Bayesian Prediction for Unobserved Data from Exponentiated Weibull Distribution based on Progressive Type II Censoring

Jinhyouk Jung
Department of Statistics, University of Connecticut
215 Glenbrook Road U-4120, Storrs, CT 06269-4120, USA

Chansoo Kim
Department of Applied Mathematics, Kongju National University, Korea
chanskim@kongju.ac.kr (Corresponding author)

Copyright © 2013 Jinhyouk Jung and Chansoo Kim. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we consider Bayesian estimation and prediction problem for the parameters and unobserved lifetimes of exponentiated Weibull distribution based on a progressively type II censored samples. By using an extended likelihood function and informative joint prior distributions, the joint posterior density for the parameters and unobserved lifetimes of units censored at the failure time is obtained. Since the closed form of Bayes estimators does not exist, we use Markov Chain Monte Carlo (MCMC) method such as Gibbs sampling and Metropolis-Hastings algorithm to generate the posterior conditional probabilities of interest. Monte Carlo simulations and real data analysis are conducted to observe the behaviour of the proposed method.

Keywords: Bayesian prediction, Extended likelihood function, Exponentiated Weibull distribution, Gibbs sampling, Metropolis-Hastings algorithm, Progressive type II censoring
1 Introduction

For the analysis of lifetime study, one could observe that many lifetime data have a bathtub shape or upside-down bathtub shape failure rates. As an extension of the well-known Weibull distribution, two-parameter Exponentiated Weibull (EW) distribution was introduced by Mudholkar and Srivastava (1993) [11] was frequently used because it has non-monotone failure rates besides a broader class of monotone failure rates. As a failure model, EW model has been applied for failure time modelling in areas of reliability, quality control, duration and so on.

The cumulative distribution function (cdf) of EW distribution with two shape parameters, $\alpha$ and $\theta$, and its corresponding probability density function (pdf) are given respectively by

$$F(t) = (1 - e^{-t^\alpha})^\theta, \quad t > 0, \quad \alpha > 0, \quad \theta > 0 \quad (1.1)$$

and

$$f(t) = \alpha \theta t^{\alpha-1} e^{-t^\alpha} (1 - e^{-t^\alpha})^{\theta-1}, \quad t > 0. \quad (1.2)$$

If the shape parameter $\theta = 1$ then the pdf is the same with Weibull distribution. Mudholkar and Hutson (1996) [12] showed that the density function of the EW distribution is decreasing when $\alpha \theta \leq 1$ and unimodal when $\alpha \theta > 1$. Parametric characterizations and statistical properties of the density function are discussed by Mudholkar and Hutson (1996) [12], Jiang and Murthy (1999) [8] and Nassar and Eissa (2003) [13].

In many life test studies, it is common that the lifetimes of test units may not be able to record exactly or the test terminates intentionally after a predetermined number of failures occurs in order to save time or cost. Furthermore, some test units may have to be removed at different stages because of various reasons. This would lead to progressive censoring. The progressively type II censored sampling is an important method of obtaining data in lifetime studies. For example, live units removed in the experiment can be readily used in other tests and used for saving costs to the experimenters. Some early works related with censoring problem can be found in Cohen (1963) [4], Viveros and Balakrishnan (1994) [16], Balakrishnan and Sandhu (1995) [2], Balakrishnan and Aggarwala (2000) [3] and Nassar and Eissa (2004) [14]. Recently, Kim et al. (2011) [9] considered the Bayesian estimation of the parameters and the reliability function of the EW distribution with progressively type II censored sampling.

In this paper, our main object is to study the Bayesian prediction of unobserved lifetimes for the two parameters of EW distribution based on progressively type II censored sample. The prediction problem will be very useful when the machine tool should be replaced or industrial process must be changed in
Bayesian prediction for unobserved data from EW distribution

the field of engineering. It is also available in the field of business when to
know the provision of warranty limits for the future performance of a specified
number of systems. Some applications of the related topics are studied by Lin
et al. (2006) [10], Huang and Wu (2012) [7] and Ahmadia et al. (2012) [1].

The rest of this article is organized as follows: In Section 2, we introduce
briefly progressive type II censoring. Bayesian prediction for the unobserved
lifetimes of units censored at the observed failure time is provided in Section 3.
Section 4 gives numerical study under the situation of two different censoring
schemes and real data analysis.

2 Progressive type II censoring

Suppose that \( n \) randomly selected items are placed on a life-testing experi-
ment and censoring scheme \((R_1, R_2, \cdots, R_m)\) is previously fixed. When the
first failure time, \( t_1 \), has occurred, \( R_1 \) surviving items are removed from the
experiment at random and immediately following the second failure \( t_2 \), \( R_2 \)
items are removed randomly from the test; this process continues until at the
time of \( m \)th observed failure time \( t_m \), the remaining \( R_m \) items are removed
from the test. Then, the \( m \) ordered failure times denoted by \((t_1, t_2, \cdots, t_m)\)
are called progressively type II censored right order statistics of size of \( m \) from
a sample of size of \( n \) with progressive censoring scheme \((R_1, R_2, \cdots, R_m)\). It
is clear that \( n = m + R_1 + R_2 + \cdots + R_m \). Note that the familiar com-
plete and type II right-censored samples are special cases of this scheme. If
\( R_1 = R_2 = \cdots = R_{m-1} = 0 \) and \( R_m = n - m \), this sampling scheme reduces to
the conventional type II censoring. Also, if \( R_1 = R_2 = \cdots = R_m = 0 \), then the
progressively type II censoring scheme reduces to the complete sampling case.

Let \( t = (t_1, t_2, \cdots, t_m) \) denote a progressively type II censored sample from
the population given in (1.1) with progressive censoring scheme \((R_1, R_2, \cdots, R_m)\).
Then the likelihood function of \((\alpha, \theta)\) based on the progressively type II cen-
sored sample can be written as:

\[
L(\alpha, \theta|t) \propto (\alpha \theta)^m \exp \left[-\varphi_1 + \varphi_2\right],
\]

where

\[
\varphi_1 = \sum_{i=1}^{m} \left[t_i^\alpha - (\alpha - 1) \ln t_i + \ln u_i\right],
\]

\[
\varphi_2 = \sum_{i=1}^{m} \left[\theta \ln u_i + R_i \ln(1 - u_i^\theta)\right],
\]

and \( u_i = (1 - e^{-t_i^\alpha}) \).

The maximum likelihood estimators (MLE) of \((\alpha, \theta)\) can not be obtained
in closed forms due to the complex of the likelihood function. It can be solved
by numerical methods in order to determine the MLE. (see Kim et al. (2011) [9].) Since the parameters \((\alpha, \theta)\) are assumed to be unknown, we consider a bivariate prior density as follows:

\[
\pi(\alpha, \theta) = \pi_2(\theta|\alpha)\pi_1(\alpha),
\]

where

\[
\pi_1(\alpha) = \frac{1}{b}e^{-\frac{\alpha}{\theta}},
\]

and

\[
\pi_2(\theta|\alpha) = \frac{\alpha^{-\nu}}{\Gamma(\nu)}\theta^{\nu-1}e^{-\frac{\alpha}{\theta}}, \theta > 0.
\]

Here, \(b\) and \(\nu\) are assumed to be known. Therefore, the bivariate prior density function of \(\alpha\) and \(\theta\) can be written as

\[
\pi(\alpha, \theta) = \left(\frac{b\Gamma(\nu)}{\Gamma(\nu)}\right)^{-1}\alpha^{-\nu}\theta^{\nu-1}e^{-\left(\frac{b}{\alpha}\right)\theta}, \alpha > 0, \theta > 0.
\]

Using (2.1) and (2.5), the joint posterior density function of \(\alpha\) and \(\theta\) given \(t\) is proportional to

\[
\pi(\alpha, \theta|t) \propto \alpha^{m-\nu}\theta^{m+\nu-1}e^{-\left(\frac{\alpha}{\theta}+\phi_1+\phi_2+\frac{\theta}{\alpha}\right)},
\]

where \(\phi_1, \phi_2\) and \(u_i\) are defined in (2.1).

Assume that \(\phi(\alpha, \theta)\) is a function of the parameters of \(\alpha\) and \(\theta\). Then Bayes estimator of a function \(\phi(\alpha, \theta)\) relative to squared error loss function (SEL) takes the posterior mean. However, the Bayes estimator of \(\phi(\alpha, \theta)\) can not be derived in an explicit form. Therefore, Kim et al. (2011) [9] obtained the Bayes estimators of the parameters using Lindley approximation. To compute the Bayes estimators of \(\alpha\) and \(\theta\) with respect to SEL, we consider the MCMC method to generate samples from the posterior distribution in next section.

3 Bayesian prediction for unobserved data

In this section, we derive the posterior predictive distribution of the unobserved lifetimes at the failure time and present a MCMC method to generate samples from its posterior distribution and in turn computing the Bayes estimators of interest. Before deriving the posterior predictive distribution of the unobserved lifetimes, we need to understand data structure referred by Figure 1.

Suppose that \(R_h = r\), \((0 \leq r \leq n - h - \sum_{i=1}^{h-1} R_i\) and \(1 \leq h \leq m)\), items are removed randomly from the test when the \(h\)th failure time \(t_h\) has occurred and \((Y_{1:h}^{[h]}, \cdots, Y_{R_h:h}^{[h]})\) are the ordered lifetimes of \(R_h\) surviving units removed at time \(t_h\). For example, \(R_h=3\) then \(Y_{1:3}^{[h]}, Y_{2:3}^{[h]}, Y_{3:3}^{[h]}\) are removed from
Bayesian prediction for unobserved data from EW distribution

Figure 1: Distribution of $Y_{h}^{(k)}$ as the $k$-th usual order statistic in a censoring scheme $R_h$ from the population $F(x)$ left truncated at $t_h$.

the experiment. It is known that $Y_{h}^{(k)}$, given $t_h$ is distributed as the $k$-th usual order statistic in a sample of size $R_h$ from the population $F(x)$ in (1.1) left truncated at $t_h$. Therefore the probability density function of $Y_{h}^{(k)} = y$ given $t_h$ can be written as follows:

$$f(y|\alpha, \theta) = \frac{R_h! \alpha \theta}{(k-1)!(R_h-k)!} [U^\theta_y - U^\theta_h]^{k-1}[1 - U^\theta_h R_h - k]^{-y} e^{-y^\alpha} U^{\theta - 1}_y, \quad y \geq t_h$$

(3.1)

where $U_h = (1 - e^{-t_h^\alpha})$ and $U_y = (1 - e^{-y^\alpha})$.

Given $t$, the joint posterior distribution of $Y_{h}^{(k)}$, $\alpha$ and $\theta$ is then

$$\pi(y, \alpha, \theta | t) = \int_0^\infty \int_0^\infty \pi(y, \alpha, \theta | t) d\alpha d\theta$$

Calculating the posterior predictive distribution, $\pi(y|t)$ is not feasible. Accordingly, its Bayes estimate under SEL cannot be obtained analytically. Therefore, we adopt the following Markov Chain Monte Carlo (MCMC) method to draw samples from the posterior density function and then to compute the Bayes estimates.

Lin et al. (2006) [10] provided an efficient approach for generating the posterior distribution and we adopt the similar manner with their method. Denote $y_i = (y_i^{(1)}:R_i, \cdots, y_i^{(j)}:R_i)$ for the ordered failure times of the $R_i$ units removed from the test at the time of the $i$-th ($1 \leq i \leq m$) failure.
The extended likelihood function of \((t, Y_1, \ldots, Y_m)\) is given by

\[
L(t, Y_1, \ldots, Y_m | \alpha, \theta) \propto \prod_{i=1}^{m} f(t_i) \prod_{j=1}^{R_i} f(y_{ij}^i)
\]

\[
= \prod_{i=1}^{m} f(t_i) \prod_{i=1}^{m} \prod_{j=1}^{R_i} f(y_{ij}^i).
\] (3.3)

Substituting (1.2) to (3.3), the extended likelihood can be obtained as follows:

\[
L(t, Y_1, \ldots, Y_m | \alpha, \theta) \propto \{ (\alpha \theta)^{(m+p)} \} (W_t W_z)^{(\alpha-1)} e^{-\sum_{i=1}^{m} t_i^\alpha + \sum_{k=1}^{p} z_k^\alpha}
\]

\[
\times \prod_{i=1}^{m} (1 - e^{-t_i^\alpha})^{\theta-1} \prod_{k=1}^{p} (1 - e^{-z_k^\alpha})^{\theta-1}.
\] (3.4)

where \(z_k = Y_{j_2}^{[h]} \), \(k = j + \sum_{l=1}^{j-1} R_l\), for \(j = 1, 2, \ldots, R_h\), \(p = \sum_{j=1}^{m} R_j\), \(z = (z_1, z_2, \ldots, z_p)\), \(W_t = \prod_{j=1}^{m} t_i\) and \(W_z = \prod_{j=1}^{m} z_i\). By (2.5) and (3.4), the joint posterior distribution can be obtained by multiplying them. Therefore, the joint posterior distribution is given by

\[
\pi(\alpha, \theta, Y_1, \ldots, Y_m | t) = \pi(\alpha, \theta, z_1, \ldots, z_p | t)
\]

\[
\propto \alpha^{(m-\nu+p)} \nu^{(m+\nu+p-1)} (W_t W_z)^{\alpha-1}
\]

\[
\times \exp \left[ - \left\{ \sum_{i=1}^{m} t_i^\alpha + \sum_{k=1}^{p} z_k^\alpha + \frac{\alpha}{\nu} + \frac{\theta}{\alpha} \right\} \right]
\]

\[
\times \prod_{i=1}^{m} (1 - e^{-t_i^\alpha})^{\theta-1} \prod_{k=1}^{p} (1 - e^{-z_k^\alpha})^{\theta-1}.
\] (3.5)

The full conditional distributions of \(Y_{kR_h}^{[h]}\), \(\alpha\) and \(\theta\) are needed to implement the Gibbs sampling algorithm.

Let \(Y_{h(k)} = (Y_{1R_h}^{[h]}, Y_{2R_h}^{[h]}, \ldots, Y_{k-1R_h}^{[h]}, Y_{k+1R_h}^{[h]}, \ldots, Y_{mR_h}^{[h]})\) be the collection of all the components of \(Y_1, \ldots, Y_m\) excluding the component \(Y_{kR_h}^{[h]}\) in \(Y_h\) for \(1 \leq h \leq m \) and \(1 \leq k \leq R_h\). From the joint posterior distribution in (3.5), the full conditional posterior distribution of \(Y_{kR_h}^{[h]}\) is as follows:

\[
\pi(Y_{kR_h}^{[h]} | \alpha, \theta, t, Y_{h(k)}) = \frac{\alpha \theta y_{kR_h}^{[h-1]} e^{-y_{kR_h}^{[h]}} (1 - e^{-y_{kR_h}^{[h]}})^{\theta-1}}{(1 - e^{-y_{kR_h}^{[h]}})^{\theta} - (1 - e^{-y_{kR_h}^{[h-1]}})^{\theta}},
\] (3.6)

\[
y_{k-1R_h}^{[h]} \equiv y_{k-1} < y_{kR_h}^{[h]} \equiv y_k < y_{k+1R_h}^{[h]} \equiv y_{k+1}.
\]

If \(k = 0\) or \(k = R_h + 1\) then we set \(Y_{0R_h}^{[h]} \equiv t_{h}\) and \(Y_{R_h+1R_h}^{[h]} \equiv \infty\), respectively. The cumulative distribution of \(y_{kR_h}^{[h]}\) is found by

\[
F(Y_{kR_h}^{[h]} | \alpha, \theta, t, Y_{h(k)}) = \frac{(1 - e^{-y_{kR_h}^{[h]}})^{\theta} - (1 - e^{-y_{kR_h}^{[h-1]}})^{\theta}}{(1 - e^{-y_{kR_h}^{[h]}})^{\theta} - (1 - e^{-y_{kR_h}^{[h-1]}})^{\theta}},
\] (3.7)
Generating $y_{k:h}^{[h]}$ from (3.7) can be used the inverse cumulative distribution function transformation method such as below

$$Y_{k:h}^{[h]} = [-\ln \{(1 - U)(1 - e^{-y_{k-1}^o})^\theta + U(1 - e^{-y_{k+1}^o})^\theta\}^{\frac{1}{\theta}}]\alpha,$$

(3.8)

where $U$ is an uniform random variable.

The full conditional distributions of $\alpha$ and $\theta$ are given by respectively

$$\pi(\alpha|\theta, z_1, z_2, \ldots, z_m, t) = D_1\alpha^{(m-\nu+p)}(W_tW_z)^{\alpha-1}\alpha e^{-\left[\sum_{i=1}^{m}t_i^\alpha + \sum_{k=1}^{p}z_k^\alpha+\frac{\alpha}{\theta}+\frac{1}{\theta}\right]}$$

$$\times \prod_{i=1}^{m} (1 - e^{-t_i^\alpha})^{\theta-1} \prod_{k=1}^{p} (1 - e^{-z_k^\alpha})^{\theta-1}$$

(3.9)

and

$$\pi(\theta|\alpha, z_1, z_2, \ldots, z_m, t) = D_2\theta^{(m-\nu+1+p)}e^{-\frac{\theta}{\alpha}}\prod_{i=1}^{m} (1 - e^{-t_i^\alpha})^{\theta-1}$$

$$\times \prod_{k=1}^{p} (1 - e^{-z_k^\alpha})^{\theta-1},$$

(3.10)

where $D_1$ and $D_2$ are normalizing constants for $\pi(\alpha|\theta, z_1, z_2, \ldots, z_m, t)$ and for $\pi(\theta|\alpha, z_1, z_2, \ldots, z_m, t)$, respectively.

For $\alpha$ and $\theta$, they are not generated from their full conditional distributions because they are not the standard distributions. Consequently, Metropolis-Hastings algorithm in Gibbs sampler and the proposal distributions are taken to be the truncated normal distributions which are

$$g_1(\alpha) = N(\hat{\alpha}, Var(\hat{\alpha}))I(\alpha > 0) \quad \text{and} \quad g_2(\theta) = N(\hat{\theta}, Var(\hat{\theta}))I(\theta > 0)$$

(3.11)

where, $\hat{\alpha}$ and $\hat{\theta}$ are the MLE of $\alpha$ and $\theta$, and $Var(\hat{\alpha})$ and $Var(\hat{\theta})$ are the asymptotic variances of the MLE obtained from the observed Fisher information matrix.

4 Numerical studies

In this section, we consider analysis of two data sets (simulated and real data) using the exponentiated Weibull model.

4.1 Simulation study

By applying the algorithm of Balakrishnan and Sandhu (1995) [2], the following steps are used to generate progressively type II censored sample and obtain Bayes estimates of interest from EW distribution.
(a) Generate \(m\) independent random variables \(Z_1, Z_2, \cdots, Z_m\) from \(U(0, 1)\).

(b) Using the pre-fixed progressive censoring schemes \(R_1, R_2, \cdots, R_m\), we set 
\[
Q_i = Z_i^{1/(i+R_m+R_{m-1}+\cdots+R_{m-i+1})}, \quad i = 1, 2, \cdots, m.
\]

(c) Set 
\[
U_i = 1 - Q_m Q_{m-1} \cdots Q_{m-i+1} \quad \text{for } i = 1, 2, \cdots, m.
\]
Then \(U_1, U_2, \cdots, U_m\) be progressively type II censored sample of size \(m\) from \(U(0, 1)\).

(d) Generate \(\alpha\) in (2.3) under a given value of \(b\) and then generate \(\theta\) in (2.4)
given \(\alpha\) and \(\nu\).

(e) For given \(\alpha\) and \(\theta\), generated a progressively type II censored sample of size \(m\) from \(EW\) distribution using the inverse cdf,
\[
t_i = \left[-\ln(1 - U_i^{1/\theta})\right]^{1/\alpha}, \quad i = 1, 2, \cdots, m
\]

(f) Calculate MLE and observed Fisher information for \(\alpha\) and \(\theta\) in order to use these as initial values for Gibbs sampler in (3.11) and also calculate \(y_1^{(0)}, \cdots, y_{m-1}^{(0)}\) and \(y_m^{(0)}\) using the truncated probability density function.

(g) Using the given initial values for \(\alpha, \theta, y_1^{(0)}, \cdots, y_{m-1}^{(0)}\) and \(y_m^{(0)}\), get \(\alpha\) from (3.9) and \(\theta\) from (3.10) as a Gibbs output at the first iteration.

(h) With Gibbs output of \(\alpha^{(1)}\) and \(\theta^{(1)}\) above, obtain \(y_1^{(1)}, \cdots, y_{m-1}^{(1)}\) and \(y_m^{(1)}\) from (3.8).

(i) Repeat this procedure from (g) to (h) until we have \((N + r)\) sample.

We consider two progressively type II censored samples of size \(m = 10\) with different censoring schemes such as \(R_1 = (0, 0, 0, 0, 0, 0, 0, 3, 2)\) and \(R_2 = (0, 0, 2, 0, 2, 0, 0, 0, 3)\). Following the above algorithm with \(\alpha = 1.90\) and \(\theta = 4.49\) as true values, the \(m\) ordered failure times with two progressive type II censoring schemes \(R_1\) and \(R_2\) are presented in Table 1, respectively.

<table>
<thead>
<tr>
<th>Time</th>
<th>(T_1)</th>
<th>(T_2)</th>
<th>(T_3)</th>
<th>(T_4)</th>
<th>(T_5)</th>
<th>(T_6)</th>
<th>(T_7)</th>
<th>(T_8)</th>
<th>(T_9)</th>
<th>(T_{10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_1)</td>
<td>0.918</td>
<td>1.093</td>
<td>1.141</td>
<td>1.167</td>
<td>1.275</td>
<td>1.299</td>
<td>1.333</td>
<td>1.352</td>
<td>1.366</td>
<td>1.934</td>
</tr>
<tr>
<td>(R_2)</td>
<td>0.873</td>
<td>1.065</td>
<td>1.110</td>
<td>1.139</td>
<td>1.253</td>
<td>1.278</td>
<td>1.322</td>
<td>1.349</td>
<td>1.368</td>
<td>1.935</td>
</tr>
</tbody>
</table>

For given above progressively type II censored samples, the MLE and asymptotic variance covariance matrix for \(\alpha\) and \(\theta\) are presented in Table
Bayesian prediction for unobserved data from EW distribution

Table 2: MLE and Asymptotic variance covariance matrix for $\alpha$ and $\theta$

<table>
<thead>
<tr>
<th>Censoring scheme</th>
<th>Parameters</th>
<th>MLE</th>
<th>Asymptotic Variance</th>
<th>Asymptotic Covariance</th>
<th>95% confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R1$</td>
<td>$\hat{\alpha}$</td>
<td>1.741</td>
<td>0.093</td>
<td>0.956</td>
<td>(1.141, 2.340)</td>
</tr>
<tr>
<td></td>
<td>$\hat{\theta}$</td>
<td>5.065</td>
<td>1.996</td>
<td></td>
<td>(2.295, 7.834)</td>
</tr>
<tr>
<td>$R2$</td>
<td>$\hat{\alpha}$</td>
<td>1.750</td>
<td>0.096</td>
<td>0.935</td>
<td>(1.141, 2.359)</td>
</tr>
<tr>
<td></td>
<td>$\hat{\theta}$</td>
<td>5.106</td>
<td>1.818</td>
<td></td>
<td>(2.463, 7.748)</td>
</tr>
</tbody>
</table>

2. It is seen that the 95% confidence intervals for $\alpha$ and $\theta$ contain the true values of the parameters.

To calculate Bayes estimates of $\alpha$, $\theta$ and $\gamma$, a Gibbs and Metropolis-Hastings algorithm were run for 10,000 iterations where 5,000 Gibbs outputs were discarded as a burn-in period. In this case, we used $g_1(\alpha) \equiv N(1.741, 0.093)I(\alpha > 0)$ and $g_2(\theta) \equiv N(5.065, 1.996)I(\theta > 0)$ as a derived function for $\alpha$ and $\theta$, respectively under the censoring scheme $R1$. We finally obtained 5,000 samples of the posterior distribution of each parameter.

Table 3: Posterior summary statistics with $R1 = (0, 0, 0, 0, 0, 0, 0, 3, 2)$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>SD</th>
<th>0.025 quantile</th>
<th>Median</th>
<th>0.975 quantile</th>
<th>95% HPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.737</td>
<td>0.089</td>
<td>1.208</td>
<td>1.737</td>
<td>1.911</td>
<td>(1.555, 1.919)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>5.012</td>
<td>0.416</td>
<td>4.206</td>
<td>5.003</td>
<td>5.857</td>
<td>(4.188, 5.838)</td>
</tr>
<tr>
<td>$Y_{1:2}^{[10]}$</td>
<td>2.242</td>
<td>0.159</td>
<td>2.081</td>
<td>2.193</td>
<td>2.657</td>
<td>(2.077, 2.561)</td>
</tr>
<tr>
<td>$Y_{2:2}^{[10]}$</td>
<td>2.543</td>
<td>0.319</td>
<td>2.136</td>
<td>2.473</td>
<td>3.329</td>
<td>(2.097, 3.166)</td>
</tr>
<tr>
<td>$Z_{1}$</td>
<td>4.785</td>
<td>0.417</td>
<td>4.237</td>
<td>4.697</td>
<td>5.811</td>
<td>(4.182, 5.602)</td>
</tr>
</tbody>
</table>

Table 4: Posterior summary statistics with $R2 = (0, 0, 2, 0, 2, 0, 0, 0, 3)$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>SD</th>
<th>0.025 quantile</th>
<th>Median</th>
<th>0.975 quantile</th>
<th>95% HPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.669</td>
<td>0.166</td>
<td>1.347</td>
<td>1.669</td>
<td>1.998</td>
<td>(1.347, 1.998)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>4.281</td>
<td>0.691</td>
<td>2.988</td>
<td>4.258</td>
<td>5.686</td>
<td>(3.056, 5.748)</td>
</tr>
<tr>
<td>$Y_{1:3}^{[10]}$</td>
<td>2.068</td>
<td>0.130</td>
<td>1.939</td>
<td>2.030</td>
<td>2.430</td>
<td>(1.936, 2.329)</td>
</tr>
<tr>
<td>$Y_{2:3}^{[10]}$</td>
<td>2.256</td>
<td>0.226</td>
<td>1.975</td>
<td>2.206</td>
<td>2.808</td>
<td>(1.945, 2.687)</td>
</tr>
<tr>
<td>$Y_{3:3}^{[10]}$</td>
<td>2.592</td>
<td>0.410</td>
<td>2.062</td>
<td>2.510</td>
<td>3.580</td>
<td>(2.012, 3.399)</td>
</tr>
<tr>
<td>$Z_{2}$</td>
<td>6.917</td>
<td>0.641</td>
<td>6.039</td>
<td>6.796</td>
<td>8.487</td>
<td>(5.929, 8.139)</td>
</tr>
</tbody>
</table>
The resulting posterior summary statistics for $\alpha$, $\theta$, and the unobserved failure time for $Y_{j,R_m}$ at time $t_m$ based on the censoring schemes $R1$ and $R2$ are presented in Table 3 and Table 4, respectively. The total of remaining time in the original sample with each censoring scheme are presented by $Z_i = \sum_{j=1}^{R_m} Y_{j,R_m}$, $i = 1,2$.

From Table 3, it is seen that the estimated posterior means of $\alpha$ and $\theta$ are fairly close to a real value of $\alpha = 1.90$ and $\theta = 4.4944$. Also, the 95% credible HPD intervals for $\alpha$ and $\theta$ contains the true values of the parameters. The lengths of 95% HPD intervals for both are narrower than those of 95% confidence intervals. In addition the unobserved failure time of $Y_{1,2}^{[10]}$ and $Y_{2,2}^{[10]}$ at the failure time $T_{10} = 1.934$ showed us a reliable prediction because both of them are greater than the last failure time $T_{10}$ and its values are also ordered. It is seen that Table 4 gives a similar result.

A graphical summary on MCMC output based on the censoring scheme $R1$ is displayed in Figure 2. The first column depicts the estimated posterior density, time series plots and autocorrelation plots are displayed at the second and the last column, respectively. The posterior density plots for $\alpha$ and $\theta$ spread relatively wider than those of $Y$s. Time series plots also seem to be distributed around posterior mean for each parameter. Autocorrelation plots became exponentially down at the early lags which indicate that the output performs quite well. Figure 2 induces that plots of the parameters of interest seem to be satisfied with convergence.

4.2 Real data analysis

Table 5 gives 100 observations on breaking stress of carbon fibres from Nichols and Padgett (2006) [15]. This data was obtained from a process producing carbon fibers to be used in constructing fibrous composite materials. To pre-
Table 5: Breaking stress of carbon fibers

<table>
<thead>
<tr>
<th></th>
<th>0.39</th>
<th>0.81</th>
<th>0.85</th>
<th>0.98</th>
<th>1.08</th>
<th>1.12</th>
<th>1.17</th>
<th>1.18</th>
<th>1.22</th>
<th>1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.36</td>
<td>1.41</td>
<td>1.47</td>
<td>1.57</td>
<td>1.57</td>
<td>1.59</td>
<td>1.59</td>
<td>1.61</td>
<td>1.61</td>
<td>1.69</td>
<td>1.69</td>
</tr>
<tr>
<td>1.69</td>
<td>1.71</td>
<td>1.73</td>
<td>1.80</td>
<td>1.84</td>
<td>1.84</td>
<td>1.87</td>
<td>1.89</td>
<td>1.92</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>2.03</td>
<td>2.03</td>
<td>2.05</td>
<td>2.12</td>
<td>2.17</td>
<td>2.17</td>
<td>2.17</td>
<td>2.35</td>
<td>2.38</td>
<td>2.41</td>
<td>2.41</td>
</tr>
<tr>
<td>2.43</td>
<td>2.48</td>
<td>2.48</td>
<td>2.50</td>
<td>2.53</td>
<td>2.55</td>
<td>2.55</td>
<td>2.56</td>
<td>2.59</td>
<td>2.67</td>
<td>2.67</td>
</tr>
<tr>
<td>2.73</td>
<td>2.74</td>
<td>2.76</td>
<td>2.77</td>
<td>2.79</td>
<td>2.81</td>
<td>2.81</td>
<td>2.82</td>
<td>2.83</td>
<td>2.85</td>
<td>2.85</td>
</tr>
<tr>
<td>2.87</td>
<td>2.88</td>
<td>2.93</td>
<td>2.95</td>
<td>2.96</td>
<td>2.97</td>
<td>2.97</td>
<td>3.09</td>
<td>3.11</td>
<td>3.11</td>
<td>3.11</td>
</tr>
<tr>
<td>3.15</td>
<td>3.15</td>
<td>3.19</td>
<td>3.19</td>
<td>3.22</td>
<td>3.22</td>
<td>3.27</td>
<td>3.28</td>
<td>3.31</td>
<td>3.31</td>
<td>3.31</td>
</tr>
<tr>
<td>3.33</td>
<td>3.39</td>
<td>3.39</td>
<td>3.51</td>
<td>3.56</td>
<td>3.60</td>
<td>3.65</td>
<td>3.68</td>
<td>3.68</td>
<td>3.68</td>
<td>3.68</td>
</tr>
<tr>
<td>3.70</td>
<td>3.75</td>
<td>4.20</td>
<td>4.38</td>
<td>4.42</td>
<td>4.70</td>
<td>4.90</td>
<td>4.91</td>
<td>5.08</td>
<td>5.56</td>
<td>5.56</td>
</tr>
</tbody>
</table>

dict the unobserved lifetime at last failure time, we consider the following progressive type II censoring scheme.

1. \( R_3 = (0 \ast 9, 8, 0 \ast 8, 2, 0 \ast 8, 5, 0 \ast 8, 5, 0 \ast 8, 3, 0 \ast 8, 2, 0 \ast 8, 2, 0 \ast 5, 3) \),

2. \( R_4 = (0 \ast 4, 10, 0 \ast 4, 2, 0 \ast 3, 5, 0 \ast 4, 3, 0 \ast 4, 4, 0 \ast 3, 6, 0 \ast 8, 7, 0 \ast 6, 8, 0 \ast 5, 5) \),

where the degree of censoring rate was set to be 30% and 50%, respectively and \( (0 \ast 4, 10) \) represents the censoring scheme \( (R_1, R_2, R_3, R_4, R_5) = (0, 0, 0, 0, 10) \) for simplicity.

Table 6: MLE and Asymptotic variance covariance matrix for \( \alpha \) and \( \theta \)

<table>
<thead>
<tr>
<th>Censoring scheme</th>
<th>Parameter</th>
<th>MLE</th>
<th>Asymptotic Variance</th>
<th>Asymptotic Covariance</th>
<th>95% confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_3 )</td>
<td>( \hat{\alpha} )</td>
<td>0.901</td>
<td>0.002</td>
<td>13.764</td>
<td>(0.805, 0.997)</td>
</tr>
<tr>
<td></td>
<td>( \hat{\theta} )</td>
<td>8.010</td>
<td>0.614</td>
<td></td>
<td>(6.173, 9.848)</td>
</tr>
<tr>
<td>( R_4 )</td>
<td>( \hat{\alpha} )</td>
<td>0.826</td>
<td>0.003</td>
<td>13.080</td>
<td>(0.719, 0.934)</td>
</tr>
<tr>
<td></td>
<td>( \hat{\theta} )</td>
<td>8.636</td>
<td>1.173</td>
<td></td>
<td>(6.512, 10.759)</td>
</tr>
</tbody>
</table>

The MLE and asymptotic variance and covariance matrix for \( \alpha \) and \( \theta \) under the censoring scheme of \( R_3 \) and \( R_4 \) are presented in Table 6.

Table 7 and 8 give the posterior summary statistics for parameters of interest under \( R_3 \) and \( R_4 \) censoring scheme, respectively. Under the censoring scheme \( R_3 \), the censoring rate is 30% and the removed observations at the last failure time are three which are \( Y_{[70]}^{[1:3]} = 4.91, Y_{[70]}^{[2:3]} = 5.08 \) and \( Y_{[70]}^{[3:3]} = 5.56 \). Therefore, we observe that the true value of total remaining time \( Z_3 = Y_{[70]}^{[1:3]} + Y_{[70]}^{[2:3]} + Y_{[70]}^{[3:3]} \) is 15.55. From Table 7, the true values of the unobserved lifetimes \( (Y_{[70]}^{[j:3]}, j = 1, 2, 3) \) and its corresponding total remaining time
under R3.

Table 7: Posterior summary statistics with R3

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>SD</th>
<th>0.025 quartile</th>
<th>Median</th>
<th>0.975 quartile</th>
<th>95% HPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.903</td>
<td>0.032</td>
<td>0.840</td>
<td>0.903</td>
<td>0.965</td>
<td>(0.841, 0.965)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>7.647</td>
<td>0.579</td>
<td>6.569</td>
<td>7.635</td>
<td>8.812</td>
<td>(6.536, 8.767)</td>
</tr>
<tr>
<td>$Y_{1:3}^{[70]}$</td>
<td>5.353</td>
<td>0.439</td>
<td>4.912</td>
<td>5.218</td>
<td>6.538</td>
<td>(4.900, 6.249)</td>
</tr>
<tr>
<td>$Y_{2:3}^{[70]}$</td>
<td>6.043</td>
<td>0.828</td>
<td>5.047</td>
<td>5.845</td>
<td>8.217</td>
<td>(4.940, 7.718)</td>
</tr>
<tr>
<td>$Y_{3:3}^{[70]}$</td>
<td>7.386</td>
<td>1.599</td>
<td>5.405</td>
<td>7.028</td>
<td>11.537</td>
<td>(5.102, 10.607)</td>
</tr>
<tr>
<td>$Z_3$</td>
<td>18.782</td>
<td>2.372</td>
<td>15.427</td>
<td>18.288</td>
<td>24.592</td>
<td>(15.171, 23.338)</td>
</tr>
</tbody>
</table>

$Z_3$ are inside the produced 95% HPD region. Under R4, the removed observations at the last failure time, $Y_{j:5}^{[50]}$, $j = 1, \cdots, 5$ are (4.70, 4.90, 4.91, 5.08, 5.56) and its corresponding total remaining time $Z_4$ is 25.15. Table 8 gives a similar result. The estimated posterior means of interest seem to be reasonable because real observations removed from original data were included in 95% HPD region.

Pictorial description for MCMC outputs under censoring scheme of R3 depicts in Figure 3. These Figures show that the estimated posterior kernel density has unimodal and time series plots are concentrated on the posterior mean. Autocorrelation plots display that MCMC outputs are very quickly reached randomness at the early lags which show for all parameters of interest perform fairy well.
Bayesian prediction for unobserved data from EW distribution

Table 8: Posterior summary statistics with \( R4 \)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>SD</th>
<th>0.025 quartile</th>
<th>Median</th>
<th>0.975 quartile</th>
<th>95% HPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.825</td>
<td>0.036</td>
<td>0.754</td>
<td>0.826</td>
<td>0.898</td>
<td>(0.752, 0.894)</td>
</tr>
<tr>
<td>( \theta )</td>
<td>8.066</td>
<td>0.702</td>
<td>6.742</td>
<td>8.058</td>
<td>9.472</td>
<td>(6.706, 9.422)</td>
</tr>
<tr>
<td>( Y_{1.5}^{[50]} )</td>
<td>4.795</td>
<td>0.379</td>
<td>4.430</td>
<td>4.680</td>
<td>5.807</td>
<td>(4.420, 5.542)</td>
</tr>
<tr>
<td>( Y_{2.5}^{[50]} )</td>
<td>5.265</td>
<td>0.616</td>
<td>4.520</td>
<td>5.114</td>
<td>6.884</td>
<td>(4.437, 6.446)</td>
</tr>
<tr>
<td>( Y_{3.5}^{[50]} )</td>
<td>5.859</td>
<td>0.873</td>
<td>4.728</td>
<td>5.670</td>
<td>8.090</td>
<td>(4.544, 7.572)</td>
</tr>
<tr>
<td>( Y_{4.5}^{[50]} )</td>
<td>6.731</td>
<td>1.273</td>
<td>5.019</td>
<td>6.473</td>
<td>9.831</td>
<td>(4.785, 9.204)</td>
</tr>
<tr>
<td>( Y_{5.5}^{[50]} )</td>
<td>8.586</td>
<td>2.390</td>
<td>5.540</td>
<td>8.041</td>
<td>14.769</td>
<td>(5.070, 13.222)</td>
</tr>
<tr>
<td>( Z_{4} )</td>
<td>31.238</td>
<td>4.349</td>
<td>25.104</td>
<td>30.497</td>
<td>41.747</td>
<td>(24.742, 40.326)</td>
</tr>
</tbody>
</table>

Acknowledgements.
This work was funded by the Korea Meteorological Administration Research and Development Program under Grant CATER 2012-3081.

References


**Received: August 16, 2013**