Fractals in G-Metric Spaces

Bhagwati Prasad and Kunti Mishra

Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida, UP-201307, India
b_prasad10@yahoo.com, bhagwati.prasad@jiit.ac.in

Copyright © 2013 Bhagwati Prasad and Kunti Mishra. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. The theory of iterated function systems (IFS) on complete metric spaces appears in almost all fractal based algorithms used for the purpose of compression of the images and their representation as well. Through a simple mathematical model, IFS technique provides an important tool for description and manipulation of the complex fractal attractors. In this paper we study the iterated function systems in G-metric spaces and obtain a collage theorem for the same.

Keywords: Iterated function systems, G-metric space, Hausdorff G-metric, G-contraction

I. INTRODUCTION

The theory of iterated function systems (IFS) was basically born as an application of the theory of discrete dynamical systems. Hutchinson [8] developed the idea of IFS in 1981. However, on the basis of the self-similarity property, this theory was practically developed by Barnsley and Demko [4]. They observed that many objects can be closely approximated by self-similar objects and thus might be generated by some IFS. Barnsley [3] used it as a tool to build fractals [10] and other similar sets. In the recent years, fractals are found to be appreciably suitable for modelling many natural objects such as clouds, trees, mountains etc. The computer modelling of
irregular patterns is a fruitful area of application of the fractal theory (see [1-2], [5-6] and several references therein). The theory of IFS is being used for computer graphics and image processing to generate and describe the images exploring the self similarity property of the fractals. Barnsley initiated the use of IFS in fractal image compression and Jacquin [9] was the first to propose an image coding algorithm. Almost all the applications of IFS use encoding and decoding processes. Encoding is based on construction of an IFS while decoding is the main application of the collage theorem. If we have an IFS, its attractor can be generated very easily using the algorithm given in literature. But to find the IFS for a given attractor was not an easy task in late seventies of twentieth century. The solution to this problem was suggested by Barnsley et al [5] in the form of collage theorems, which provides a way to find the IFS whose attractor is close to or looks like a given set. Thereafter, IFS theory is extended, generalized and enriched in numerous directions in different settings by a number of authors (see, for example [1-2], [14-15], [17] and references thereof). In 2004, Mustafa et al. [11] modified the notion of $D$-metric spaces of Dhage [7] by introducing an appropriate notion of generalized metric space called a $G$-metric space. Our aim is to obtain the attractor (also called a fractal) of an iterated function systems in $G$-metric spaces. A collage theorem is also given in this setting.

II. PRELIMINARIES

**Definition 2.1** [11]. Let $X$ be a nonempty set and $G: X \times X \times X \to R^+$ satisfies the following conditions, for all $x, y, z, a \in X$
(i) $G(x, y, z) = 0$ if $x = y = z$,
(ii) $0 < G(x, x, y)$ with $x \neq y$,
(iii) $G(x, x, y) \leq G(x, y, z)$ with $z \neq y$,
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality).

Then the function $G$ is called a generalized metric or $G$-metric on $X$ and $(X, G)$ is called a $G$-metric space.

**Definition 2.2** [12]. Let $(X, G)$ be a $G$-metric space and $\{x_n\}$ be a sequence of points of $X$, if there is an $M \in N$ (set of natural numbers) for $n, m, l \geq M$, then the sequence $\{x_n\}$
(i) is said to be $G$-convergent to $x$, if $\lim_{n,m \to \infty} G(x_n, x_m) = 0$ and
(ii) is called $G$-Cauchy if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

**Definition 2.3** [12-13]. A $G$-metric space $(X, G)$ is $G$-complete if every $G$-Cauchy sequence in $(X,G)$ is $G$-convergent in $(X, G)$. 
**Definition 2.4.** Let $H(X)$ be the collection of compact subsets of $X$, other than the empty set. We define, Hausdorff $G$-metric $H_g$, for $X, Y, Z \in H(X)$ as

$$H_g(X, Y, Z) = d_g(X, Y) \lor d_g(Y, Z) \lor d_g(Z, X);$$

where, $d_g(X, Y) = \max_{x \in X} \min_{y \in Y} \max_{x \in X} G(x, y, y)$.

The notation $a \lor b$ denotes the maximum of the two real numbers $a$ and $b$.

**Theorem 2.5 [12].** Let $(X, G)$ be a complete $G$-metric space and $T : X \to X$ satisfy any one of the following contractive conditions:

(i) $G(T(x), T(y), T(z)) \leq k \{ G(x, T(x), T(x)) \lor G(y, T(y), T(y)) \lor G(z, T(z), T(z)) \}$

(ii) $G(T(x), T(y), T(z)) \leq k \{ G(x, x, T(x)) \lor G(y, y, T(y)) \lor G(z, z, T(z)) \}$

for all $x, y, z \in X$ where $0 \leq k < 1$, then $T$ has a unique fixed point (say $f$) and $T$ is $G$-continuous at $f$.

Throughout the paper, we call $T$ a $G$-contraction if it satisfies any one of the conditions (i) and (ii).

Now we define a $G$-iterated function system.

**Definition 2.6.** A $G$-iterated function system consists of a complete $G$-metric space $(X, G)$ together with a finite set of $G$-contractions $T_n : X \to X$ with contractivity factor $0 \leq k_n < 1$ for $n = 1, 2, \ldots, N$. It is represented by $\{X, T_n, n = 1, 2, \ldots, N\}$.

### III. MAIN RESULTS

**Lemma 3.1.** Let $(X, G)$ be a complete $G$-metric space and $T : X \to X$ a $G$-contraction with contractivity factor $0 \leq k < 1$. Then $T : H_g(X) \to H_g(X)$ defined by $T(B) = \{T(x) : x \in B\}$ for every $B \in H_g(X)$ will also be a $G$-contraction with same contractivity factor $k$.

**Proof.** Let $U, V, W \in H_g(X)$ then by the definition of a Hausdorff $G$-metric, we have

$$H_g(TU, TV, TW) = d_g(TU, TV) \lor d_g(TV, TW) \lor d_g(TW, TU)$$

Using Bianchini contractive condition (5) of Rhoades [16], we get

$$H_g(TU, TV, TW) \leq k_1 \{ d_g(U, TU) \lor d_g(V, TV) \} \lor k_2 \{ d_g(V, TV) \lor d_g(W, TW) \} \lor k_3 \{ d_g(W, TW) \lor d_g(U, TU) \}$$
Lemma 3.2. Let $X$ be a $G$-metric space and $\{T_n\}$ be sequence of $G$-contractions with the contractivity factor $0 \leq k_n < 1$ for each $T_n$, then $T : H(X) \to H(X)$ defined by

$$T(B) = T_1(B) \cup T_2(B) \cup \ldots T_N(B) = \bigcup_{n=1}^{N} T_n(B), \forall B \in H(X)$$

is also a $G$-contraction with the contractivity factor $k = \max_n \{k_n\}$.

Proof. Let $U, V, W \in H(X)$. For $N = 2$, we have

$$H_g(TU, TV, TW) = H_g(T_1(U), T_1(V), T_1(W)) \leq H_g(T_1(U) \cup T_2(V), T_1(W)) \leq (k_1 \vee k_2)(H_g(U, U, TU) \vee H_g(V, V, TV) \vee H_g(W, W, TW)).$$

Similarly, it can be proved for any natural number $N$.

Lemma 3.3. Let $X$ be a complete $G$-metric space and $T$ be a contraction on $X$ with contractivity factor $0 \leq k < 1$. Then for any natural number $n$, we have

$$G(T^n x, T^{n+1} x) \leq k^n G(x, Tx, Tx).$$

Proof. It follows immediately from [11].

Lemma 3.4. Let $T$ be a contraction with contractivity factor $0 \leq k < 1$ on a complete $G$-metric space $X$. Let fixed point of $T$ be denoted by $f$, then

$$G(x, f, f) \leq \frac{1}{1 - k} G(x, Tx, Tx).$$

Proof. Let $x \in X$, then we have $\lim_{n \to \infty} T^n x = f$. Now, using the continuity of $T$ and Lemma 3.3 we have

$$G(x, f, f) = G\left(\lim_{n \to \infty} T^n x, \lim_{n \to \infty} T^n x\right) = \lim_{n \to \infty} G\left(x, T^n x, T^n x\right) \leq G(x, Tx, Tx) + G(Tx, T^2 x, T^2 x) + \ldots + G(T^{n-1} x, T^n x, T^n x) + \ldots$$

$$\leq G(x, Tx, Tx) + kG(x, Tx, Tx) + \ldots + k^{n-1}G(x, Tx, Tx) + \ldots = (1/1 - k)G(x, Tx, Tx).$$
On the basis of above results, we can easily deduce the following existence theorem in a $G$-metric space.

**Theorem 3.5.** Let $\{X, (T_n), T_1, \ldots T_N\}$ be a $G$-IFS with a contractivity factor $k_n$ for each $T_n$, where $n \in N$. Then the mapping $T : H(X) \to H(X)$, where $T = \bigcup_{n=1}^{N} T_n$ is a contraction on a complete Hausdorff $G$-metric space $(H(X), H_g)$ and $T$ has a unique fixed point (also called an attractor or fractal) $A \in H(X)$. Then we have $A = TA = \bigcup_{n=1}^{N} T_n(A)$ and also $A = \lim_{n \to \infty} T^n(B)$ for any $B \in H(X)$.

The following collage theorem in G-metric spaces is obvious.

**Theorem 3.6.** Let $X$ be a complete $G$-metric space and $T$ be a contraction on $X$, where $T = \bigcup_{n=1}^{N} T_n$. Let $L \in H(X)$ and $\varepsilon > 0$ be given. Choose a $G$-IFS $\{X, (T_0), T_1, \ldots T_N\}$, where $T_0$ is the condensation mapping with contractivity factor $0 \leq k < 1$ so that $H_g(L, \bigcup_{n=1}^{N} T_n(L), \bigcup_{n=1}^{N} T_n(L)) \leq \varepsilon$, then $H_g(L, A, A) \leq \varepsilon / 1 - k$, where $A$ is the attractor of the $G$-IFS. Equivalently,

$$H_g(L, A, A) \leq \frac{1}{1 - k} H_g(L, \bigcup_{n=1}^{N} T_n(L), \bigcup_{n=1}^{N} T_n(L)).$$

We illustrate it by following example.

**Example 3.1.** Let $X = [0, 1]$. The self maps $T_1$, $T_2$ are defined as $T_1 = x / 3$ and $T_2 = (x + 2) / 3$. Let $T = T_1 \cup T_2$, then $T$ is a $G$-contraction. Starting the iteration from any arbitrary set $L = [0, 0.5]$, we obtain

$H_g(L, A, A) = 0.166666666666667$, $H_g(TL, A, A) = 0.055555555555556$, ...

$H_g(T^0L, A, A) = 8.467543904266961 \times 10^{-6}$ and $H_g(L, TL) = 0.833333333333333$

It is clear from the above calculated values that $L$ converges to the attractor of the IFS $\{T_1, T_2; X\}$, which is the well known Cantor set (a fractal). Thus for every $\frac{1}{3} \leq k < 1$, there exists an $\varepsilon > 0$ such that $H_g(L, TL, TL) \leq \varepsilon$ implies $H_g(L, A, A) \leq \varepsilon \frac{1}{1 - k}$.

**References**


Received: April 15, 2013