Probabilistic Properties of Discrete Mean and Variance Reversed Residual Lifetime Functions

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Abstract

The random variable \( T_{(t)} = \min(T \leq t) \), which is called reversed residual lifetime random variable, will gather the attention of most researchers in reliability and economic studies. The mean and the variance of this variable in continuous distribution have been studied by several authors. But, in discrete case, only in recent years, some studies have been done for the mean of this variable. In this paper, we will define and study the properties of the mean and variance of \( X_{(a)} = \{x \in X | x \leq a\} \) where \( X \) is a discrete random variable. Besides similar results for discrete and continuous lifetime distributions, relationships with its mean, monotonicity and the associated ageing classes of distributions are obtained for discrete cases. Furthermore, some characterization results about the class of increasing variance reversed residual lifetime distributions based on the mean reversed residual lifetime and the reversed residual coefficient of variation, are presented and the lower and upper bound for them are achieved. Also, we will investigate the connection between discrete increasing variance reversed residual lifetimes and other classes of distributions. I will provide two simple characterizations of the increasing discrete variance reversed residual lifetime class of distributions based on discrete mean and variance reversed residual lifetime. Since Geometric, Weibull, Shift Geometric distributions belong to increasing variance reversed residual lifetime class distributions, therefore the

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present results will provide characterizations of all these special cases. Application, examples and simulation results are provided. Finally, nonparametric procedure, empirical power and critical values for testing increasing discrete variance reversed residual lifetime are obtained.

**Keywords:** Conditional variance, reversed hazard rate, mean reversed residual lifetime, variance reversed residual lifetimes, reversed residual coefficient of variation

1. Introduction

In many realistic situations, the random variables are not necessarily related to the future only, but they can also refer to the past. Such a random variable can be called inactivity time, \( T(t) = \{\tau \mid T \leq t\} \) for fixed \( t > 0 \), or the time elapsed after failure till time \( t \), given that the unit has already failed by time \( t \). It is also called as the reversed residual lifetime. The reversed residual lifetime random variable was introduced by many researchers recently (c.f. [1], [4], [5], [10], [11], [13], [14]).

The mean and the variance of the random variable \( T(t) \) has received considerable attention in reliability. The concepts of inactivity time based on the current age is effectively used to infer properties of the underlying life distributions. The mean reversed residual time function (MRRL) has many applications, for example, in life insurance, maintenance and product quality control, economics and social studies. MRRL was studied by various authors such as [1], [2], [7], [13] and [14].

Another quantity of interest which has generated interest in recent years is the variance reversed residual life (VRRL). [13] has studied increasing variance reversed residual life (IVRRL). Also, they demonstrated the relationship of this class to other well-known classes of life distributions, and closure properties under some reliability operations such as mixture, convolution and formation of coherent systems, are obtained. It remains an open problem to find similar characterization and properties for discrete distributions.

An important aspect of lifetime analysis is to find a lifetime distribution that can adequately describe the ageing behavior of the device concerned. Most of the lifetimes are continuous in nature and hence many continuous life distributions have been proposed in the literature. On the other hand, discrete failure data arise in several common situations, for example:

- Reports on field failures are collected weekly, monthly, and the observations are the number of failures, without specification of the failure times;
- A piece of equipment operates in cycles and the experimenter observes the number of cycles successfully completed prior to failure. A frequently referred example is a copier whose life length would be the total number of copies it produces. Another example is the number of on/off cycles of a switch before failure occurs.
Interests in the discrete failure data came relatively late in comparison to its continuous analogue. The subject matter has to some extent been neglected. It was only briefly mentioned by [3]. For earlier works on discrete lifetime distributions, see [6], [8], [9], [15], [17], [18], [20] and [21].

To this end, the objective of this paper is to study \textit{MRRL} and \textit{VRRL} of discrete lifetime distributions which the results are different as those in the continuous case. Its relationship with mean reversed residual life and reversed residual coefficient of variation are obtained. Also, its monotonicity and the associated ageing classes of distributions are discussed. Some characterization results of the class of increasing discrete variance reversed residual life (\textit{IDVRRRL}), are presented and the upper bounds for discrete variance reversed residual life under some conditions are obtained. Application, examples and simulation results are studied. Finally, nonparametric procedure, empirical power and critical values for testing \textit{IVRR} are obtained.

\section{Definitions and Notions}

Let $X$ be a discrete random variable defined on the set of non-negative integers with distribution function $F(x)$ and probability mass function $p(x)$. In many reliability problems, there are interesting to consider variables of the following kind

$$X_{(a)} = [a \leq X \leq a], \ a \geq 0.$$ 

Such a random variable can be called reversed residual lifetime, or the time elapsed after failure till time $a$, given that the unit has already failed by time $a$. In such situations, this random variable was found to be more adequate than the residual random variable ([10] and [19]). This paved the way of studying many reliability concepts in the reversed time scale. These measures gained attention as they were not just "duals" of the existing probability and reliability measures but they found use and applications in the field of actuaries, biometry, maintenance theory and economics etc. in their own right.

Now, we introduce new aging classes in discrete failure data:

\textbf{Definition 2.1} \textit{The reversed hazard rate function (RHR) is defined as:}

$$v(a) = \lim_{\varepsilon \to 0} \frac{\Pr(a - \varepsilon \leq X \leq a | X \leq a)}{\varepsilon},$$

$$= \frac{F(a) - F(a - 1)}{F(a)}, \quad (2.1)$$

\textbf{Definition 2.2} \textit{The mean reversed residual lifetime (MRR) is defined as:}

$$\delta(a) = E[a - X | X \leq a], \ a \geq 0,$$

$$= \sum_{j=0}^{a} (a - j) \Pr(X = j) = \sum_{j=0}^{a - 1} F(j) \frac{F(a)}{F(a)}, \ a \geq 0. \quad (2.2)$$
Remark 2.1 By using relation in (2.2) and recurrence relation we can get:
\[
\delta(a + 1)F(a + 1) = F(a)(1 + \delta(a)), a \geq 1. \tag{2.3}
\]
From the equation (2.1) we can get
\[
p(a-1) = \nu(a-1)p(a)(1 - \nu(a))/\nu(a).
\]
Thus we get by iteration on \(a\)
\[
p(a) = \nu(a)(1-\nu(a+1))(1-\nu(a+2))..., \tag{2.4}
\]
and
\[
F(a) = \prod_{x=a}^{\infty} (1 - \nu(x+1)). \tag{2.5}
\]
Equations (2.4) and (2.5) show that \(\nu(a)\) determines the distribution of \(X\) uniquely.

Combining (2.3) and (2.5) we get the relationship between \(RHR\) and \(MRR\) of \(X\) as
\[
1 - \nu(a) = \frac{\delta(a - 1)}{\delta(a - 2) + 1}, a \geq 2. \tag{2.6}
\]

For defining \(VRRL\) we need the following lemma:

**Lemma 2.1** If \(p(x)\) is the probability mass function of discrete non-negative random variable \(X\) and \(F(x)\) be its distribution function, then we have the following relations,
\[
E[X|X \leq a] = a - \frac{\sum_{j=0}^{a-1} F(j)}{F(a)}.
\]
and
\[
E[X^2|X \leq a] = a^2 - \frac{\sum_{j=0}^{a-1} (2j + 1)F(j)}{F(a)}.
\]

**Proof.** According to \(p(j) = F(j) - F(j - 1)\), one can verify the relations.

**Definition 2.3** The variance reversed residual lifetime (VRR) is defined as
\[
\beta^2(t) = Var[a - X|X \leq a], 0 \leq X < a; a \geq 0,
\]
\[
= a^2 + E[X^2|X \leq a] - 2aE[X|X \leq a] - E^2[a - X|X \leq a]. \tag{2.7}
\]
\[
= \frac{2a\sum_{j=0}^{a-1} F(j)}{F(a)} - \frac{\sum_{j=0}^{a-1} (2j + 1)F(j)}{F(a)} - \left[\frac{\sum_{j=0}^{a-1} F(j)}{F(a)}\right]^2.
\]

**Definition 2.4** The reversed residual coefficient of variation (RRC) is
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\[ \eta(a) = \frac{\beta(a)}{\delta(a)} = \left[ \frac{2aF(a)}{\sum_{j=0}^{a} F(j)} - \frac{F(a) \sum_{j=0}^{a-1} (2j+1)F(j)}{\left( \sum_{j=0}^{a-1} F(j) \right)^2} - 1 \right]^{1/2} \]  

(2.8)

3. New Classes of Distributions and Their Characterizations

Here, we define new classes of distributions for discrete random life variable reversed residual lifetime

**Definition 3.1** A nonnegative discrete random variable \( X \), having distribution \( F(\cdot) \), is said to be:

(i). Decreasing reversed hazard rate (DRHR) if \( \nu(a) \) is decreasing in \( a \geq 0 \).

(ii). Increasing mean reversed residual lifetime (IMRR) if \( \delta(a) \) is increasing in \( a \geq 0 \).

(iii). IDVRRL if \( \beta(a) \) is increasing in \( a \geq 0 \).

(iv). Increasing reversed residual coefficient of variation (IRRC) if \( \eta(a) \) is increasing in \( a \geq 0 \).

**Theorem 3.1** There exists no nonnegative random variable \( X \) for which \( \nu(a) \) increases over the entire domain \( \{0,1,2,\ldots\} \).

**Proof:** Suppose \( \nu(a) \) is increasing function in \( a \), i.e., for \( a > 0 \):

\[ \frac{p(a)}{F(a)} \geq \lim_{a \to \infty} \frac{p(a)}{F(a)} = \infty, \]

which is a contradiction.

**Remark 3.1:**

1. Using the same arguments, one can prove that there exists no nonnegative random variable \( X \) for which \( \delta(a) \) and \( \beta(a) \) decreases over the entire domain \( \{0,1,2,\ldots\} \).

2. By using (2.6) we imply that if \( X \) has the DRHR property then \( X \) has the IMRR property.

In the next theorem, we obtain a lower bound for \( \beta(a) \), when \( \delta(a) \) is increasing in \( a \).

**Theorem 3.2** If the non-negative discrete random variable \( X \), has IMRR property, then if the non-negative discrete random variable \( X \), has IMRR property, then

\[ \beta^2(a) > \delta(a)(1+\delta(a)) \]

and for \( a = 0 \) the above inequality becomes equality.

**Proof:** According to (2.7), we have,
Thus, it implies,
\[ \frac{2}{F(a)} \sum_{i=0}^{a-1} F(i) \delta(i) = \beta^2(a) + \delta(a)(\delta(a) - 1). \]
Since \( \delta(a) \) is increasing, \( \delta(i) - \delta(a) > 0 \) for \( i \geq a \). Hence the required result is followed.

Now, we investigate the connection between IVRR and other classes of distributions.

**Theorem 3.3** If \( \delta(a) \) is increasing in \( a \), then \( \beta^2(a) \) is increasing in \( a \), i.e., the IMRR property is stronger than the IVRR property.

**Proof.** Using (2.7) we have,
\[
\beta^2(a) - \beta^2(a-1) = \delta(a)[2a-1] - \delta^2(a) - \frac{\sum_{j=0}^{a-1} 2jF(j)}{F(a)}
\]
\[
- \delta(a-1)[2a-3] + \delta^2(a-1) + \frac{\sum_{j=0}^{a-1} 2jF(j)}{F(a-1)}. \tag{3.1}
\]
On the other hand, one can show that,
\[
\frac{\sum_{j=0}^{a-1} 2jF(j)}{F(a-1)} - \frac{\sum_{j=0}^{a-1} 2jF(j)}{F(a)} = v(a) \frac{\sum_{j=0}^{a-1} 2jF(j)}{F(a-1)} - (2a-2)\frac{F(a-1)}{F(a)}, \tag{3.2}
\]
and
\[
\delta(a-1) - \delta(a) = v(a)\delta(a-1) - \frac{F(a-1)}{F(a)}, \tag{3.3}
\]
then, by replacing the equations (3.2) and (3.3) into (3.1), we have,
\[
\beta^2(a) - \beta^2(a-1) = v(a)[\delta(a) - \beta^2(a-1) + \delta(a) \delta(a)]. \tag{3.4}
\]
Since, \( \delta(a) \) is increasing,
\[
\beta^2(a) - \beta^2(a-1) \geq v(a)[\delta(a) + \delta(a)] - \beta^2(a),
\]
then by using Theorem 3.2, we get the required results.

**Proposition 3.1** If \( F \) has DRHR property, then it has IVRR property. Since, it is well-known in discrete distributions that DRHR property implies IMRR property, i.e.

\[
DRHR \Rightarrow IMRR \Rightarrow IVRR \Rightarrow IRRC.
\]

Here, another characterization of the IVRR classes of distributions based on \( \eta(a) \) are obtained.

**Proposition 3.2** For non-negative discrete random variable \( X \) we have, \( X \) has IVRR property if and only if,
\[
\eta(a-1) > \frac{[\delta(a) + \delta(a-1)]^2}{\delta(a-1)}. \]
Proof. On using (3.4) and (2.8), the proof is completed.

In the next theorem, we present a characterization about $\beta^2(a)$ which is not quit similar in continuous case.

**Theorem 3.4** $\beta^2(a)$ is increasing with respect to $a$ if and only if,

$$\psi(a) < 1,$$

where

$$\psi(a) = \frac{\beta^2(a)}{\delta(a+1)\delta'(a+1)},$$

and $\delta'(a) = E[a - X|X < a]$. 

**Proof.** Using (3.4) and $\delta(a) = \delta'(a+1)-1$ the required result is obvious.

### 4. Application, Examples and Simulation Results

#### 4.1. Application with Uncensored Data

In a standard survival analysis application, individuals are followed over time for the occurrence of a specific event. If the event is observed to occur, the data is recorded as the time the event occurred, $x$. In some situations, however, the times of the events of interest may only be known to have occurred within an interval of time, $[0, a]$, where $0 \leq X < a$. This can occur in a clinical trial, for example, when patients are assessed often only at pre-scheduled visits, but the event of interest may occur in between visits. These data are known as interval-censored data.

Listed below are the survival times of 29 patients from a chronic myelocytic leukemia (CML) study conducted by [16]. The survival times, say $x_i$, are given in weeks from date of diagnosis. They are listed sequentially in order of entry into the study.

14, 43, 61, 29, 16, 70, 35, 45, 5, 10, 14, 0, 62, 15, 72, 93, 58, 26, 72, 4, 10, 73, 23, 22, 61, 95, 24, 34, 5.

Here, ‘0’ indicates the survival time for a patient who died at the same day of diagnosis. Using this data, the empirical estimate of $MRR(\delta(a))$ for a person who was found dead 50 weeks after his/her diagnosis ($a$), is 33 weeks. Having this result, the estimate of $VRR$ is $\hat{\beta}^2(a) = 73.962$ and the estimate of $RRC$ is $\eta(a) = 0.261$.

#### 4.2 Examples

The following are examples of an increasing IVRR functions of some well-known distributions:

**Example 4.1:** The random variable $X$ has the probability miss function

$$f_{w_i(a, \beta)}(t) = q^{(a-1)\beta} - q^\beta,$$

i.e. $X$ follows as the type I discrete Weibull distribution $w_i(q, \beta)$ with $q$ (the probability of surviving the first demand), if any one of the following is true for all $t > 0$:
Example 4.2: The random variable $X$ has the probability mass function

$$f_{G(p)}(t) = p(1-p)^{t-1}; t = 1,2,3,...,$$

i.e. $X$ follows as the geometric distribution, $G(p)$, if any one of the following is true for all $t > 0$:

i. $\delta_{G(p)}(t) = \frac{1 - tp - (1-p)^t}{p(1-p)^t}$;

ii. $\beta^2_{G(p)}(t) = \frac{(2t-1)(1 - tp - (1-p)^t) - [1 - pt - (1-p)^t]}{p^2(1-p)^{2t} - 2[1 - p - t(1-p)^t + (t-1)(1-p)^{t-1}]}$.

Note that the geometric distribution is the analogous in discrete time of the exponential distribution, since it has the lack of memory property (no ageing, no burn-in): the system failure probabilities on each demand are independent and all equal to $p \in [0,1]$. Equivalently, the failure rate is constant.

Example 4.3: Let $K_\theta$ be a shifted geometric random variable with probability mass function as follows

$$f_{K_\theta}(k) = \theta(1-\theta)^{k-1}, \ k \in \{1,2,...\},$$

then, the reversed hazard rate function of $K_\theta$ is given by

$$v_{K_\theta}(k) = \frac{\theta(1-\theta)^{k-1}}{1 - (1-\theta)^k}, \ k \in \{1,2,...\}.$$

We have the mean reversed residual time function of $K_\theta$ is given by

$$\delta(k) = \frac{\theta(k+1) + (1-\theta)^{k+1} - 1}{\theta - \theta(1-\theta)^k}, \ k \in \{1,2,...\}.$$

Example 4.4: Let $K_{\theta_1}$ and $K_{\theta_2}$ be independent shift geometric random variables with respective parameters $\theta_1$ and $\theta_2$. Then the probability mass function of $Z = K_{\theta_1} + K_{\theta_2}$ can be written as

$$f_{\theta_1,\theta_2}(Z = z) = \begin{cases} \theta_1 \theta_2 \frac{(1-\theta_1)^{z} - (1-\theta_2)^{z}}{(1-\theta_1)(\theta_2 - \theta_1)}, & \theta_1 = \theta_2, \\ z \theta_1 (1-\theta_1)^{z-2}, & \theta_1 = \theta_2 = \theta, \end{cases}$$

where $z \in \{2,3,...\}$. Then

(i). The reversed hazard rate function of $Z$ is given by
Probabilistic properties of discrete mean and variance

\[
\begin{aligned}
\nu_{n, \theta}(z) &= \begin{cases} 
\alpha_1 \alpha_2 \left[ (1-\theta)^2 - (1-\theta)^{\gamma} \right] 
& ; \quad \theta_1 = \theta_2, \\
\alpha_1 \left[ (1+\theta)^2 - (1-\theta)^{\gamma+1} \right] - \alpha_2 \left[ (1+\theta)^2 - (1-\theta)^{\gamma+1} \right] 
& \quad \frac{z \theta^2 (1-\theta)^{\gamma-2}}{1 + \theta + (\theta+1)(1-\theta)^{\gamma+1}}; \quad \theta_1 = \theta_2 = \theta.
\end{cases}
\end{aligned}
\]

(ii). MRR of Z is given by

\[
\delta_{\alpha, \beta}(z) = \sum_{u=2}^{z} \frac{F_{n, \alpha}(u)}{F_{n, \beta}(z)}, \quad z \in \{2, 3, \ldots\}
\]

where,

\[
\begin{align*}
\alpha_1 &= \theta_1 \theta_2 (z-1)(\theta_2 (1-\theta) + \theta_1 (1-\theta_2)) \\
&\quad - \theta_1^2 \left( (1-\theta)^{\gamma} - (1-\theta_2)^{\gamma+1} \right) + \theta_2^2 \left( (1-\theta_1)^{\gamma} - (1-\theta)^{\gamma+1} \right), \\
\alpha_2 &= \theta_1 \theta_2 \left[ (1-\theta_2 - (1-\theta)^{\gamma+1}) + \theta_1 (1-\theta_2)^{\gamma+1} + \theta_2 - 1 \right], \\
\alpha_3 &= (z-1) + \frac{1}{\theta} \left[ (\theta+2)(1-\theta)^{\gamma} - (1-\theta)(\theta+2) \right], \\
\alpha_4 &= 1 - (1-\theta)^{\gamma+1}(\theta(z+1) - \theta + 1).
\end{align*}
\]


5.1. Nonparametric Procedure

Based on a sample \(X_1, X_2, \ldots, X_n\) with distribution function \(F\). In order to test for the IVRR class, one observe that there is no boundary distribution at all (i.e., there is no distribution where VRR is constant). Hence, to test the IVRR, we set

\[
H_0 : F = F_0,
\]

against

\[
H_1 : F \text{ is IVRR and not } F_0,
\]

where \(F_0\) is known (up to a set of parameters). It is obvious to choose \(F_0\) as the type I discrete Weibull distribution, \(W_1(q, \beta)\). Thus, we address

\[
H_0 : F = W_1(q, \beta) = F_0,
\]

against

\[
H_1 : F \text{ is IVRR and not } W_1(q, \beta).
\]
Note that $F \in IVRR$ if and only if equation (2.7) is increasing.
We present an empirical approach which is derived from the property of the IVRR class that if $F$ is IVRR, then
\[
\frac{2g(a)}{F(a)} - \left( \frac{s(a)}{F(a)} \right)^2 \text{ is increasing in } a \geq 0,
\]
where $g(a) = \sum_{i=0}^{a-1} \sum_{j=0}^{i} F(j)$ and $s(a) = \sum_{j=0}^{a-1} F(j)$. If we set
\[
\Lambda = 2g(x)F(x)F^2(a) - F^2(a)s(x) - 2g(a)F(a)F^2(x) + F^2(x)s(a), \text{ for all } a \leq x.
\]
We thus take a measure of departure for this test, $\tau$, as:
\[
\tau = \sum \sum \Lambda p(a)p(x)I(a-x),
\]
where $p(a)$ is probability mass function and
\[
I(a-x) = \begin{cases} 1 & \text{if } a-x \geq 0 \\ 0 & \text{o.w.} \end{cases}
\]
Using the empirical distribution function, one can find an unbiased estimate of $\tau$ (see [1]) in the form of
\[
\hat{\tau} = \frac{1}{(n)^4} \sum \sum \sum \sum_{i_1,i_2,i_3,i_4} \left( \frac{1}{2} x_{i_4} - 2x_{i_2}x_{i_3} + 3x_{i_1}^2 + x_{i_2}x_{i_3} - 2x_{i_2}x_{i_3} + x_{i_4}x_{i_5} - 2x_{i_2}x_{i_3} + \frac{1}{2} X_i^2 \right)
\times I(X_{i_4} - X_{i_2}I(X_{i_3} - X_{i_3}) \sum \sum \Lambda p(a)p(x)I(a-x).
\]
Setting
\[
\phi(x_1,x_2,x_3,x_4) = \frac{1}{(n)^4} \sum \sum \sum \sum_{i_1,i_2,i_3,i_4} \left( \frac{1}{2} x_{i_4} - 2x_{i_2}x_{i_3} + 3x_{i_1}^2 + x_{i_2}x_{i_3} - 2x_{i_2}x_{i_3} + x_{i_4}x_{i_5} - 2x_{i_2}x_{i_3} + \frac{1}{2} X_i^2 \right)
\times I(X_{i_4} - X_{i_2}I(X_{i_3} - X_{i_3}) \sum \sum \Lambda p(a)p(x)I(a-x).
\]
and define $\Phi(x_1,x_2,x_3,x_4) = \frac{1}{4!} \sum \sum \sum \sum_{i_1,i_2,i_3,i_4} \phi(x_1,x_2,x_3,x_4)$ where the sum is extended over all permutations $(i_1,i_2,i_3,i_4)$ of $(1,2,3,4)$, a simplified form of the above symmetric kernel is in the form of
\[
\Phi(x_1,x_2,x_3,x_4) = \frac{1}{4} \left[ \phi(x_1,x_2,x_3,x_4) + \phi(x_2,x_1,x_3,x_4) + \phi(x_2,x_1,x_3,x_4) + \phi(x_2,x_1,x_3,x_4) \right].
\]
Then, an equivalent U-Statistic type of $\hat{\tau}$ is
\[
U = \sum_{i_1,i_2,i_3,i_4} \Phi(x_1,x_2,x_3,x_4).
\]
Having the standard U-statistics theory, [19] proved the following result:
Theorem 5.1 as $n \to \infty$, $\frac{1}{n^2}(U - \tau)$ is asymptotically distributed normal with zero mean and variance $\sigma^2$ where

$$\sigma^2 = \text{Var}\left\{E(\Phi(X_1, X_2, X_3, X_4) \mid X_1) + E(\Phi(X_2, X_3, X_4, X_5) \mid X_2) + E(\Phi(X_2, X_3, X_4, X_5) \mid X_3)\right\}$$

$$= \text{Var}\left(\sum_{i=1}^{4} \varphi_i(X_i)\right)$$

Under the null hypothesis, standard the type I discrete Weibull distribution, the variance is $\sigma_0^2 = 103.46$.

To perform a test hypothesis, we reject $H_0$ if $0.098 n^2 U_n \geq Z_a$ where $Z_a$ is the standard normal variate.

5.2. Empirical Power and Critical Values

For empirical studies on the performance of IVRR test procedure, we carried out a series of 1000 simulations of size $n=10$, $n=50$ and $n=100$ from the following alternative probability mass functions:

1. The Weibull I discrete distribution with probability mass function

$$f_{W(I)}(t) = q^{(t-1)} - q^t, \quad t > 0, 1 \geq q \geq 0.$$  

2. The geometric distribution with probability mass function

$$f_{G(q)}(t) = (1 - q)^{t-1}; \quad t = 1, 2, 3, \ldots; 1 \geq q \geq 0.$$  

For small to moderate sample sizes, we use Monte Carlo methods with 1000 replicates to obtain empirical critical values and power estimates of our procedure. The results of simulation for this test procedure are shown in Table 1, Table 2 and Table 3.

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<th>$q$</th>
<th>Critical values for the following percentage points:</th>
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References


### TABLE II

EMPIRICAL CRITICAL OF $\frac{1}{\sigma_{\hat{\theta}}^2 n^2 U_{\theta}}$ FOR TESTING THE IVRR (THE GEOMETRIC DISTRIBUTION)

<table>
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<tr>
<th>$n$</th>
<th>$q$</th>
<th>0.90</th>
<th>0.95</th>
<th>0.97</th>
<th>0.90</th>
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</thead>
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<td>10</td>
<td>0.25</td>
<td>0.007</td>
<td>0.286</td>
<td>3.002</td>
<td>4.005</td>
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<td></td>
<td>0.50</td>
<td>1.158</td>
<td>1.258</td>
<td>3.215</td>
<td>4.889</td>
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<tr>
<td></td>
<td>0.75</td>
<td>2.081</td>
<td>2.589</td>
<td>3.857</td>
<td>5.005</td>
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<tr>
<td>50</td>
<td>0.25</td>
<td>0.112</td>
<td>0.652</td>
<td>3.112</td>
<td>4.627</td>
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<td></td>
<td>0.50</td>
<td>1.511</td>
<td>1.895</td>
<td>3.895</td>
<td>4.991</td>
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<td>0.75</td>
<td>2.890</td>
<td>2.486</td>
<td>4.002</td>
<td>5.226</td>
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<td>100</td>
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<td>1.005</td>
<td>0.879</td>
<td>3.698</td>
<td>4.725</td>
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<td>0.50</td>
<td>2.089</td>
<td>2.008</td>
<td>3.999</td>
<td>4.889</td>
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<td>0.75</td>
<td>2.871</td>
<td>2.997</td>
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### TABLE III

POWER ESTIMATES OF THE TEST AGAINST IVRR

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<th>$n$</th>
<th>$q$</th>
<th>Power estimate for the following sample size:</th>
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<td></td>
<td></td>
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<tr>
<td>Weibull</td>
<td>0.25</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.112</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.236</td>
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<tr>
<td>Geometric</td>
<td>0.25</td>
<td>0.211</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.369</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.421</td>
</tr>
</tbody>
</table>
Probabilistic properties of discrete mean and variance


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