On Some General Aspects of Forming Fuzzy Concept Lattices

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Abstract

Fuzzy concept lattices represent generalization of the classical concept lattices (FCA). In this paper we describe some algebraic aspects of forming fuzzy concept lattices, based on Galois connections.

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1 Introduction

Formal Concept Analysis (FCA) is a theory of data analysis for identification of conceptual structures among data sets. Mathematical theory of FCA is
based on the notion of concept lattices and is well developed in the monograph of Ganter and Wille [4]. In practice, there are natural examples of object-attribute models where relationships between objects and attributes are represented by many-valued (fuzzy) relations. Therefore several attempts to apply FCA for these situations have been proposed. We mention an approach of Bělohlávek [2] based on the logical framework of complete residuated lattices, an approach of Krajčí [5] which generalizes the previous one, work on multi-adjoint concept lattices [6] and other approaches [1, 3, 7, 8, 9].

The theory of fuzzy concept lattices can be characterized as searching for a Galois connection between powers of complete lattices. In this paper we describe algebraic aspects of forming fuzzy concept lattices. Since concept lattices are closely related to Galois connections between complete lattices, in the preliminary section we give a basic overview of the algebraic notions needed for our purposes. Next we describe our main result, i.e., the general method for constructing Galois connection between the direct products of complete lattices.

2 Preliminary Notes

In this section we give a basic overview of the Galois connection and residuated mappings, cf. [10]. We assume that the reader is familiar with the basic notions of the lattice theory.

**Definition 2.1.** Let \((P, \leq)\) and \((Q, \leq)\) be ordered sets and let \(\varphi: P \to Q\), \(\psi: Q \to P\) be two mappings between these ordered sets. Such a pair \((\varphi, \psi)\) of mappings is called a Galois connection (residuated mappings) between the ordered sets if

\[
\begin{align*}
    p \leq \psi(q) & \iff \varphi(p) \geq q \\
    (p \leq \psi(q)) & \iff \varphi(p) \leq q.
\end{align*}
\]

Galois connections between complete lattices fulfill the following important property, see [10] or [4].

**Lemma 2.2.** A map \(\varphi: L \to M\) between complete lattices \(L\) and \(M\) is a part of Galois connection (has a dual adjoint) if and only if

\[
\varphi\left( \bigvee_{i \in I} x_i \right) = \bigwedge_{i \in I} \varphi(x_i)
\]

holds for any subset \(\{x_i : i \in I\}\) of \(L\).

Note, that in this case the dual adjoint \(\psi\) is uniquely determined by

\[
\psi(y) = \bigvee\{x \in L : \varphi(x) \geq y\}.
\]
Galois connections between complete lattices are closely related to the notion of closure system. Let \( L \) be a complete lattice. A subset \( X \) of the complete lattice \( L \) is called closure system in \( L \) if \( X \) is closed under arbitrary meets. We note that this condition guarantees that \((X, \leq)\) is a complete lattice, in which the infima are the same as in \( L \), but the suprema in \( X \) may not coincide with those from \( L \).

There is a well-known characterization of Galois connections as dually isomorphic closure systems. Let \( L, M \) be complete lattices and \((\varphi, \psi)\) be a Galois connection between \( L \) and \( M \). Then the image \( \psi(M) \) is a closure system in \( L \) and similarly \( \varphi(L) \) is a closure system in \( M \). Moreover these closure systems are dually isomorphic.

Conversely, suppose that \( X_1 \) and \( X_2 \) are closure systems in \( L \) and \( M \) respectively, and \( f : X_1 \to X_2 \) is a dual isomorphism between complete lattice \((X_1, \leq)\) and complete lattice \((X_2, \leq)\). Then a pair \((c_{L_1} \circ f, c_{L_2} \circ f^{-1})\), where \( c_{L_1} \), \( c_{L_2} \) are closure operators corresponding to \( X_1 \) and to \( X_2 \), forms a Galois connection between \( L \) and \( M \). Hence, any Galois connection between complete lattices induces dually isomorphic closure systems on these lattices and vice-versa.

The properties of Galois connections allow us to construct complete lattices commonly known as concept (Galois) lattices. Formally, let \((\varphi, \psi)\) be a Galois connection between complete lattices \( L \) and \( M \). Denote by \( G_{\varphi, \psi} \) a subset of \( L \times M \) consisting of all pairs \((x, y)\) with \( \varphi(x) = y \) and \( \psi(y) = x \). Define a partial order on \( G_{\varphi, \psi} \) as follows:

\[
(x_1, y_1) \leq (x_2, y_2) \quad \text{if} \quad x_1 \leq x_2 \quad \text{or equivalently if} \quad y_1 \geq y_2.
\]

Since the order structure of \( G_{\varphi, \psi} \) is fully determined by first coordinates and any Galois connection determines dually isomorphic closure systems in \( L \) and \( M \) respectively, we obtain the following characterization of the set \( G_{\varphi, \psi} \), cf. [4].

**Theorem 2.3.** Let \((\varphi, \psi)\) be a Galois connection between complete lattices \( L \) and \( M \). Then \((G_{\varphi, \psi}, \leq)\) forms a complete lattice, where

\[
\bigwedge_{i \in I} (x_i, y_i) = \left( \bigwedge_{i \in I} x_i, \varphi \left( \bigvee_{i \in I} y_i \right) \right), \quad \bigvee_{i \in I} (x_i, y_i) = \left( \psi \left( \bigvee_{i \in I} x_i \right), \bigwedge_{i \in I} y_i \right)
\]

for each family \((x_i, y_i)_{i \in I}\) of elements from \( G_{\varphi, \psi} \).

Finally we recall the definition of biresiduated mappings ([10]), which are used as fuzzy logic conjunctors in the theory of fuzzy concept lattices.

**Definition 2.4.** Let \( L, M \) be complete lattices and \( P \) be an ordered set. A mapping \( T : L \times M \to P \) is called biresiduated if for all \( x \in L \) and for all \( y \in M \) the mappings \( T_x : M \to P \) and \( T_y : L \to P \) where \( T_x(y) = T_y(x) = T(x, y) \), are parts of some residuated mappings.
Hence for all \( x \in L \) there is \( T^*_x: P \to M \) such that \((T_x, T^*_x)\) is a residuated mapping between \( M \) and \( P \). Similarly, for all \( y \in M \) there exists \( T^*_y: P \to L \) such that \((T_y, T^*_y)\) is residuated mapping between \( L \) and \( P \). The following diagram describes this situation (let us note that this diagram is not commutative in general).

\[
\begin{array}{ccc}
L & \xrightarrow{i_y} & L \times M \xrightarrow{i_x} M \\
& \searrow & \downarrow T & \swarrow \\
& & T_y & \\
& \searrow & \downarrow & \swarrow \\
& & T^*_y & \\
& \downarrow & & \\
& P & & \end{array}
\]

Let us note, that for all \( x \in L \) and for all \( y \in M \) the mappings \( i_x \) and \( i_y \) denote embeddings given by \( i_x(y) = (x, y) = i_y(x) \).

## 3 Main Results

In this section we describe the general method for constructing Galois connections between direct products of complete lattices. Consequently we illustrate how to define concept forming operators within this framework. First we prove an important result concerning biresiduated mappings.

Let \( L, M \) be complete lattices, \( P \) be a poset and \( T: L \times M \to P \) be a biresiduated mapping. For all \( p \in P \) define a pair of mappings \( \varphi_p: L \to M \) and \( \psi_p: M \to L \) by

\[
\varphi_p(x) = T^*_x(p) \quad \text{and} \quad \psi_p(y) = T^*_y(p). \tag{3}
\]

**Lemma 3.1.** The pair \((\varphi_p, \psi_p)\) forms a Galois connection between \( L \) and \( M \) for all \( p \in P \).

*Proof.* We show that for each \( p \in P \) and for all \( x \in L, y \in M \) the assertion \( x \leq \psi_p(y) \) if and only if \( y \leq \varphi_p(x) \) holds. Since for any \( y \in M \) a pair \((T_y, T^*_y)\) forms residuated mappings between \( L \) and \( P \), due to (2) of 2.1 we have \( x \leq T^*_y(p) \) if and only if \( p \geq T_y(x) \) for all \( x \in L, p \in P \). Similarly \( y \leq T^*_x(p) \) if and only if \( p \geq T_x(y) \) for all \( y \in M, p \in P \). Using this facts we obtain the following equivalent assertions.

\[
x \leq \psi_p(y) = T^*_y(p) \iff p \geq T_y(x) = T(x, y) = T_x(y) \iff y \leq T^*_x(p) = \varphi_p(x).
\]

Hence, for any \( p \in P \) the pair \((\varphi_p, \psi_p)\) satisfies (1) of 2.1 and it forms a Galois connection between \( L \) and \( M \). \(\square\)
Further we give a characterization of Galois connections between direct powers of lattices.

First we recall the definition of direct product of lattices. If \( L_i \) for \( i \in I \) is a family of lattices the direct product \( \prod_{i \in I} L_i \) is defined as the set of all functions \( f : I \to \bigcup_{i \in I} L_i \) such that \( f(i) \in L_i \) for all \( i \in I \) with the “componentwise” order, i.e., \( f \leq g \) if \( f(i) \leq g(i) \) for all \( i \in I \). Also the lattice operations are calculated componentwise, i.e., for any subset \( \{ f_j : j \in J \} \subseteq \prod_{i \in I} L_i \) we obtain

\[
\bigvee_{j \in J} f_j(i) = \bigvee_{j \in J} f_j(i) \quad \text{and} \quad \bigwedge_{j \in J} f_j(i) = \bigwedge_{j \in J} f_j(i),
\]

where these equalities hold for each index \( i \in I \).

Let \( (L_i : i \in I) \) and \( (M_j : j \in J) \) be two systems of complete lattices and \( (\varphi_{i,j}, \psi_{i,j})_{(i,j) \in I \times J} \) be a system of Galois connections such that for all \( i \in I \), \( j \in J \) the pair \( (\varphi_{i,j}, \psi_{i,j}) \) forms a Galois connection between \( L_i \) and \( M_j \).

Now we define a pair of mappings between the direct product of complete lattices \( \prod_{i \in I} L_i \) and \( \prod_{j \in J} M_j \) by

\[
\uparrow(f)(j) = \bigwedge_{i \in I} \varphi_{i,j}(f(i)) \quad \text{for all} \quad j \in J, \quad \text{and} \quad f \in \prod_{i \in I} L_i \tag{4}
\]

\[
\downarrow(g)(i) = \bigvee_{j \in J} \psi_{i,j}(g(j)) \quad \text{for all} \quad i \in I, \quad \text{and} \quad g \in \prod_{j \in J} M_j. \tag{5}
\]

The following theorem characterize the properties of the mappings \( \uparrow \) and \( \downarrow \).

**Theorem 3.2.** The pair \( (\uparrow, \downarrow) \) forms a Galois connection between \( \prod_{i \in I} L_i \) and \( \prod_{j \in J} M_j \). If \( (\Phi, \Psi) \) is a Galois connection between \( \prod_{i \in I} L_i \) and \( \prod_{j \in J} M_j \), then there exists a system \( (\varphi_{i,j}, \psi_{i,j})_{(i,j) \in I \times J} \) such that \( \uparrow = \Phi \) and \( \downarrow = \Psi \).

**Proof.** Let \( f \in \prod_{i \in I} L_i \) and \( g \in \prod_{j \in J} M_j \) be arbitrary elements. We show that \( f \leq \downarrow(g) \) if and only if \( \uparrow(f) \geq g \). From the basic properties of infimum we obtain

\[
f(i) \leq \bigwedge_{j \in J} \psi_{i,j}(g(j)), \; \forall i \in I \quad \text{iff} \quad f(i) \leq \psi_{i,j}(g(j)), \; \forall i \in I, \forall j \in J,
\]

Since each \( (\varphi_{i,j}, \psi_{i,j}) \) forms a Galois connection, this holds if and only if

\[
\varphi_{i,j}(f(i)) \geq g(j), \; \forall i \in I, \forall j \in J \quad \text{iff} \quad \bigwedge_{i \in J} \varphi_{i,j}(f(i)) \geq g(j), \; \forall j \in J.
\]
In order to prove the second statement, denote by \(0^i_x\) an element of \(\prod_{i \in I} L_i\), such that \(0^i_x(i) = x\) and \(0^i_x(i_1) = 0_{L(i)}\) for all \(i_1 \neq i\). Note that \(0_L\) denotes the smallest element of the complete lattice \(L\). Similarly, \(0^j_y\) will denote an element of \(\prod_{j \in J} M_j\) with \(0^j_y(j) = y\) and \(0^j_y(j_1) = 0_{M(j)}\) for \(j_1 \neq j\).

Further we define a system \((\varphi_{i,j}, \psi_{i,j})_{(i,j) \in I \times J}\) by

\[
\varphi_{i,j}(x) = \Phi(0^i_x)(j), \quad \psi_{i,j}(y) = \Psi(0^j_y)(i).
\]

We show that \((\varphi_{i,j}, \psi_{i,j})\) forms a Galois connection between \(L_i\) and \(M_j\). This follows from the following series of equivalent assertions:

\[
x \leq \psi_{i,j}(y) \iff 0^i_x \leq \Psi(0^j_y) \iff 0^j_y \leq \Phi(0^i_x) \iff y \leq \varphi_{i,j}(x).
\]

Further, let \(f \in \prod_{i \in I} L_i\) be an arbitrary element. Then \(f = \bigvee_{i \in I} 0^i_{f(i)}\) and according to Lemma 2.2 we obtain that:

\[
\Phi(f)(j) = \Phi \left( \bigvee_{i \in I} 0^i_{f(i)} \right)(j) = \bigwedge_{i \in I} \Phi(0^i_{f(i)})(j) = \bigwedge_{i \in I} \varphi_{i,j}(f(i)) = \uparrow(f)(j).
\]

This shows that \(\uparrow = \Phi\) and the equality \(\downarrow = \Psi\) can be proved similarly. \(\square\)

Now we are able to describe a formal context and the concept forming operators in the terms of mappings defined by (4) and (5). Formally, a formal context is a triple \((B, A, R)\) where \(B \neq \emptyset\) represents the set of objects, \(A \neq \emptyset\) represents the set of attributes and \(R: B \times A \to P\) represents fuzzy relation (in general \(P\) is a poset). The triple \((B, A, R)\) formalizes the notion of object-attribute model, where \(R(b, a)\) represents the value of attribute \(a\) for the object \(b\). Objects are evaluated in the \(L\)-fuzzy sets represented by \(L^B\). In fact \(L^B\) represents the direct power of the lattice \(L\) and \(L^B = \prod_{b \in B} L\). Attributes are evaluated in the \(M\)-fuzzy sets represented by \(M^B\). Let us note that based on Theorem 3.2, concept lattices from different types of contexts can be defined too, cf. [7] or [9].

Concept forming operators are defined by usage of some class of biresiduated mappings. In fact for each entry of the object-attribute model, there can be defined some biresiduated mapping \(b^a T: L \times M \to P\) for all \(b \in B, a \in A\). In this case each biresiduated mapping \(b^a T\) represents the product in fuzzy logic. Now concept forming operators \(\uparrow: L^B \to M^A\) and \(\downarrow: M^A \to L^B\) are defined as follows:

\[
f^\uparrow(a) = \bigwedge_{b \in B} b^a \varphi_{R(b,a)}(f(b)) = \bigwedge_{b \in B} b^a T^*_f(R(b,a)) \tag{6}
\]

\[
g^\downarrow(b) = \bigwedge_{a \in A} b^a \psi_{R(b,a)}(g(a)) = \bigwedge_{a \in A} b^a T^*_g(R(b,a)). \tag{7}
\]
According to Lemma 3.1 and Theorem 3.2 operators $\uparrow$ and $\downarrow$ forms a Galois connection between fuzzy subsets $L^B$ and $M^A$. The corresponding set $G_{\uparrow, \downarrow} \subseteq L^B \times M^A$ of fixed points forms a complete lattice, usually called fuzzy concept lattice, cf. Theorem 2.3.

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