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# Some Steady-State Thermoelastic Problems of Rectangular Plates

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**Abstract**

Two steady-state thermoelastic problems of rectangular plates are studied in the paper. The first problem is the inverse steady-state thermoelastic problem of determining the temperature, displacement and stress functions of a thin rectangular plate. On the two horizontal edges and one vertical edge the temperature is maintained at zero. On a vertical line the temperature is prescribed which serves as the interior condition. At the end a discussion on the inverse thermoelastic problem is included.

The second problem is the direct steady-state thermoelastic problem of determining temperature, displacement and stress functions in a thin rectangular plate with the stated boundary conditions, which are special cases of third kind boundary conditions.

**Keywords**: Thermoelastic problem, Thermal Stresses

## 1. INTRODUCTION

Tanigawa and Komatsurbara [1], Vihak, Vyuzvyak and Yasinkij [2] and Adams and Bert [3] have studied the direct problem of thermoelasticity in a rectangular plate under thermal shock. Khobragade and Wankhede [4] have studied a two dimensional inverse steady-state thermoelastic problem of determining the temperature, displacement function and thermal stresses at the boundary of a thin rectangular plate occupying the space $D_1$: $0 \le x \le a$; $0 \le y \le b$. The temperature $T(\xi, y) = f(y)$ for a fixed value of $\xi; 0 \le \xi < a$ is a known function of y and the temperature is maintained at zero on the two horizontal edges y = 0, b and a vertical edge x = 0 of the thin rectangular plate.

The displacement components $u_x$ and $u_y$ in the x and y directions are represented in the integral form

$$u_x = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \tag{1.1}$$

$$u_y = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \tag{1.2}$$

where ν and α are Poisson's ratio and linear coefficient of thermal expansion of the material of the plate respectively and U(x,y) is the Airy stress function which satisfies the following relations

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 U = -\alpha E \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 T \tag{1.3}$$

In Eqns (1.1)–(1.3), E is Young's modulus of elasticity and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.4}$$

subject to the interior condition and the boundary condition stated above.

We can find the temperature T by solving the steady-state heat conduction problem described above. Then we have to determine U satisfying Eqn. (1.3) or rather the following equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\alpha E T \tag{1.5}$$

as, for example, in [5]. The Airy stress function U in [4] is not determined correctly. Indeed, U in [4] satisfies the equation

$$\frac{\partial^2 U}{\partial x^2} = -\frac{\partial^2 U}{\partial y^2} = -\alpha E T \tag{1.6}$$

which implies that U is a harmonic function and therefore (1.5) cannot be satisfied by U unless $T \equiv 0$. Also, if we insist on U satisfying (1.3) where T satisfies (1.4), then U becomes a biharmonic function having no connection with the temperature T. Therefore we need to find U satisfying (1.5).

A similar situation exists in a paper by Singru and Khobragade [6]. The paper investigates a direct steady-state thermoelastic problem of determining the temperature, displacement and stress functions in a thin rectangular plate occupying the space $D_2 : 0 \le x \le a$ ; $-b \le y \le b$ with the stated boundary conditions which are special cases of the third kind boundary conditions. The temperature T in [6] satisfies Eqn. (1.4) in $D_2$. We can find the temperature T by solving the steady-state heat conduction problem defined by (1.4) and the prescribed boundary conditions. Singru and Khobragade [6] then take the Airy stress function U satisfying the equation

$$U(x,y) = -\alpha E T(x,y) \tag{1.7}$$

In view of Eqn. (1.4) the above equation implies that U is harmonic and hence it cannot satisfy (1.5) unless $T \equiv 0$. Also, if we insist on U satisfying (1.3) where T satisfies (1.4) then U becomes a biharmonic function having no connection with temperature T. Therefore we need to find U satisfying (1.5).

A three dimensional analogue of the above mentioned situation exists in a paper by Dange and Khobragade [7]. An attempt is made in [7] to determine the temperature, displacement function and the thermal stresses on upper plane surface of a three dimensional rectangular plate occupying the region $D_0$: $0 \le x \le a$ ; $-b \le y \le b$, $0 \le z \le h$ with known interior and boundary conditions. The displacement components $u_x$, $u_y$ and $u_z$ in the x, y and z directions respectively are in the integral form given by

$$u_x = \int \left[ \frac{1}{E}\left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \tag{1.8}$$

$$u_y = \int \left[ \frac{1}{E}\left( \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \tag{1.9}$$

$$u_z = \int \left[ \frac{1}{E}\left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial z^2} \right) + \alpha T \right] dz \tag{1.10}$$

where E, ν and α are Young's modulus, Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively, and U(x,y,z) is Airy's stress function which satisfies the differential equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 U = -\alpha\, E \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T \tag{1.11}$$

where T(x,y,z) denotes the temperature of the rectangular plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1.12}$$

an interior condition and boundary conditions which are special cases of the third kind boundary conditions. Also, a three dimensional analogue of the condition (1.5) may be written

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=-\alpha E T. \tag{1.13}$$

We can find the temperature T by solving the steady-state heat conduction problem described by the differential equation (1.12) the interior and the boundary conditions. Dange and Khobragade [7] then determine the Airy stress function U(x, y, z) which satisfies the equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0 \tag{1.14}$$

This implies that U has no connection with the temperature T. If we insist on U satisfying the condition (1.11) where T satisfies (1.12) then also U becomes a biharmonic function having no connection with the temperature T. Therefore we need to find U satisfying the condition (1.13).

Summarizing the above discussion we may say that the steady-state heat conduction problems are solved correctly in [4], [6] and [7], but the subsequent determination of thermal stresses and displacements is not correct. In the present paper we carry out these calculations correctly. Also, in [6] – [7] the Marchi-Fasulo transform is used to suppress one of the independent variables y. We show in the present paper that one can suppress the independent variable y by making use of a finite Fourier transform. The equivalence of the two transforms in the special case under consideration is established in the Appendix.

## 2. AN INVERSE THERMOELASTIC PROBLEM

This is the first problem described in Section 1 through the Eqns.(1.1) – (1.4) together with the following boundary conditions

$$T(x, 0) = 0; \quad T(x, b) = 0, \quad 0 < x < a \tag{2.1}$$

$$T(0, y) = 0; \quad T(\xi, y) = f(y), \quad 0 < y < b \tag{2.2}$$

Applying the finite Fourier sine transform defined by

$$\overline{T}(x,p)=\int_0^b T(x,y)\sin(py)dy;\ p=\frac{m\pi}{b} \tag{2.3}$$

to Eqns.(1.4), (2.2) and using (2.1) we get

$$\frac{\partial^2 \overline{T}}{\partial x^2}-p^2\overline{T}=0 \tag{2.4}$$

$$\overline{T}(0,p)=0\,;\quad \overline{T}(\xi,p)=\overline{f}(p) \tag{2.5}$$

Solving (2.4) subject to the conditions (2.5) gives

$$\overline{T}(x,p) = \overline{f}(p)\sinh(px)/\sinh(p\xi). \tag{2.6}$$

Inverting the finite Fourier sine transform yields

$$T(x,y) = \frac{2}{b}\sum_{m=1}^{\infty}\overline{f}(p)\frac{\sinh(px)}{\sinh(p\xi)}\sin(py);\ p = \frac{m\pi}{b} \tag{2.7}$$

Having determined the temperature T we have to find the Airy stress function U satisfying (1.5). Assuming that

$$U(x,0) = U(x,b) \equiv 0 \tag{2.8}$$

the finite Fourier sine transform of (1.5) gives

$$\frac{\partial^2 \overline{U}}{\partial x^2} - p^2 \overline{U} = -\alpha E \overline{T} \tag{2.9}$$

A particular solution of this equation is

$$\overline{U}(x,p) = \frac{\alpha E}{p}\int_0^x \overline{T}(r,p)\sinh[p(r-x)]dx \tag{2.10}$$

Substituting $\overline{T}(r,p)$ from (2.6) into (2.10), carrying out the required integration and inverting the finite Fourier sine transform gives

$$U(x,y) = -\frac{\alpha E}{b}\sum_{m=1}^{\infty}\frac{\overline{f}(p)\sin(py)}{p^2\sinh(p\xi)}\left[px\cosh(px) - \sinh(px)\right], \tag{2.11}$$

The stress components given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2};\quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2};\quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \tag{2.12}$$

together with (2.11) become

$$\sigma_{xx} = \frac{\alpha E}{b}\sum_{m=1}^{\infty}\frac{\overline{f}(p)\sin(py)}{\sinh(p\xi)}\left[px\cosh(px) - \sinh(px)\right], \tag{2.13}$$

$$\sigma_{yy} = -\frac{\alpha E}{b}\sum_{m=1}^{\infty}\frac{\overline{f}(p)\sin(py)}{\sinh(p\xi)}\left[px\cosh(px) + \sinh(px)\right] \tag{2.14}$$

$$\sigma_{xy} = -\frac{\alpha E}{b}\sum_{m=1}^{\infty}\frac{\overline{f}(p)\cos(py)}{\sinh(p\xi)}\left[px\sinh(px)\right], \tag{2.15}$$

where p is given by (2.7).

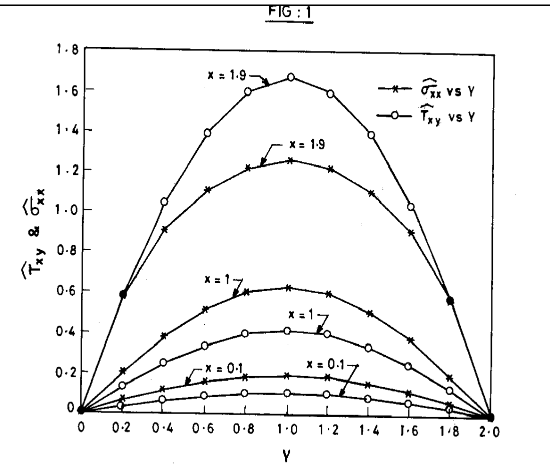


Fig1:Variation of $\hat{T}_{xy} = T(x,y)$ and $\hat{\sigma}_{xx} = (0.8\alpha E)^{-1}\ \sigma_{xx}$ given by equation (2.7) , (2.13) with (2.17) with respect to Y for X= 0.1,1.0, 1.9 . The values of other parameters are a=b=2

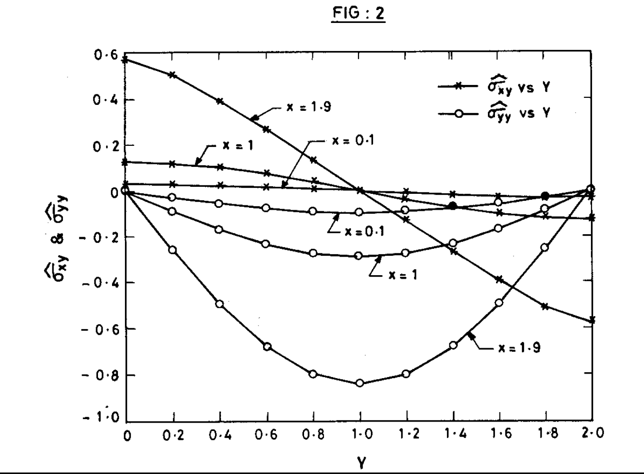


Fig 2 : Variation of $\hat{\sigma}_{xy} = (4\alpha E)^{-1}\ \sigma_{xy}$ and $\hat{\sigma}_{yy} = (0.4\alpha E)^{-1}\ \sigma_{yy}$ given by equation (2.14) ,(2.15) with (2.17) with respect to Y for X=0.1,1.0,1.9 . The values of other parameters are same as fig. 1

## SPECIAL CASES

Let the prescribed function f(y) in (2.2) be given by

$$f(y) = (b - y)y\xi \qquad (2.16)$$

with the finite Fourier sine transform

$$\overline{f}(p) = \frac{2\xi}{p^3}\left[1 - (-1)^m\right] \qquad (2.17)$$

so that the temperature and the thermal stresses are given by (2.7), (2.13) – (2.15) and (2.17).

## COMMENTS ON THE INVERSE PROBLEM

The series in the expression (2.7) for temperature is convergent only for $0 \le x \le \xi$, which may be seen more clearly in the special case (2.17). For $x > \xi$ the series diverges and hence it cannot represent any solution. Indeed, the steady-state heat conduction problem is well posed only for the region $D_{11}$ : $0 \le x \le \xi$, $0 \le y \le b$ ($0 < \xi < a$) and not for whole of the region $D_1 : 0 \le x \le a, 0 \le y \le b$ ($0<\xi < a$) . In particular, if we single out the rectangular portion $D_{12}$: $\xi \le x \le a$, $0 \le y \le b$ of $D_1$ then we find that the boundary conditions are given only for $y = 0$, b and $x = \xi$. In the absence of any condition on the boundary x = a of $D_{12}$ we cannot determine T on $D_{12}$.

However, the corrections of the solution can be restored by taking, $\xi = a$ and dropping the word 'inverse'.

## NUMERICAL RESULTS

We carried out some numerical calculations for the special case (2.16). The relevant formulae for the temperature T(x,y) and the stress components $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ are given by Eqns. (2.7), (2.13) – (2.15) together with (2.17) and, of course, $\xi = a$. The variation of T(x,y) with y is shown in Fig.1 for x = 0.1 and x = 1.9 by taking a = b = 2. The variation of $(0.8\ \alpha\ E)^{-1}\ \sigma_{xx}$ with y is also shown in Fig. 1 for x = 1.0 and x = 1.9. By carrying out the calculations for some other values of x, we find that T as well as $\sigma_{xx}$ is an increasing function of x. The variation of $(4\ \alpha\ E)^{-1}\sigma_{xy}$ with y is shown in Fig. 2 for x = 1.0 and x = 1.9. By carrying out the calculations for some other values of x, we find that $\sigma_{xy}$ increases for $0 < x < 1$ and decreases for $1 < x < 2$. The variation of $(0.4\ \alpha\ E)^{-1}\ \sigma_{yy}$ with y is also shown in Fig.2 for x = 0.1 and x = 1.9. By carrying out the calculations for some other values of x, we that $\sigma_{yy}$ is a decreasing function of x. Fig. 1 as well as in Fig. 2 we have taken a = b = 2.

## 3. A DIRECT THERMOELEASTIC PROBLEM

This is the second problem described in Section 1 wherein the rectangular plate occupies the space $D_2$ : $0 \le x \le a$; $-b \le y \le b$. The temperature T(x,y) in this problem satisfies Eqn. (1.4) in $D_2$ subject to the boundary conditions

$$\left[T + \frac{\partial T}{\partial x}\right]_{x=0} = h(y); \quad \left[T + \frac{\partial T}{\partial x}\right]_{x=a} = f(y) \tag{3.1}$$

$$\left[T + k_1 \frac{\partial T}{\partial y}\right]_{y=b} = 0; \quad \left[T + k_2 \frac{\partial T}{\partial x}\right]_{y=-b} = 0 \tag{3.2}$$

where $k_1$, $k_2$ are the radiation constants on the two edges $y = \pm b$ of the rectangular plate.

To solve the steady-state heat conduction problem we use the finite Fourier transform

$$\overline{T}(x, a_n) = \int_{-b}^{b} T(x, y) K(y, a_n) dy \tag{3.3}$$

together with the inversion formula

$$T(x,y)=\sum_{n=1}^{\infty}\frac{4a_nK(y,a_n)\overline{T}(x,a_n)}{N_n} \tag{3.4}$$

defined in the Appendix by Eqns. (13) – (16) where $a_n$ is the $n^{th}$ positive root of Eqn.(8). Taking finite Fourier transform of Eqns.(1.4), (3.1) and making use of the conditions(3.2) we find

$$\frac{\partial^2\overline{T}}{\partial x^2}-a_n^{\ 2}\overline{T}=0 \tag{3.5}$$

$$\left[\overline{T}+\frac{\partial\overline{T}}{\partial x}\right]_{x=0}=\overline{h}(a_n);\quad \left[\overline{T}+\frac{\partial\overline{T}}{\partial x}\right]_{x=a}=\overline{f}(a_n) \tag{3.6}$$

Solving the differential equation (3.5) subject to the conditions (3.6) gives

$$\overline{T}(x,a_n)=\frac{\overline{f}(a_n)}{(1-a_n^{\ 2})}\left[\frac{\sinh(a_nx)}{\sinh(a_na)}-\frac{a_n\cosh(a_nx)}{\sinh(a_na)}\right]$$
$$-\frac{\overline{h}(a_n)}{(1-a_n^{\ 2})}\left[\frac{\sinh[a_n(x-a)]}{\sinh(a_na)}-\frac{a_n\cosh[a_n(x-a)]}{\sinh(a_na)}\right]. \tag{3.7}$$

Finally, taking the inverse finite Fourier transform gives the temperature T(x,y) as

$$T(x,y)=\sum_{n=1}^{\infty}\frac{4a_n\overline{f}(a_n)K(y,a_n)}{(1-a_n^{\ 2})N_n}\left[\frac{\sinh(a_nx)}{\sinh(a_na)}-\frac{a_n\cosh(a_nx)}{\sinh(a_na)}\right]$$
$$-\sum_{n=1}^{\infty}\frac{4a_n\overline{h}(a_n)K(y,a_n)}{(1-a_n^{\ 2})N_n}\left[\frac{\sinh[a_n(x-a)]}{\sinh(a_na)}-\frac{a_n\cosh[a_n(x-a)]}{\sinh(a_na)}\right] \tag{3.8}$$

In view of the relations (12) and (18) in the Appendix the temperature T given by (3.8) is the same as that obtained by Singru and Khobragade [6]. However, we have to find the Airy stress function U(x,y) satisfying (1.5). Following the analysis of Section 2 we take

$$U(x,y)=-4\alpha E\sum_{n=1}^{\infty}\frac{K(y,a_n)}{N_n}\int_0^x\overline{T}(r,a_n)\sinh[a_n(x-r)]dr. \tag{3.9}$$

where $\overline{T}(r,a_n)$ is the finite Fourier transform of T(r,y). Substituting $\overline{T}$ from (3.7) into (3.9) and carrying out the integration gives

$$U(x,y)=-2\alpha E\sum_{n=1}^{\infty}\frac{\overline{f}(a_n)k(y,a_n)}{(1-a_n^{\ 2})N_n\sinh(a_na)}\left[x\cosh(a_nx)-xa_n\sinh(a_nx)-\frac{\sinh(a_nx)}{a_n}\right]$$
$$+2\alpha E\sum_{n=1}^{\infty}\frac{\overline{h}(a_n)K(y,a_n)}{(1-a_n^{\ 2})Nn}\sinh(a_na)\left[x\cosh[a_n(x-a)-xa_n\sinh[a_n(x-a)\right]. \tag{3.10}$$

It may be noted that if U is a solution of (1.5) then so is U + $U_0$ where $U_0$ is a harmonic function. This fact has been used to arrive at (3.10) where we have retained only the terms which produce convergent series.

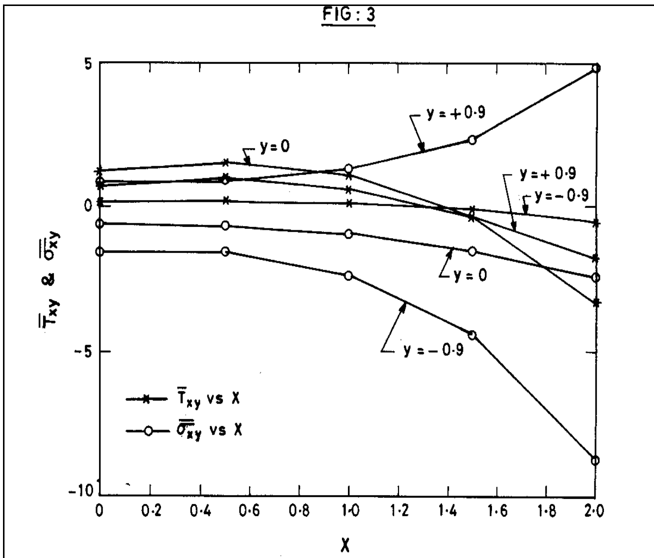


Fig3: Variation of $\bar{T}_{xy} = T(x,y)$ and $\bar{\sigma}_{xx} = (2\alpha E)^{-1}\ \sigma_{xx}$ given by equation (3.8), (3.11) with (3.17) as well as equations (8), (13) and (16) in appendix with respect to X for Y=-0.9,0.0,0.9 The values of other parameters are a=2, b=1, $k_1$ =1, $k_2$ =0.

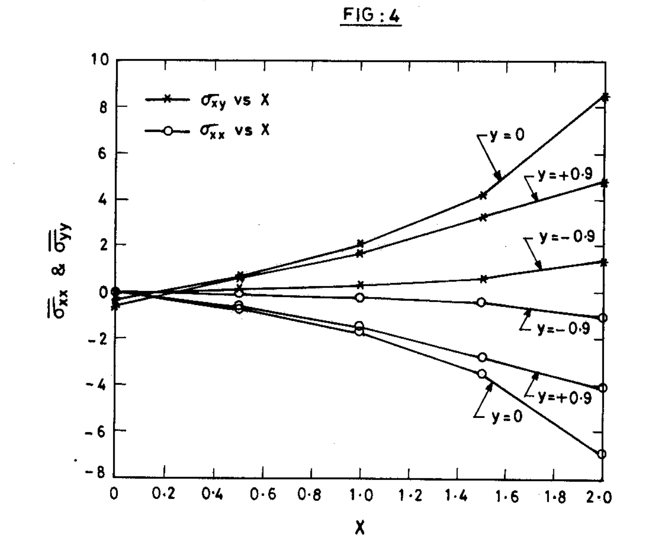


Fig 4 : Variation of $\bar{\sigma}_{xy} = (4\alpha E)^{-1}\ \sigma_{xy}$ and $\bar{\sigma}_{yy} = (2\alpha E)^{-1}\ \sigma_{yy}$ given by equation (3.12) ,(3.13) with (3.17) as well as equations (8), (13) and (16) in appendix with respect to X for Y=-0.9,0.0,0.9 . The values of other parameters are same as fig.3

In terms of U in (3.10) the stress components are given by Eqns. (2.12). Carrying out the require differentiations we have

$$\sigma_{xx} = 2\alpha E \sum_{n=1}^{\infty} \frac{\bar{f}(a_n)K(y,a_n)a_n}{(1-a_n^2)N_n \sinh(a_n a)} \left[x a_n \cosh(a_n x) - x a_n^2 \sinh(a_n x) - \sinh(a_n x)\right]$$

$$- 2\alpha E \sum_{n=1}^{\infty} \frac{\bar{h}(a_n)K(y,a_n)a_n}{(1-a_n^2)N_n \sinh(a_n a)} \left[x a_n \cosh[(a_n(x-a)] - x a_n^2 \sinh[a_n(x-a)]\right]$$

(3.11)

$$\sigma_{yy} = -2\alpha E\sum_{n=1}^{\infty}\frac{\bar{f}(a_n)K(y,a_n)a_n}{(1-a_n^2)N_n\sinh(a_n a)}\left[xa_n\cosh(a_n x) - xa_n^2\sinh(a_n x) + \sinh(a_n x) - 2a_n\cosh(a_n x)\right]$$
$$+2\alpha E\sum_{n=1}^{\infty}\frac{\bar{h}(a_n)K(y,a_n)a_n}{(1-a_n^2)N_n\sinh(a_n a)}\left[xa_n\cosh[a_n(x-a)] - xa_n^2\sinh[a_n(x-a)]\right.$$
$$\left.+\,2\sinh[a_n(x-a)] - 2a_n\cosh[a_n(x-a)]\right], \quad (3.12)$$

$$\sigma_{xy} = -2\alpha E\sum_{n=1}^{\infty}\frac{\bar{f}(a_n)L(y,a_n)a_n}{(1-a_n^2)N_n\sinh(a_n a)}\left[xa_n\sinh(a) - xa_n^2\cosh(a_n x) - a_n\sinh(a_n x)\right]$$
$$+2\alpha E\sum_{n=1}^{\infty}\frac{\bar{h}(a_n)L(y,a_n)a_n}{(1-a_n^2)N_n\sinh(a_n a)}\left[xa_n\sinh[a_n(x-a)] - xa_n^2\cosh[a_n(x-a)]\right.$$
$$\left.+\cosh[a_n(x-a)] - a_n\sinh[a_n(x-a)]\right], \quad (3.13)$$

$$L(y,a_n) = k_2 a_n\sin[a_n(y+b)] + \cos[a_n(y+b)]. \quad (3.14)$$

Using (3.11)-(3.12), (2.12) and (3.8) it is easily verified that

$$\sigma_{xx} + \sigma_{yy} = \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2} = -\alpha ET. \quad (3.15)$$

## SPECIAL CASE AND NUMERICAL RESULTS

Let h(y) and f(y) in (3.1) be given by

$$h(y) = (y^2 - b^2); \quad f(y) = (y^2 - b^2)e^a \quad (3.16)$$

The finite Fourier transform of the functions may be written

$$\bar{f}(a_n) = e^a\bar{h}(a_n) = \frac{4e^a}{a_n^3}\left[a_n^2 k_2 b\cos^2(a_n b) + \sin^2(a_n b) - a_n(b+k_2)\cos(a_n b)\sin(a_n b)\right]. \quad (3.17)$$

We have carried out some numerical calculations for the special case given by Eqn.(3.16). The temperature T(x,y) and the stress components $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ are given by Eqns. (3.8), (3.11)-(3.14), (3,17) and equations (8), (13), (16) in the Appendix. The numerical values of various parameters are fixed as follows

$$a = 2; \quad b = 1; \quad k_1 = 1; \quad k_2 = 0 \quad (3.18)$$

The variation of T(x,y) with x is shown in Fig. 3 for y = - 0.9 and y = 0.0. We carried out the calculations for y = 0.9 as well. The curve for y = 0.9 lies for most part between the two curves shown in Fig. 3. The variation of $(2\alpha E)^{-1}\,\sigma_{xx}$ with x is also shown in Fig. 3 for y = - 0.9 and y = 0.0. The calculations for y = 0.9 indicate that $\sigma_{xy}$ is an increasing function of y.

The variations of $(4\alpha E)^{-1}\sigma_{xy}$ and $(2\alpha E)^{-1}\sigma_{yy}$ with x are shown in Fig. 4 for y = - 0.9 and y = 0.0.We carried out calculations for y = 0.9 as well. We observe that the curves for y = 0.9 lie for most part between the curves for y = - 0.9 and y = 0.0.

## APPENDIX

### A SPECIAL CASE OF MARCHI-FASULO TRANSFORM

The Marchi-Fasulo transform used in Singru and Khobragade [6] is defined by the pair

$$\hat{f}(n) = \int_{-b}^{b} f(y)P_n(y)dy; \quad f(y) = \sum_{n=1}^{\infty} \frac{\hat{f}(n)}{\lambda_n} P_n(y) \tag{1}$$

where

$$P_n(y) = Q_n \cos(a_n y) - W_n \sin(a_n y) \tag{2}$$

$$Q_n = a_n(\alpha_1 + \alpha_2)\cos(a_n b) + (\beta_1 - \beta_2)\sin(a_n b) \tag{3}$$

$$W_n = (\beta_1 + \beta_2)\cos(a_n b) + (\alpha_2 - \alpha_1)\sin(a_n b) \tag{4}$$

and $a_n$ are the positive roots of the equation

$$[\alpha_1 a_n \cos(a_n b) + \beta_1 \sin(a_n b)][\beta_2 \cos(a_n b) + \alpha_2 a_n \sin(a_n b)] = [\alpha_2 a_n \cos(a_n b) - \beta_2 \sin(a_n b)][\beta_1 \cos(a_n b) - \alpha_1 a_n \sin(a_n b)], \tag{5}$$

together with

$$\lambda_n = \int_{-b}^{b} P_n^{\,2}(y) = b[Q_n^{\,2} + W_n^{\,2}] + \frac{\sin(2a_n b)}{2a_n}[Q_n^{\,2} - W_n^{\,2}]\,. \tag{6}$$

Now consider the special case

$$\beta_1 = \beta_2 = 1; \qquad \alpha_1 = k_1, \quad \alpha_2 = k_2 \tag{7}$$

With these values Eqn.(5) reduces to

$$(1 + k_1 k_2 a_n^{\,2})\sin(2a_n b) + (k_1 - k_2)a_n \cos(2a_n b) = 0 \tag{8}$$

and $P_n(y)$ in Eqn.(2) becomes

$$P_n(y) = a_n k_2 \cos[a_n(y+b)] - \sin[a_n(y+b)] + R_n(y), \tag{9}$$

where

$$R_n(y) = a_n k_1 \cos[a_n(y-b)] - \sin[a_n(y-b)]. \tag{10}$$

Writing (y - b) = (y + b) – 2b, rearranging the terms in Eqn.(10) and using (8) we can show that

$$R_n(y) = [\cos(2a_n b) - k_1 a_n \sin(2a_n b)][a_n k_2 \cos[a_n(y+b)] - \sin[a_n(y+b)]].$$

(11)

In view of Eqns.(9) and (11) we write

$$P_n(y) = A_n K(y, a_n); \quad A_n = 1 + \cos(2a_n b) - k_1 a_n \sin(2a_n b), \tag{12}$$

where

$$K(y, a_n) = k_2 a_n \cos[a_n(y+b)] - \sin[a_n(y+b)] \tag{13}$$

The relation (12) permits us to define a simpler transform

$$\bar{f}(a_n) = \int_{-b}^{b} f(y) K(y, a_n) dy \tag{14}$$

with the inversion formula

$$f(y) = \sum_{n=1}^{\infty} \frac{4a_n K(y, a_n) \bar{f}(a_n)}{N_n} \tag{15}$$

where $a_n$ is the $n^{th}$ positive root of Eqn.(8) and we have

$$N_n = 2a_n\left[2b(1 + k_2^2 a_n^2) - k_2\right] + 2k_2 a_n \cos(4a_n b) + (k_2^2 a_n^2 - 1)\sin(4a_n b). \tag{16}$$

The transform pair (14) – (15) is equivalent to the special case (7) of the Marchi-Fasulo transform (1). Transforms of this type have been used in several research papers by Deshmukh and his co-authors (see, for example, [9]). As in [9] the transform pair (14) – (15) may be called finite Fourier transform. The kernel function $K(y, a_n)$ in (13) satisfies the conditions

$$\left[K + k_2 \frac{\partial K}{\partial y}\right]_{y=-b} = 0; \quad \left[K + k_1 \frac{\partial K}{\partial y}\right]_{y=b} = 0 \tag{17}$$

The second of these conditions is equivalent to Eqn. (8). Also from (1), (6), (12) and (14) it follows that

$$\hat{f}(n) = A_n \bar{f}(a_n); \quad \lambda_n = A_n^2 \int_{-b}^{b} \left[K(y, a_n)\right]^2 dy = \frac{N_n}{4a_n} A_n^2. \tag{18}$$

which establish a relationship between the two transforms.

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