Note that a chain $(Z, \prec)$ is Dedekind-complete if and only if it is compact Hausdorff in the order topology, in particular $Z$ has endpoints. Observe also that the lexicographic product of any well-ordered family of Dedekind-complete chains is a Dedekind-complete chain. Finally, note that any homomorphism from a subchain of a chain $L$ into a Dedekind-complete chain can be lifted to $L$.

Theorem 3.2 Let $\alpha$ be an ordinal, $\kappa$ a cardinal, $Z$ a Dedekind-complete chain, and $(L_{\eta,\xi})_{(\eta,\xi)\in\alpha\times\kappa}$ a family of chains such that for each fixed $\eta < \alpha$, the subfamily $(L_{\eta,\xi})_{\xi < \kappa}$ is increasing. If $\alpha < \text{cf} \kappa$, then

$$\text{repr}_Z \left( \bigcup_{\eta < \alpha} \prod_{\xi < \kappa} L_{\eta,\xi} \right) \leq \sum_{\xi < \kappa} \sum_{\eta < \alpha} \text{repr}_Z (L_{\eta,\xi})$$ (1)

where $\sum$ denotes ordinal sum. In particular, if $\alpha < \kappa$, $\kappa$ is regular and $\text{repr}_Z(L_{\eta,\xi}) < \kappa$ for each $(\eta,\xi) \in \alpha \times \kappa$, then

$$\text{repr}_Z \left( \bigcup_{\xi < \kappa} \prod_{\eta < \alpha} L_{\eta,\xi} \right) \leq \kappa.$$ (2)

Proof. First we prove the following inequality:

$$\text{repr}_Z \left( \bigcup_{\eta < \alpha} \prod_{\xi < \kappa} L_{\eta,\xi} \right) \leq \sum_{\xi < \kappa} \sum_{\eta < \alpha} \text{repr}_Z (L_{\eta,\xi}).$$ (3)

For each $\xi < \kappa$, let $L_\xi := \prod_{\eta < \alpha} L_{\eta,\xi}$. Since $(L_\xi)_{\xi < \kappa}$ is an increasing family of chains, $L := \bigcup_{\xi < \kappa} L_\xi$ is a well-defined chain. For each $\eta < \alpha$ and $\xi < \kappa$, denote $\beta_{\eta,\xi} := \text{repr}_Z(L_{\eta,\xi})$, $\beta_\xi := \sum_{\eta < \alpha} \beta_{\eta,\xi}$ and $\beta := \sum_{\xi < \kappa} \beta_\xi$. Now we show that $L$ embeds into $Z^\beta_{\text{lex}}$, this will prove (3). Fix $\xi < \kappa$. Since $L_{\eta,\xi}$ embeds into $Z^\beta_{\text{lex}}$ for each $\eta < \alpha$, it follows that $L_\xi$ embeds into $Z^\beta_{\text{lex}}$ for each $\xi < \kappa$; let $\iota_\xi : L_\xi \hookrightarrow Z^\beta_{\text{lex}}$ be an embedding. By the Dedekind-completeness of $Z$, each embedding $\iota_\xi$ can be lifted to a homomorphism $f_\xi : L \to Z^\beta_{\text{lex}}$. Then the function $f : L \to \prod_{\xi < \kappa} Z^\beta_{\text{lex}}$, defined by $f(x) = (f_\xi(x))_{\xi < \kappa}$ for all $x \in L$, is an embedding. Since $\prod_{\xi < \kappa} Z^\beta_{\text{lex}} \cong Z^\beta_{\text{lex}}$, we obtain $L \hookrightarrow Z^\beta_{\text{lex}}$, as claimed. Next we prove that if $\alpha < \text{cf} \kappa$, then the equality (of sets)

$$\prod_{\eta < \alpha} \bigcup_{\xi < \kappa} L_{\eta,\xi} = \bigcup_{\xi < \kappa} \prod_{\eta < \alpha} L_{\eta,\xi}$$ (4)
holds; then (1) follows from (3) and (4). Let \( x = (x_\eta)_{\eta < \alpha} \in \prod_{\eta < \alpha} \bigcup_{\xi < \kappa} L_{\eta, \xi} \). Then for each \( \eta < \alpha \), there exists \( \xi_\eta < \kappa \) such that \( x_\eta \in L_{\eta, \xi_\eta} \). Set \( \xi(x) := \sup\{\xi_\eta : \eta < \alpha\} \). Note that \( \xi(x) < \kappa \), because \( \alpha < \text{cf} \kappa \). Since \( (L_{\eta, \xi})_{\xi < \kappa} \) is increasing, we obtain that \( x_\eta \in L_{\eta, \xi_\eta}(x) \) for each \( \eta < \alpha \), whence \( x \in \prod_{\eta < \alpha} L_{\eta, \xi_\eta}(x) \). This proves one inclusion in (4); the other inclusion is obvious.

Finally observe that if \( \alpha < \kappa = \text{cf} \kappa \) and \( \text{repr}_Z(L_{\eta, \xi}) < \kappa \) for each \( (\eta, \xi) \in \alpha \times \kappa \), then (using the same notation as before) \( \beta_\xi < \kappa \) for each \( \xi < \kappa \), and \( \beta \leq \kappa \). Thus (2) holds. \( \square \)

The following consequences of Theorem 3.2, in which the Dedekind-complete base chain \( Z \) is the linear ordering with two elements only, are of some interest.

**Corollary 3.3** Let \( \alpha \) be an ordinal, \( \kappa \) a regular cardinal such that \( \alpha < \kappa \), and \( (L_\eta)_{\eta < \alpha} \) a family of thin chains. If \( \text{wo}(L_\eta) < \kappa \) for each \( \eta < \alpha \), then \( \text{repr}_2(\prod_{\eta < \alpha} L_\eta) \leq \kappa \).

**Proof.** Fix \( \eta < \alpha \). Since \( L_\eta \) is thin and \( \kappa \) is a regular cardinal \( \geq \text{wo}(L_\eta)^+ \), Proposition 2.5 implies that there exists an increasing family of chains \( (L_{\eta, \xi})_{\xi < \kappa} \) such \( \bigcup_{\xi < \kappa} L_{\eta, \xi} = L_\eta \) and \( d(L_{\eta, \xi}, \tau_{\text{ord}}) < \kappa \) for each \( \xi < \kappa \). By Lemma 1.1(i), \( \text{repr}_2(L_{\eta, \xi}) \leq d(L_{\eta, \xi}, \tau_{\text{ord}}) + 1 \) for all \( \eta < \alpha \) and \( \xi < \kappa \). Now apply Theorem 3.2 to conclude \( \text{repr}_2(\prod_{\eta < \alpha} L_\eta) \leq \kappa \). \( \square \)

**Corollary 3.4** Let \( \alpha \) be an ordinal, \( \kappa \) a cardinal, and \( (L_\eta)_{\eta < \alpha} \) a family of thin chains.

(i) If \( \alpha < \kappa^+ \) and for each \( \eta < \kappa \), \( \text{wo}(L_\eta) \leq \kappa \), then \( \text{repr}_2(\prod_{\eta < \alpha} L_\eta) \leq \kappa^+ \).

(ii) If \( \alpha \) is countable and each chain \( L_\eta \) is short, then \( \text{repr}_2(\prod_{\eta < \alpha} L_\eta) \leq \omega_1 \).

(iii) If \( \alpha \) is countable \( \neq 0 \) and each chain \( L_\eta \) is either a Souslin line or an Aronszajn line, then \( \text{repr}_2(\prod_{\eta < \alpha} L_\eta) = \text{repr}_\aleph(\prod_{\eta < \alpha} L_\eta) = \omega_1 \).

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**References**

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