On Simpson-like Type Integral Inequalities for Differentiable Preinvex Functions

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Abstract

In this article, we establish some estimates of Simpson-like type integral inequalities for functions whose first derivatives in absolute value at certain powers are preinvex and obtain some inequality for the product of logarithmic preinvex.

Mathematics Subject Classification: 26A51, 26D15

Keywords: Hermite-Hadamard-type inequality, Hölder’s inequality, preinvexity, logarithmic preinvexity

1 Introduction

The following definition is well known in the literature: Let $I$ be on interval in $R$. Then $f : I \to R$ is said to be convex, if

$$f \left( tx + (1 - t) y \right) \leq tf \left( x \right) + (1 - t) f \left( y \right)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [7, 10, 13, 14, 15] and references therein.
In recent years, several extensions and generalizations have been considered for classical convexity [3, 10, 15, 16, 19, 20]. A significant generalization of convex functions is that of invex functions introduced by Hanson in [6]. Weir and Mond [23] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [16] introduced the concept of preinvex function as a generalization of invex functions. Later Mohan and Neogy [11] obtained some properties of generalized preinvex functions.

For an interval $I$ on the real line $R$, let $f : I \to R$ be a convex function and let $a, b \in I$ with $a < b$. We consider the well-known Hadamard’s inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.$$ 

Both inequalities hold in the reversed direction if $f$ is concave. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality [1].

Noor [1, 2, 3, 4, 5, 8, 19, 23, 24, 25] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers, Noor and Barani et al. in [1, 2, 3, 5] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

**Definition 1.** A set $S \subseteq R$ is said to be invex with respect to the map $\eta : S \times S \to R$, if $x + t\eta(y, x) \in S$ for any $x, y \in S$ and $t \in [0, 1]$.

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = y - x$, but there exist invex sets which are not convex [11].

**Definition 2.** Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$. Then the function $f : S \to R$ is said to be preinvex with respect to $\eta$, if for any $x, y \in S$ and $t \in [0, 1]$,

$$f(y + t\eta(x, y)) \leq tf(x) + (1-t)f(y).$$

In [5], Barani et al. introduced some generalizations of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are preinvex. In recent years many authors have established error estimations for the Simpson’s inequality; for refinements, counterparts, generalizations and new Simpson-type inequalities, you may see [17, 21, 22].

**Theorem 1.1.** Suppose that $f : [a, b] \to R$ is a differentiable function whose derivative is continuous on $[a, b]$ and $f' \in L([a, b])$. Then the following inequality holds: for $\| f' \|_1 = \int_a^b | f(x) | dx$,

$$\left| \frac{1}{3} \left( \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{3} \| f' \|_1.$$
Theorem 1.2. Suppose that $f : [a, b] \rightarrow R$ is an absolutely continuous differentiable function on $[a, b]$ and $f' \in L^p([a, b])$. Then the following inequality holds: for $\| f' \|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$ for nonnegative $p, q$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left| \frac{1}{3} \left\{ \frac{f(a) + f(b)}{2} + 2f\left( \frac{a + b}{2} \right) \right\} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^{\frac{1}{q}}}{6} \left\{ \frac{2^{q+1} + 1}{3(q+1)} \right\}^{\frac{1}{q}} \| f' \|_p .$$

In [4], Noor proved the following inequality for logarithmic preinvex functions:

Theorem 1.3. Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \rightarrow R$ and, $\eta(b, a) \neq 0$ with $0 \leq a \leq a + \eta(b, a) < \infty$ for all $a, b \in S$ with $a < b$. Suppose that $f, g : S \rightarrow R^+$ are logarithmically preinvex functions on the interval $[a, a + \eta(b, a)]$ in $K$. Then the following inequality holds:

$$\frac{4}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x)g(x) \, dx \leq [f(a) + f(b)]L(f(a), f(b)) + [g(a) + g(b)]L(g(a), g(b)),$$

where $L(a, b) = \frac{b-a}{m \cdot b-m} a$ for two nonnegative numbers $a, b$ with $a < b$.

In this article we consider a real interval $I \subset R$, and we denote that $S^0$ is the interior of $S$.

The main aim of this paper is to establish new similar inequalities concerning Simpson-like type inequality for the class of differentiable functions whose derivatives at certain powers are preinvex functions and to prove some applications for special means of real numbers. Also we obtain some inequality for the product of two log-preinvex functions.

2 Some new Hermite-Hadamard-type inequalities

To establish some new Hermite-Hadamard type inequalities for $s$-convex functions in the second sense, we need the following lemma.

Lemma 1. Let $S \subseteq R$ be an invex set with respect to the map $\eta : K \times K \rightarrow R$ and, $\eta(b, a) \neq 0$ with $0 \leq a \leq a + \eta(b, a) < \infty$ for all $a, b \in S$ with $a < b$. Suppose that $f : S \rightarrow R$ is a differentiable function on $K^0$ such that $f' \in$
\( L([a, a + \eta(b, a)]) \). Then, for \( h \in (0, 1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} \), the following identity holds:

\[
I(f : a, b, n, h) = \frac{1}{\eta(b, a)} \int_{a}^{a + \eta(b, a)} f(x)dx
- \frac{1}{n} \left\{ f(a) + (n - 2)f(a + h\eta(b, a)) + f(a + \eta(b, a)) \right\}
= \eta(b, a) \int_{0}^{1} p(t, n, h) f'(a + t\eta(b, a))dt,
\]
where

\[
p(t, n, h) = \begin{cases} t - \frac{1}{n} & t \in [0, h] \\ t - \frac{n-1}{n} & t \in (h, 1] \end{cases}
\]
for \( n \geq 2 \).

**Proof.** By integration by parts, this equality (1) is proved. \( \square \)

Now we turn our attention to establish inequalities of Hermit-Hadamard type for twice differentiable \( s \)-convex functions.

**Theorem 2.1.** Let \( S \subseteq R \) be an invex set with respect to the map \( \eta : S \times S \rightarrow R \) and, \( \eta(b, a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b, a) < \infty \) for all \( a, b \in S \) with \( a < b \). Suppose that \( f : S \rightarrow R \) is a differentiable function on \( S \) such that \( f' \in L([a, a + \eta(b, a)]) \). If \( |f'| \) is preinvex with respect to \( \eta \) on \( S \), then for \( h \in (0, 1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} (n \geq 2) \), the following inequality holds:

\[
I(f : a, b, n, h) \leq \eta(b, a) \left\{ (\mu_{11} - \nu_{11}) |f'(a)| + \nu_{11} |f'(b)| \right\},
\]
where

\[
\mu_{11} = \frac{1}{2} - h + h^2 + \frac{2 - n}{n^2}, \\
\nu_{11} = \frac{2 - n}{2n^2} + \frac{2 - 6h + 9h^2 - 4h^3}{6}.
\]

**Proof.** Suppose that \( a, a + \eta(b, a) \in S \). Since \( S \) is invex with respect to \( \eta \), for any \( t \in [0, 1] \), we have \( a + t\eta(b, a) \in S \). By Lemma 1, we have

\[
|I(f : a, b, n, h)| \leq |\eta(b, a)| \left\{ \int_{0}^{h} |(t - \frac{1}{n})| |f'(a + t\eta(b, a))| dt \\
+ \int_{h}^{1} |(t - \frac{n-1}{n})| |f'(a + t\eta(b, a))| dt \right\}.
\]
By Lemma 1 and Hölder’s inequality, we get

\[
|I(f : a, b, n, h)|
\leq |\eta(b, a)| \left\{ \int_0^{\frac{1}{n}} (\frac{1}{n} - t)|f'(a + t\eta(b, a))|dt + \int_h^{\frac{1}{n}} (t - \frac{1}{n})|f'(a + t\eta(b, a))|dt + \int_{\frac{1}{n}}^{\frac{n-1}{n}} (\frac{n-1}{n} - t)|f'(a + t\eta(b, a))|dt + \int_{\frac{n-1}{n}}^{1} (t - \frac{n-1}{n})|f'(a + t\eta(b, a))|dt \right\}.
\]

Making use of the preinvexity of $|f'|$ on $[a, a + \eta(b, a)]$, we get

\[
|I(f : a, b, n, h)|
\leq |\eta(b, a)| \times \left\{ \int_0^{\frac{1}{n}} (\frac{1}{n} - t)dt + \int_h^{\frac{1}{n}} (\frac{n-1}{n} - t)dt + \int_{\frac{1}{n}}^{\frac{n-1}{n}} (\frac{n-1}{n} - t)dt + \int_{\frac{n-1}{n}}^{1} (t - \frac{n-1}{n})dt \right\} f'(a) + \left\{ \int_0^{\frac{1}{n}} (\frac{1}{n} - t)dt + \int_{\frac{1}{n}}^{\frac{1}{h}} (t - \frac{1}{n})dt + \int_h^{\frac{1}{h}} (\frac{n-1}{n} - t)dt + \int_{\frac{n-1}{n}}^{1} (t - \frac{n-1}{n})dt \right\} f'(b) \right\}
\]

\[
= \eta(b, a) \left\{ (\mu_{11} - \nu_{11}) |f'(a)| + \nu_{11} |f'(b)| \right\},
\]

which completes the proof.

\[\square\]

**Theorem 2.2.** Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \rightarrow R$ and, $\eta(b, a) \neq 0$ with $0 \leq a \leq a + \eta(b, a) < \infty$ for all $a, b \in S$ with $a < b$. Suppose that $f: S \rightarrow R$ is a differentiable function on $S^0$ such that $f' \in L([a, a + \eta(b, a)])$. Assume $p \in R$ with $p > 1$. If $|f'|^p$ is preinvex with respect to $\eta$ on $S$, then for $h \in (0, 1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$ $(n \geq 2)$, the following inequality holds:

\[
|I(f : a, b, n, h)|
\leq |\eta(b, a)| \left( \frac{1}{p + 1} \right)^{\frac{1}{p}}
\]
Proof. Suppose that \(a, a + \eta(b, a) \in S\). Since \(S\) is invex with respect to \(\eta\), for any \(t \in [0, 1]\), we have \(a + t\eta(b, a) \in S\).

By Lemma 1 and Hölder’s inequality, we get

\[
|I(f : a, b, n, h)| \\
\leq |\eta(b, a)| \left\{ \int_0^{\frac{1}{n}} \left( \frac{1}{n} - t \right)^p dt \right\} \left\{ \int_0^{\frac{1}{n}} |f'(a + t\eta(b, a))| \cdot \frac{p}{p-1} dt \right\}^{\frac{p}{p-1}} \\
+ \left\{ \int_0^{\frac{1}{n}} (t - \frac{1}{n})^p dt \right\} \left\{ \int_0^{\frac{1}{n}} |f'(a + \eta(b, a))| \cdot \frac{p}{p-1} dt \right\}^{\frac{p}{p-1}} \\
+ \left\{ \int_{\frac{n-1}{n}}^{\frac{1}{n}} \left( \frac{n-1}{n} - t \right)^p dt \right\} \left\{ \int_{\frac{n-1}{n}}^{\frac{1}{n}} |f'(a + \eta(b, a))| \cdot \frac{p}{p-1} dt \right\}^{\frac{p}{p-1}} \\
+ \left\{ \int_{\frac{n-1}{n}}^{1} (t - \frac{n-1}{n})^p dt \right\} \left\{ \int_{\frac{n-1}{n}}^{1} |f'(a + \eta(b, a))| \cdot \frac{p}{p-1} dt \right\}^{\frac{p}{p-1}}. \tag{2}
\]

Making use of the preinvexity of \(|f'|\) on \([a, a + \eta(b, a)]\), we get

(a) \(\int_0^{\frac{1}{n}} |f'(a + t\eta(b, a))| \cdot \frac{p}{p-1} dt \leq \left( \frac{2n-1}{2n^2} \right) |f'(a)| \cdot \frac{p}{p-1} + \left( \frac{1}{2n^2} \right) |f'(b)| \cdot \frac{p}{p-1}\), \(\tag{3}\)

(b) \(\int_{\frac{n-1}{n}}^{\frac{1}{n}} |f'(a + \eta(b, a))| \cdot \frac{p}{p-1} dt \leq \left( \frac{2 - h}{2} \right) |f'(a)| \cdot \frac{p}{p-1} + \left( \frac{1 - 2n}{2n^2} \right) |f'(b)| \cdot \frac{p}{p-1}\), \(\tag{4}\)

(c) \(\int_{\frac{n-1}{n}}^{1} |f'(a + t\eta(b, a))| \cdot \frac{p}{p-1} dt \leq \left( \frac{1 - h}{2n^2} \right) |f'(a)| \cdot \frac{p}{p-1} + \left( \frac{1 - h^2 n^2 + 2n - 1}{2n^2} \right) |f'(b)| \cdot \frac{p}{p-1}\), \(\tag{5}\)
Suppose that inequality holds: \( \eta \in R \) and, \( \eta(b,a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b,a) < \infty \) for all \( a, b \in S \) with \( a < b \). Suppose that \( f : S \to R \) is a differentiable function on \( S^n \) such that \( f' \in L([a, a + \eta(b,a)]) \). Assume \( q \in R \) with \( q \geq 1 \). If \( |f'|^q \) is preinvex with respect to \( \eta \) on \( S \), then for \( h \in (0, 1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} (n \geq 2) \), the following inequality holds:

\[
|I(f : a, b, n, h)| \\
\leq |\eta(b,a)| \left\{ \frac{1}{2n^2} \left( \frac{1}{3n} \right)^{\frac{1}{q}} \right\} \times \left\{ (3n - 1)|f'(a)|^q + |f'(b)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ (nh - 1)^2 \right\}^{\frac{1}{q}} \left\{ (1 - nh)^2((3 - 2h)n - 1)|f'(a)|^q \right. \\
+ (1 - nh)^2(1 + 2nh)|f'(b)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ (n - nh - 1)^2 \right\}^{\frac{1}{q}} \left\{ (1 - n + nh)^2(1 + 2n - nh)|f'(a)|^q \right. \\
+ (1 - n + nh)^2(2nh + n - 1)|f'(b)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ |f'(a)|^q + (3n - 1)|f'(b)|^q \right\}^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** Suppose that \( a, a + \eta(b,a) \in S \). Since \( S \) is invex with respect to \( \eta \), for any \( t \in [0, 1] \), we have \( a + t\eta(b,a) \in S \).

By Lemma 1 and Hölder’s inequality, we get

\[
|I(f : a, b, n, h)| \\
\leq |\eta(b,a)| \left\{ \int_0^1 \left( \frac{1}{n} - t \right) dt \right\}^{\frac{1}{p}} \left\{ \int_0^1 \left( \frac{1}{n} - t \right) |f'(a + t\eta(b,a))|^q dt \right\}^{\frac{1}{q}} \\
+ \left\{ \int_0^{\frac{1}{n}} (t - \frac{1}{n}) dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n}} (t - \frac{1}{n}) |f'(a + t\eta(b,a))|^q dt \right\}^{\frac{1}{q}} \\
+ \left\{ \int_{\frac{1}{n}}^{n-1} \left( \frac{n-1}{n} - t \right) dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n}}^{n-1} \left( \frac{n-1}{n} - t \right) |f'(a + t\eta(b,a))|^q dt \right\}^{\frac{1}{q}} \\
+ \left\{ \int_{n-1}^n \left( \frac{n-1}{n} - t \right) dt \right\}^{\frac{1}{p}} \left\{ \int_{n-1}^n \left( \frac{n-1}{n} - t \right) |f'(a + t\eta(b,a))|^q dt \right\}^{\frac{1}{q}}.
\]
Making use of the preinvexity of $|f'|^q$ on $[a, a + \eta(b, a)]$, we get

\[(a) \int_0^1 \left( \frac{1}{n} - t \right) |f'(a + \eta(b, a))|^q dt \]
\[
\leq \left( \frac{3n - 1}{6n^3} \right) |f'(a)|^q + \left( \frac{1}{6n^3} \right) |f'(b)|^q, \tag{9}
\]

\[(b) \int_0^n \left( t - \frac{1}{n} \right) |f'(a + \eta(b, a))|^q dt \]
\[
\leq \left( \frac{1 - nh}{6n^3} \right) (3n - 2nh - 1) |f'(a)|^q
+ \left( \frac{1 - nh}{6n^3} \right) (1 + 2nh) |f'(b)|^q, \tag{10}
\]

\[(c) \int_h^n \left( \frac{n - 1}{n} - t \right) |f'(a + \eta(b, a))|^q dt \]
\[
\leq \left( \frac{1 - n + nh}{6n^3} \right) (2n - n - 1) |f'(a)|^q
+ \left( \frac{1 - n + nh}{6n^3} \right) (2nh + n - 1) |f'(b)|^q, \tag{11}
\]

\[(d) \int_n^1 \left( t - \frac{n - 1}{n} \right) |f'(a + \eta(b, a))|^q dt \]
\[
\leq \left( \frac{1}{6n^3} \right) |f'(a)|^q + \left( \frac{3n - 1}{6n^3} \right) |f'(b)|^q. \tag{12}
\]

By substituting (9)-(12) in (8), we get the desired result. \qed

**Theorem 2.4.** Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$ and, $\eta(b, a) \neq 0$ with $0 \leq a \leq a + \eta(b, a) < \infty$ for all $a, b \in S$ with $a < b$. Suppose that $f : S \to R$ is a differentiable function on $S^0$ such that $f' \in L([a, a + \eta(b, a)])$. Assume $p \in R$ with $p \geq 1$. If $|f'|^q$ is preinvex with respect to $\eta$ on $S$, then for $h \in (0, 1)$ with $\frac{1}{n} \leq h \leq \frac{n - 1}{n} (n \geq 2)$, the following inequality holds:

\[
|I(f : a, b, n, h)|
\leq |\eta(b, a)| \left( \frac{1}{p + 1} \right)^\frac{1}{p} \left\{ \frac{1}{n^{p+1}} \right\}^{\frac{1}{p}} \left\{ \frac{1}{n} |f'(a)|^{q(1 - \frac{1}{p})} L\left( |f'(a)|^{\frac{q}{p}} ; |f'(b)|^{\frac{q}{p}} \right) \right\}^{\frac{1}{p}}
+ \left\{ \frac{(nh - 1)^{p+1}}{n^{p+1}} \right\}^{\frac{1}{p}}.
\]
where $L(a, b) = \frac{b-a}{n b - La}$ for $a \neq b$, $a, b \neq 0$ for $a, b \in R^+$.

**Proof.** Suppose that $a, a + \eta(b, a) \in S$. Since $S$ is invex with respect to $\eta$, for any $t \in [0, 1]$, we have $a + t \eta(b, a) \in S$.

Making use of the preinvexity of $| f' |^q$ on $[a, a + \eta(b, a)]$, Lemma 1 and H"older’s inequality, we get

\[
|I(f : a, b, n, h)| \\
\leq |\eta(b, a)| \left\{ \int_0^{\frac{1}{n}} (\frac{1}{n} - t)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n}} |f'(a + t \eta(b, a))|^q dt \right\}^{\frac{1}{q}} \\
+ \left\{ \int_0^{\frac{n-1}{n}} (t - \frac{1}{n})^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{n-1}{n}}^{\frac{1}{n}} |f'(a + t \eta(b, a))|^q dt \right\}^{\frac{1}{q}} \\
+ \left\{ \int_{\frac{n-1}{n}}^{\frac{n-1}{n}} (\frac{n-1}{n} - t)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{n-1}{n}}^{\frac{n-1}{n}} |f'(a + t \eta(b, a))|^q dt \right\}^{\frac{1}{q}} \\
+ \left\{ \int_{\frac{n-1}{n}}^{1} (t - \frac{n-1}{n})^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{n-1}{n}}^{1} |f'(a + t \eta(b, a))|^q dt \right\}^{\frac{1}{q}} \\
= |\eta(b, a)| \left( \frac{1}{p} + 1 \right)^\frac{1}{p} \\
\times \left\{ \left\{ \frac{1}{n^{p+1}} \right\}^{\frac{1}{p}} \left\{ \int_{n} |f'(a)|^{q(1-h)} L \left| f'(a) \right|^{\frac{1}{n}}, \left| f'(b) \right|^{\frac{1}{n}} \right\}^{\frac{1}{q}} + \left\{ \frac{(nh - 1)^{p+1}}{n^{p+1}} \right\}^{\frac{1}{p}} \right. \\
\times \left\{ \left\{ \left| f'(a) \right|^{\frac{1}{n}}, \left| f'(b) \right|^{\frac{1}{n}}, \left| f'(a) \right|^{h} \right\}^{\frac{1}{n}}, \left| f'(b) \right|^{\frac{1}{n}} \right\}^{\frac{1}{q}} \\
+ \left\{ \frac{(n - nh - 1)^{p+1}}{n^{p+1}} \right\}^{\frac{1}{p}} \left\{ \left\{ \left| f'(a) \right|^{\frac{1}{n}}, \left| f'(b) \right|^{\frac{1}{n}} \right\}^{\frac{1}{q}} \right. \\
\times \left\{ \left\{ \left| f'(a) \right|^{\frac{1}{n}}, \left| f'(b) \right|^{\frac{1}{n}} \right\}^{\frac{1}{q}} \right\}^{\frac{1}{q}} \\
+ \left\{ \frac{1}{n^{p+1}} \right\}^{\frac{1}{p}} \left\{ \frac{\left| f'(b) \right|^{q(1-h)}}{n^{p+1}} \left| f'(a) \right|^{\frac{1}{n}}, \left| f'(b) \right|^{\frac{1}{n}} \right\}^{\frac{1}{q}}. 
\]
where we have used the fact that

\[
\begin{align*}
(a) & \int_0^1 |f'(a + t\eta(b,a))|^{q} dt \leq \frac{1}{n} |f'(a)|^{q(1-\frac{1}{n})} L\left(|f'(a)|^{\frac{2}{n}}, |f'(b)|^{\frac{2}{n}}\right), \\
(b) & \int_1^n |f'(a + t\eta(b,a))|^{q} dt \\
& \leq \left\{ |f'(a)|^{\frac{2}{n}} |f'(b)|^{qh} - |f'(a)|^{bq} |f'(b)|^{\frac{2}{n}} \right\} L\left(|f'(a)|^{q}, |f'(b)|^{q}\right), \\
(c) & \int_1^{\frac{a-h}{n}} |f'(a + t\eta(b,a))|^{q} dt \\
& \leq (1 - h - \frac{1}{n}) |f'(a)|^{\frac{2}{n}} |f'(b)|^{qh} L\left(|f'(a)|^{q(1-\frac{1}{n})}, |f'(b)|^{q(1-\frac{1}{n})}\right), \\
(d) & \int_{\frac{a-h}{n}}^1 |f'(a + t\eta(b,a))|^{q} dt \\
& \leq \frac{1}{n} |f'(b)|^{q(1-\frac{1}{n})} L\left(|f'(a)|^{\frac{2}{n}}, |f'(b)|^{\frac{2}{n}}\right).
\end{align*}
\]

\( \Box \)

The following theorem is an another expression of Theorem 1.3:

**Theorem 2.5.** Let \( S \subseteq R \) be an invex set with respect to the map \( \eta : S \times S \to R \) and, \( \eta(b,a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b,a) < \infty \) for all \( a, b \in S \) with \( a < b \). Suppose that \( f, g : S \to R \) are logarithmically preinvex functions on the interval \([a, a + \eta(b,a)]\) in \( K \). Then the following inequality holds:

\[
\begin{align*}
\int_{[a, a + \eta(b,a)]} f(x)g(x) dx \\
& \leq \frac{\eta(b,a)}{6} \left\{ 2G^2(f(a), f(b)) + 2G^2(g(a), g(b)) \\
& \quad + A(f^2(a), f^2(b)) + A(g^2(a), g^2(b)) \right\}.
\end{align*}
\]

where the arithmetic mean \( A(a,b) \) and the geometric mean \( G(a,b) \) are respectively defined by

\[
A(a,b) = \frac{a + b}{2}, \quad G(a,b) = \sqrt{ab}
\]

for two nonnegative numbers \( a, b \) with \( a < b \).

**Proof.** Let \( f, g : S \to R \) be logarithmically preinvex functions on the interval \([a, a + \eta(b,a)]\) in \( K \). Then for \( t \in [0,1] \) we have

\[
f(a + t\eta(b,a)) \leq [f(a)]^{1-t}[f(b)]^t \quad g(a + t\eta(b,a)) \leq [g(a)]^{1-t}[g(b)]^t, \quad (14)
\]
which implies that
\[ f(a + t\eta(b,a))g(a + t\eta(b,a)) \leq \frac{1}{2} \{ f^2(a + t\eta(b,a)) + g^2(a + t\eta(b,a)) \}. \quad (15) \]

By (15), we have
\[
\int_{a}^{a+\eta(b,a)} f(x)g(x)dx \\
= \eta(b,a) \int_{0}^{1} f(a + t\eta(b,a))g(a + t\eta(b,a))dt \\
\leq \frac{\eta(b,a)}{2} \int_{0}^{1} \{ f^2(a + t\eta(b,a)) + g^2(a + t\eta(b,a)) \} dt.
\]

By (14), we get
\[
\int_{a}^{a+\eta(b,a)} f(x)g(x)dx \\
\leq \frac{\eta(b,a)}{2} \int_{0}^{1} \{ f^2(a + t\eta(b,a)) + g^2(a + t\eta(b,a)) \} dt \\
\leq \frac{\eta(b,a)}{2} \left\{ \int_{0}^{1} \{ [f(a)]^{1-t}[f(b)]^t \}^2 + \int_{0}^{1} \{ [g(a)]^{1-t}[g(b)]^t \}^2 \right\} dt \\
= \frac{\eta(b,a)}{2} \left\{ \int_{0}^{1} [f^2(a)]^{1-t}[f^2(b)]^t dt + \int_{0}^{1} [g^2(a)]^{1-t}[g^2(b)]^t dt \right\}.
\]

By the classical Polya inequality, we have
\[
\int_{a}^{a+\eta(b,a)} f(x)g(x)dx \\
\leq \frac{\eta(b,a)}{2} \left\{ \int_{0}^{1} [f^2(a)]^{1-t}[f^2(b)]^t dt + \int_{0}^{1} [g^2(a)]^{1-t}[g^2(b)]^t dt \right\} \\
\leq \frac{\eta(b,a)}{6} \left( \left\{ 2\sqrt{|f^2(a)||f^2(b)|} + \frac{|f^2(a)| + |f^2(b)|}{2} \right\} \\
+ \left\{ 2\sqrt{|g^2(a)||g^2(b)|} + \frac{|g^2(a)| + |g^2(b)|}{2} \right\} \right) \\
= \frac{\eta(b,a)}{6} \left( \left\{ 2\{ f(a)[f(b)] + [g(a)][g(b)] \} \\
+ \frac{1}{2} \left\{ [f^2(a)] + [f^2(b)] + [g^2(a)] + [g^2(b)] \right\} \right) \\
= \frac{\eta(b,a)}{6} \left\{ 2G^2(f(a), f(b)) + 2G^2(g(a), g(b)) \\
+ A(f^2(a), f^2(b)) + A(g^2(a), g^2(b)) \right\}.
\]
References


Received: September 5, 2013