Hermite-Hadamard and Simpson-like Type Integral Inequalities for Differentiable Preinvex Functions

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Abstract
In this article, we establish some estimates of Hermite-Hadamard-like type and Simpson-like type integral inequalities for functions whose first derivatives in absolute value at certain powers are preinvex.

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1 Introduction
The following definition is well known in the literature: Let $I$ be an interval in $\mathbb{R}$. Then $f : I \rightarrow \mathbb{R}$ is said to be convex on $I$ if

$$f (tx + (1 - t) y) \leq tf (x) + (1 - t) f (y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-hadamard inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a
convex function and \( a, b \in I \) with \( a < b \). Then

\[
 f\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Hadamard's inequality and Simpson's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [2, 9, 11, 14, 15, 20, 22, 23] and references therein.

In recent years, several refinements and generalizations have been considered for classical convexity [20, 21]. A significant generalization of convex functions is that of invex functions introduced by Hanson in [8]. Weir and Mond [24] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [19] introduced the concept of preinvex function as a generalization of invex functions. Later, Mohan and Neogy [13] obtained some properties of generalized preinvex functions.

Noor [2, 3, 4] has established some Hermite-Hadamard type inequalities for preinvex and logarithmic preinvex functions. In recent papers, Noor and Barani et al. in [2, 3, 4, 7] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

**Definition 1.** A set \( K \subseteq \mathbb{R} \) is said to be **invex** with respect to the map \( \eta : K \times K \to \mathbb{R} \), if for any \( x, y \in K \) and \( t \in [0, 1] \), \( x + t \eta(y, x) \in K \).

It is obvious that every convex set is invex with respect to the map \( \eta(x, y) = y - x \), but there exist invex sets which are not convex [13].

**Definition 2.** Let \( K \subseteq \mathbb{R} \) be an invex set with respect to the map \( \eta : K \times K \to \mathbb{R} \). Then the function \( f : K \to \mathbb{R} \) is said to be **preinvex** with respect to \( \eta \), if

\[
 f(y + t \eta(x, y)) \leq tf(x) + (1 - t)f(y)
\]

for any \( x, y \in K \) and \( t \in [0, 1] \).

**Definition 3.** Let \( K \subseteq \mathbb{R} \) be an invex set with respect to the map \( \eta : K \times K \to \mathbb{R} \). Then the function \( f : K \to \mathbb{R} \) is said to be **logarithmic preinvex** with respect to \( \eta \), if

\[
 f(y + t \eta(x, y)) \leq [f(x)]^{1-t} [f(y)]^t
\]

for any \( x, y \in K \) and \( t \in [0, 1] \).

For the refinement and generalizations of preinvex function and logarithmic preinvex functions, you may see [1, 10, 12, 13, 18, 20, 25, 26]. In recent paper, Noor [4] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:
Theorem 1.1. For an interval \([a, a + \eta(b, a)]\) on the real line \(R\), let \(f : [a, a + \eta(b, a)] \to R_+\) be a preinvex function on an interior \(K^0\) of the interval \(K\) and 
\(a, b \in K^0\) with \(a < a + \eta(b, a)\). Then the inequality holds:
\[
\frac{f(a) + f(a + \eta(b, a))}{2} \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(t) \, dt \leq \frac{f(a) + f(a + \eta(b, a))}{2}.
\]

In [7], Barani et al. introduced some generalizations of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are preinvex.

Theorem 1.2. Let \(K \subseteq R\) be an open invex subset with respect to \(\eta : K \times K \to R\). Suppose that \(f : K \to R\) is a differentiable function. If \(|f'|\) is preinvex on \(K\), then for any \(a, b \in K\) with \(\eta(b, a) \neq 0\) the following inequality holds:
\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(t) \, dt \right| \leq \frac{\eta(b, a)}{8} \left\{ \left| f'(a) \right| + \left| f'(a) \right| \right\}.
\]  

(1)

Theorem 1.3. Let \(K \subseteq R\) be an open invex subset with respect to \(\eta : K \times K \to R\). Suppose that \(f : K \to R\) is a differentiable function. Assume \(p \in R\) with \(p > 1\). If \(|f'|^{p-1}\) is preinvex on \(K\), then for any \(a, b \in K\) with \(\eta(b, a) \neq 0\) the following inequality holds:
\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(t) \, dt \right| \leq \frac{\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left[ \frac{1}{2} \left\{ \left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right\} \right].
\]  

(2)

Theorem 1.4. Let \(K \subseteq R\) be an open invex subset with respect to \(\eta : K \times K \to R\). Suppose that \(f : K \to R\) is a differentiable function. Assume \(p \in R\) with \(p > 1\). If \(|f'|^{p-1}\) is preinvex on \(K\), then for any \(a, b \in K\) with \(\eta(b, a) \neq 0\) the following inequality holds:
\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(t) \, dt \right| \leq \frac{\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left[ \sup \left\{ \left| f'(a) \right|^{\frac{p}{p-1}}, \left| f'(b) \right|^{\frac{p}{p-1}} \right\} \right]^{\frac{p-1}{p}}.
\]  

(3)
In recent article, Latif [12] proved the following theorem:

**Theorem 1.5.** Let \( K \subseteq R \) be an open invex subset with respect to \( \eta : K \times K \rightarrow R \). Suppose that \( f : K \rightarrow R \) is a differentiable function on \( K \) such that \( f' \in L([a, a + \eta(b,a)]) \). If \( |f'| \) is preinvex on \( K \), then for any \( a, b \in K \) with \( \eta(b, a) \neq 0 \) the following inequality holds:

\[
\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)dt \right| \\
\leq \frac{\eta(b,a)}{8} \left[ \sup \left\{ \left| f'(a) \right|, \left| f'(a + \frac{1}{2}\eta(b,a)) \right| \right\} \right] \\
+ \sup \left\{ \left| f'(a + \frac{1}{2}\eta(b,a)) \right|, \left| f'(a + \eta(b,a)) \right| \right\}. 
\]  

(4)

The main aim of this paper is to establish new generalized similar inequalities concerning Hermite-Hadamard-like and Simpson-like type inequality for the class of differentiable functions whose derivatives at certain powers are preinvex functions.

2 Some new Hermite-Hadamard-type inequalities

To establish some new Hermite-Hadamard type inequalities for \( s \)-convex functions in the second sense, we need the following lemma.

**Lemma 1.** Let \( K \subseteq R \) be an invex set with respect to the map \( \eta : K \times K \rightarrow R \) and, \( \eta(b, a) \neq 0 \) with 0 \( \leq a \leq a + \eta(b,a) < \infty \) for all \( a, b \in K \) with \( a < b \). Suppose that \( f : K \rightarrow R \) is a differentiable function on \( K^0 \) such that \( f' \in L([a, a + \eta(b,a)]) \). Then the following identity holds:

\[
I(f : \lambda, \mu) \\
= \lambda \left\{ \mu f(a) + (1 - \mu) f(a + \eta(b,a)) \right\} \\
+ (1 - \lambda) f(a + \mu \eta(b,a)) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)dt \\
= \eta(b,a) \int_0^1 p(\lambda, \mu, t) f'(a + t\eta(b,a))dt, 
\]  

(5)
where

\[
p(\lambda, \mu, t) = \begin{cases} 
\lambda(1 - \mu) - t, & t \in [0, \mu] \\
(1 - \lambda\mu) - t, & t \in (\mu, 1]
\end{cases}
\]

for \( \lambda, \mu \in [0, 1] \).

**Proof.** By integration by parts, this equality (5) is proved. \( \square \)

Now we turn our attention to establish inequalities of Hermit-Hadamard type for differentiable preinvex functions and logarithmic preinvex functions.

**Theorem 2.1.** Let \( K \subseteq \mathbb{R} \) be an invex set with respect to the map \( \eta : K \times K \to \mathbb{R} \) and, \( \eta(b, a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b, a) < \infty \) for all \( a, b \in K \) with \( a < b \). Suppose that \( f : K \to \mathbb{R} \) is a differentiable function on \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \). If \( |f'| \) is preinvex with respect to \( \eta \) on \( S \), then for \( \lambda, \mu \in [0, 1] \), the following inequality holds:

\[
|I(f : \lambda, \mu)| \leq |\eta(b, a)| \begin{cases} 
\{ (\delta_1 + \delta_2) - (\varepsilon_1 + \varepsilon_3) \} |f'(a)| + (\varepsilon_1 + \varepsilon_3) |f'(b)|, & \mu \leq \lambda(1 - \mu) \leq 1 - \lambda\mu \\
\{ (\delta_2 + \delta_3) - (\varepsilon_2 + \varepsilon_3) \} |f'(a)| + (\varepsilon_2 + \varepsilon_3) |f'(b)|, & \lambda(1 - \mu) \leq \mu \leq 1 - \lambda\mu \\
\{ (\delta_2 + \delta_4) - (\varepsilon_2 + \varepsilon_4) \} |f'(a)| + (\varepsilon_2 + \varepsilon_4) |f'(b)|, & \lambda(1 - \mu) \leq 1 - \lambda\mu \leq \mu
\end{cases}
\]

where

\[
\delta_1 = \frac{1}{2} \mu(2(1 - \mu)\lambda - \mu), \\
\delta_2 = (\frac{1}{2} + \lambda + \lambda^2)\mu^2 - \lambda(1 + \lambda)\mu + \lambda^2, \\
\delta_3 = \frac{1}{2}\{ 1 + 2\lambda + 2\lambda^2 \} \mu^2 - 2(1 + \lambda)\mu + 1, \\
\delta_4 = \frac{1}{2}(1 - \mu)\{(1 + 2\lambda)\mu - 1\}, \\
\varepsilon_1 = \frac{1}{6}\{ 3\lambda - (2 + 3\lambda)\mu \} \mu^2, \\
\varepsilon_2 = \frac{1}{6}\{ (2 + 3\lambda - 2\lambda^3)\mu^3 + 6\lambda\mu^2 - 4\lambda^3\mu + 2\lambda^3 \}, \\
\varepsilon_3 = \frac{1}{6}\{ (2 + 3\lambda - 2\lambda^3)\mu^3 - 3(1 - 2\lambda^2)\mu^2 - 3\lambda\mu + 1 \}, \\
\varepsilon_4 = \frac{1}{6}\{ 3\mu(\mu + \lambda) - (1 + 3\lambda)\mu^3 - 1 \}.
\]
Proof. Suppose that \( a, a + \eta(b, a) \in S \). Since \( S \) is an invex set with respect to \( \eta \), we have \( a + t\eta(b, a) \in S \) for any \( t \in [0, 1] \).

By Lemma 1, we have

\[
|I(\lambda, \mu)|
= \left| \lambda \left\{ \mu f(a) + (1 - \mu) f(a + \eta(b, a)) \right\} 
+ (1 - \lambda) f(a + \mu \eta(b, a)) - \frac{1}{\eta(b, a)} \int_{a}^{a + \eta(b, a)} f(t) dt \right|
\leq |
\eta(b, a)| \int_{0}^{1} |p(\lambda, \mu, t)| \left| f'(a + t\eta(b, a)) \right| dt
= |\eta(b, a)| \left\{ \int_{0}^{\mu} \left| \lambda(1 - \mu) - t \right| \left| f'(a + t\eta(b, a)) \right| dt 
+ \int_{\mu}^{1} \left| (1 - \lambda\mu) - t \right| \left| f'(a + t\eta(b, a)) \right| dt \right\}.
\]

Note that

\[
\int_{0}^{\mu} \left| \lambda(1 - \mu) - t \right| (1 - t) dt = \begin{cases} 
\delta_1 - \varepsilon_1, & \mu \leq \lambda(1 - \mu) \\
\delta_2 - \varepsilon_2, & \mu \geq \lambda(1 - \mu),
\end{cases}
\]

(8)

\[
\int_{\mu}^{1} \left| (1 - \lambda\mu) - t \right| (1 - t) dt = \begin{cases} 
\delta_3 - \varepsilon_3, & \mu \leq 1 - \lambda\mu \\
\delta_4 - \varepsilon_4, & \mu \geq 1 - \lambda\mu,
\end{cases}
\]

(9)

\[
\int_{0}^{\mu} \left| \lambda(1 - \mu) - t \right| t dt = \begin{cases} 
\varepsilon_1, & \mu \leq \lambda(1 - \mu) \\
\varepsilon_2, & \mu \geq \lambda(1 - \mu),
\end{cases}
\]

(10)

\[
\int_{\mu}^{1} \left| (1 - \lambda\mu) - t \right| t dt = \begin{cases} 
\varepsilon_3, & \mu \leq 1 - \lambda\mu \\
\varepsilon_4, & \mu \geq 1 - \lambda\mu,
\end{cases}
\]

(11)

\[
\int_{0}^{\mu} \left| \lambda(1 - \mu) - t \right| dt = \begin{cases} 
\delta_1, & \mu \leq \lambda(1 - \mu) \\
\delta_2, & \mu \geq \lambda(1 - \mu),
\end{cases}
\]

(12)

\[
\int_{\mu}^{1} \left| (1 - \lambda\mu) - t \right| dt = \begin{cases} 
\delta_3, & \mu \leq 1 - \lambda\mu \\
\delta_4, & \mu \geq 1 - \lambda\mu,
\end{cases}
\]

(13)
Making use of the preinvexity of \( f' \) on \([a, a + \eta(b, a)]\), we get

\[
(a) \int_0^\mu |(1 - \mu) - t||f'(a + t\eta(b, a))|dt \\
\leq \left\{ \int_0^\mu |(1 - \mu) - t|(1 - t)dt \right\} |f'(a)| \\
+ \left\{ \int_0^\mu |(1 - \mu) - t|tdt \right\} |f'(b)| \\
= \left\{ (\delta_1 - \varepsilon_1) |f'(a)| + \varepsilon_1 |f'(b)|, \quad \mu \leq \lambda(1 - \mu) \right\} \\
\left\{ (\delta_2 - \varepsilon_2) |f'(a)| + \varepsilon_2 |f'(b)|, \quad \mu \geq \lambda(1 - \mu) \right\} \\
(14)
\]

\[
(b) \int_\mu^1 |f'(a + t\eta(b, a))|dt \\
\leq \left\{ \int_0^\mu |(1 - \lambda\mu) - t|(1 - t)dt \right\} |f'(a)| \\
+ \left\{ \int_0^\mu |(1 - \lambda\mu) - t|tdt \right\} |f'(b)| \\
= \left\{ (\delta_3 - \varepsilon_3) |f'(a)| + \varepsilon_3 |f'(b)|, \quad \mu \leq \lambda(1 - \mu) \right\} \\
\left\{ (\delta_4 - \varepsilon_4) |f'(a)| + \varepsilon_4 |f'(b)|, \quad \mu \geq \lambda(1 - \mu) \right\} \\
(15)
\]

By substituting (8)-(15) in (7), we get

\[
|I(\lambda, \mu)| \leq |\eta(b, a)| \left\{ \begin{array}{ll}
(\delta_1 + \delta_2) - (\varepsilon_1 + \varepsilon_3) & |f'(a)| + (\varepsilon_1 + \varepsilon_3) |f'(b)|, \\
\mu \leq \lambda(1 - \mu) \leq 1 - \lambda\mu
\end{array} \right.

\[
(\delta_2 + \delta_3) - (\varepsilon_2 + \varepsilon_3) & |f'(a)| + (\varepsilon_2 + \varepsilon_3) |f'(b)|, \\
\lambda(1 - \mu) \leq \mu \leq 1 - \lambda\mu
\end{array} \right.

\[
(\delta_2 + \delta_4) - (\varepsilon_2 + \varepsilon_4) & |f'(a)| + (\varepsilon_3 + \varepsilon_4) |f'(b)|, \\
\lambda(1 - \mu) \leq 1 - \lambda\mu \leq \mu
\end{array} \right.
\]

which completes the proof. \(\square\)

**Theorem 2.2.** Let \( K \subseteq R \) be an invex set with respect to the map \( \eta : K \times K \to R \) and, \( \eta(b, a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b, a) < \infty \) for all \( a, b \in S \) with \( a < b \). Suppose that \( f : K \to R \) is a differentiable function on \( K^0 \) such that \( f' \in L([a, a + \eta(b, a)]) \). Assume \( p \in R \) with \( p > 1 \). If \( f' \) is preinvex with respect to \( \eta \) on \( K \), for \( q \geq 1 \), then, for \( \lambda, \mu \in [0, 1] \) the following inequality
holds:

$$|I(\lambda, \mu)| \leq \left| \eta(b, a) \right| \left\{ \int_0^{\mu} \left| \lambda(1 - \mu) - t \right| dt \right\}^{1 - \frac{1}{q}} \times \left\{ \int_0^{\mu} \left| \lambda(1 - \mu) - t \right| |f'(a + t\eta(b, a))|^{\frac{q}{q}} dt \right\}^{\frac{1}{q}}$$

$$+ \left\{ \int_{\mu}^1 \left| (1 - \lambda\mu) - t \right| dt \right\}^{1 - \frac{1}{q}} \times \left\{ \int_{\mu}^1 \left| (1 - \lambda\mu) - t \right| |f'(a + t\eta(b, a))|^{\frac{q}{q}} dt \right\}^{\frac{1}{q}}.$$

Making use of the preinvexity of $|f'|^q$ on $[a, a + \eta(b, a)]$, for any $t \in [0, 1]$ we know that

$$|f'(a + t\eta(b, a))|^q \leq (1 - t) |f'(a)|^q + t |f'(b)|^q,$$
which implies that

\[
(a) \int_0^\mu |\lambda(1 - \mu) - t| |f'(a + t\eta(b, a))|^q \, dt \\
\leq \left\{ \int_0^\mu |\lambda(1 - \mu) - t|(1 - t)dt \right\} |f'(a)|^q \\
+ \left\{ \int_0^\mu |\lambda(1 - \mu) - t|t\, dt \right\} |f'(b)|^q \\
= \left\{ (\delta_1 - \varepsilon_1) |f'(a)|^q + \varepsilon_1 |f'(b)|^q, \quad \mu \leq \lambda(1 - \mu) \right\} \\
\left\{ (\delta_2 - \varepsilon_2) |f'(a)|^q + \varepsilon_2 |f'(b)|^q, \quad \mu \geq \lambda(1 - \mu) \right\},
\]

(17)

\[
(b) \int_\mu^1 |(1 - \lambda\mu) - t| |f'(a + t\eta(b, a))|^q \, dt \\
\leq \left\{ \int_0^\mu |(1 - \lambda\mu) - t|(1 - t)dt \right\} |f'(a)|^q \\
+ \left\{ \int_0^\mu |(1 - \lambda\mu) - t|t\, dt \right\} |f'(b)|^q \\
= \left\{ (\delta_3 - \varepsilon_3) |f'(a)|^q + \varepsilon_3 |f'(b)|^q, \quad \mu \leq \lambda(1 - \mu) \right\} \\
\left\{ (\delta_4 - \varepsilon_4) |f'(a)|^q + \varepsilon_4 |f'(b)|^q, \quad \mu \geq \lambda(1 - \mu) \right\},
\]

(18)

By substituting (17)-(18) in (16), we get the desired result.

\[\square\]

**Theorem 2.3.** Let \( K \subseteq R \) be an invex set with respect to the map \( \eta : K \times K \to R \) and, \( \eta(b, a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b, a) < \infty \) for all \( a, b \in K \) with \( a < b \). Suppose that \( f : K \to R \) is a differentiable function on \( K^0 \) such that \( f' \in L([a, a + \eta(b, a)]) \). Assume \( q \in R \) with \( q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( |f'|^q \) is preinvex with respect to \( \eta \) on \( K \), then for \( \lambda, \mu \in [0, 1] \), the following inequality holds:
where

\[
\begin{align*}
|I(\lambda, \mu)| & \leq |\eta(b, a)| \left\{ \begin{array}{l}
\delta_5 \left\{ \frac{(2-\mu)^2}{2} \left| f'(a) \right|^q + \frac{\mu^2}{2} \left| f'(b) \right|^q \right\}^{\frac{1}{q}} \\
+ \delta_7 \left\{ \frac{(1-\mu)^2}{2} \left| f'(a) \right|^q + \frac{1-\mu^2}{2} \left| f'(b) \right|^q \right\}^{\frac{1}{q}}, \\
\mu \leq \lambda(1-\mu) \leq 1 - \lambda \mu \\
\delta_6 \left\{ \frac{(2-\mu)^2}{2} \left| f'(a) \right|^q + \frac{\mu^2}{2} \left| f'(b) \right|^q \right\}^{\frac{1}{q}} \\
+ \delta_8 \left\{ \frac{(1-\mu)^2}{2} \left| f'(a) \right|^q + \frac{1-\mu^2}{2} \left| f'(b) \right|^q \right\}^{\frac{1}{q}}, \\
\lambda(1-\mu) \leq \mu \leq 1 - \lambda \mu \\
\delta_5 \left\{ \frac{(2-\mu)^2}{2} \left| f'(a) \right|^q + \frac{\mu^2}{2} \left| f'(b) \right|^q \right\}^{\frac{1}{q}} \\
+ \delta_7 \left\{ \frac{(1-\mu)^2}{2} \left| f'(a) \right|^q + \frac{1-\mu^2}{2} \left| f'(b) \right|^q \right\}^{\frac{1}{q}}, \\
\lambda(1-\mu) \leq 1 - \lambda \mu \leq \mu,
\end{array} \right.
\end{align*}
\]

Proof. Suppose that \(a, a + \eta(b, a) \in S\). Since \(S\) is invex with respect to \(\eta\), for any \(t \in [0, 1]\), we have \(a + t \eta(b, a) \in S\).

Making use of the preinvexity of \(|f'|\) on \([a, a + \eta(b, a)]\), Lemma 1 and Hölder’s inequality, we get

\[
\begin{align*}
|I(\lambda, \mu)| & \leq |\eta(b, a)| \int_0^\mu |\lambda(1-\mu) - t||f'(a + t \eta(b, a))| \, dt \\
& \quad + |\int_\mu^1 |(1-\lambda \mu) - t||f'(a + t \eta(b, a))| \, dt \leq |\eta(b, a)| \left\{ \int_0^\mu |\lambda(1-\mu) - t| \, dt \right\}^{\frac{1}{p}} \left\{ \int_0^\mu |f'(a + t \eta(b, a))|^q \, dt \right\}^{\frac{1}{q}} \\
& \quad + \left\{ \int_\mu^1 |(1-\lambda \mu) - t| \, dt \right\}^{\frac{1}{p}} \left\{ \int_\mu^1 |f'(a + t \eta(b, a))|^q \, dt \right\}^{\frac{1}{q}}. \tag{19}
\end{align*}
\]
By the simple calculations, we get
\[
\begin{align*}
\int_0^\mu |t - \lambda(1 - \mu)|^p dt &= \delta_5, \quad \text{if } \mu \leq \lambda(1 - \mu), \\
\int_0^\mu |t - \lambda(1 - \mu)|^p dt &= \delta_6, \quad \text{if } \mu \geq \lambda(1 - \mu), \\
\int_\mu^1 |(1 - \lambda \mu) - t|^p dt &= \delta_7, \quad \text{if } \mu \leq 1 - \lambda \mu, \\
\int_\mu^1 |(1 - \lambda \mu) - t|^p dt &= \delta_8, \quad \text{if } \mu \geq 1 - \lambda \mu.
\end{align*}
\]  
(20) (21) (22) (23)

Making use of the preinvexity of $|f'|^q$ on $[a, a + \eta(b, a)]$, for any $t \in [0, 1]$ we know that
\[
|f'(a + t\eta(b, a))|^q \leq (1 - t)|f'(a)|^q + t|f'(b)|^q,
\]
which implies that
\[
\begin{align*}
(a) \int_0^\mu |f'(a + t\eta(b, a))|^q dt &\leq \left\{ \frac{(2 - \mu)\mu}{2} \right\} \frac{f'(a)|^q + \left\{ \frac{\mu^2}{2} \right\} f'(b)|^q}{}, \\
(b) \int_\mu^1 |(1 - \lambda \mu) - t||f'(a + t\eta(b, a))|^q dt &\leq \left\{ \frac{(1 - \mu)^2}{2} \right\} f'(a)|^q + \left\{ \frac{1 - \mu^2}{2} \right\} f'(b)|^q.
\end{align*}
\]  
(24) (25)

By substituting (20)-(25) in (19), we have
\[
|I(\lambda, \mu)| \leq \left| \eta(b, a) \right| \left\{ \begin{array}{l}
\frac{1}{5} \left\{ \frac{(2 - \mu)\mu}{2} |f'(a)|^q + \frac{\mu^2}{2} |f'(b)|^q \right\}^\frac{1}{5} \\
+ \frac{1}{5} \left\{ \frac{(1 - \mu)^2}{2} |f'(a)|^q + \frac{1 - \mu^2}{2} |f'(b)|^q \right\}^\frac{1}{5}, \\
\quad \mu \leq \lambda(1 - \mu) \leq 1 - \lambda \mu \\
\frac{1}{5} \left\{ \frac{(2 - \mu)\mu}{2} |f'(a)|^q + \frac{\mu^2}{2} |f'(b)|^q \right\}^\frac{1}{5} \\
+ \frac{1}{5} \left\{ \frac{(1 - \mu)^2}{2} |f'(a)|^q + \frac{1 - \mu^2}{2} |f'(b)|^q \right\}^\frac{1}{5}, \\
\quad \lambda(1 - \mu) \leq \mu \leq 1 - \lambda \mu \\
\frac{1}{5} \left\{ \frac{(2 - \mu)\mu}{2} |f'(a)|^q + \frac{\mu^2}{2} |f'(b)|^q \right\}^\frac{1}{5} \\
+ \frac{1}{5} \left\{ \frac{(1 - \mu)^2}{2} |f'(a)|^q + \frac{1 - \mu^2}{2} |f'(b)|^q \right\}^\frac{1}{5}, \\
\quad \lambda(1 - \mu) \leq 1 - \lambda \mu \leq \mu.
\end{array} \right.
\]
For nonnegative real numbers $a$ and $b$, the Heinz mean in the parameter $t$ with $0 \leq t \leq 1$, is defined as

$$H_t(a, b) = \frac{a^t b^{1-t} + a^{1-t} b^t}{2}.$$ 

Note that $H_0(a, b) = H_1(a, b) = \frac{a+b}{2}$ and $H_{\frac{1}{2}}(a, b) = \sqrt{ab}$. It is easy to show that as a function of $t$, $H_t(a, b)$ is convex, attains its minimum at $t = 1$ and attains its maximum at $t = 0$ and $t = 1$. Moreover, $H_t(a, b) = H_{1-t}(a, b)$ and $\sqrt{ab} \leq H_t(a, b) \leq \frac{a+b}{2}$.

**Theorem 2.4.** Let $K \subseteq R$ be an invex set with respect to the map $\eta : K \times K \rightarrow R$. If a function $f : K \rightarrow R$ is logarithmic preinvex with respect to $\eta$, then for any $a, b \in K$ and $t \in [0, 1]$ the following inequality holds:

$$\eta(b, a) \int_a^{a+\eta(b,a)} f(x)dx + \eta(a, b) \int_b^{b+\eta(a,b)} f(x)dx \leq \frac{2}{3} \left\{ 2H_{\frac{1}{2}}([f(a)], [f(b)]) + H_0([f(a)], [f(b)]) \right\}.$$ 

**Proof.** By the definition of logarithmic preinvexity of $f : K \rightarrow R$, for any $x, y \in K$ and $t \in [0, 1]$ we have

$$f(a + t\eta(b,a)) \leq [f(a)]^{1-t}[f(b)]^t,$$

$$f(b + t\eta(a,b)) \leq [f(a)]^t[f(b)]^{1-t},$$

which implies that

$$f(a + t\eta(b,a)) + f(b + t\eta(a,b)) \leq [f(a)]^{1-t}[f(b)]^t + [f(a)]^t[f(b)]^{1-t}.$$ 

Integrating this inequality over $t$ on $[0, 1]$ and using Polya inequality

$$\int_0^1 a^{1-t}b^t dt \leq \frac{1}{3} \left\{ 2\sqrt{ab} + \frac{a+b}{2} \right\}$$

for $a, b \geq 0$ and the property $\int_0^1 a^{1-t}b^t dt = \int_0^1 a^t b^{1-t} dt$, this is proved. \qed

**References**


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