Computing Performance Measures of Fuzzy Non-Preemptive Priority Queues Using Robust Ranking Technique

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Abstract

This paper presents a procedure to find the various performance measures in terms of crisp values for fuzzy non-preemptive priority queues with 3-priorities. Here, the inter-arrival time and service time are trapezoidal as well as triangular fuzzy numbers. The basic idea is to convert the fuzzy inter-arrival and service rates into crisp values by applying robust ranking technique. Then, apply the crisp values in the traditional queuing performance measure formulas. Ranking fuzzy numbers is an important aspect of decision making in a fuzzy environment. This ranking technique is a systematic procedure, easy to apply and can be utilized for all types of queuing problems. An illustration is given to establish the performance measures for 3-priority queues.

Keywords: Fuzzy sets (normal and convex), Trapezoidal fuzzy number, Triangular fuzzy number, fuzzy ranking, Membership functions, Non-preemptive Priority Queues, Performance measures.

1. Introduction

Most of the queuing models studied in the last three decades had the service discipline that units proceed to service on a first come- first served basis. This is obviously not only the manner of services and there are many alternatives such as last come-first served, selection in random order and selection by priority. In priority schemes, customers with the highest priority are selected for service ahead of those with lower priority, independent of their time of arrival into the system. There are two
further refinements possible in priority situation, namely preemption and non-preemption. In preemptive cases the customer with the highest priority is allowed to enter service immediately even if another with lower priority is already present in service when the higher customer arrives to the system.

In addition to that a decision has to be made whether to continue the preempted customer’s service from point of preemption when resumed or to start a new. The priority discipline is said to be non-preemptive if there is no interruption and the highest priority customer just goes to the head of the queue to wait his turn. In practical, the priority queuing model, the input data arrival rate, service rate are uncertainly known. Uncertainty is resolved by using fuzzy set theory. Hence the classical queuing model with non-preemptive priority discipline will have more application if it is expanded using fuzzy models. Fuzzy non-preemptive priority queues has been described by Devaraj and Jayalakshmi [1]. Here, fuzzy problem has been converted into crisp problem by Robust ranking technique. Fuzzy queuing models have been described by such researchers like Li and Lee [2], Negi and Lee [3], Kao et al [4]. Chen [5,6] have analyzed fuzzy queues using Zadeh’s extension principle [7]. Ranking technique has been analyzed by such researchers like F.Choobinesh and H.Li[8], R.R.Yager[9], S.H.Chen[10] and Buckley [11]. W.Ritha and L.Robert have analyzed Fuzzy Queues with Priority Discipline[12]

2. Preliminaries


2.1 Definition
A fuzzy set is characterized by a membership function mapping elements of a domain space or universe of discourse X to the unit interval [0,1]. (i.e) A = {(x, \( \mu_A(x) \)); x \in X}, Here \( \mu_A: X \rightarrow [0,1] \) is a mapping called the degree of membership function of the fuzzy set A and \( \mu_A(x) \) is called the membership value of x \in X in the fuzzy set A. These membership grades are often represented by real numbers ranging from [0,1].

2.2 Definition
A fuzzy set A of the universe of discourse X is called a normal fuzzy set implying that there exist at least one x \in X such that \( \mu_A(x) = 1 \).

2.3 Definition
The fuzzy set A is convex if and only if, for any \( x_1, x_2 \in X \), membership function of A satisfies the inequality \( \mu_A((1-\lambda)x_1 + \lambda x_2) \geq \min \{\mu_A(x_1), \mu_A(x_2)\}, 0 \leq \lambda \leq 1 \).

2.4 Definition (Trapezoidal fuzzy number):
For a trapezoidal number \( A(x) \), it can be represented by \( A(a,b,c,d;1) \) with membership function \( \mu(x) \) given by
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\[ \mu(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{d-x}{d-c}, & c \leq x \leq d \\ 0, & \text{otherwise} \end{cases} \]

2.5 Definition: (Triangular fuzzy number):
For a triangular number \( A(x) \), it can be represented by \( A(a,b,c;1) \) with membership function \( \mu(x) \) given by

\[ \mu(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases} \]

2.6 Definition: (\( \alpha \)-cut of a fuzzy number)
The \( \alpha \)-cut of a fuzzy number \( A(x) \) is defined as

\[ A(\alpha) = \{ x : \mu(x) \geq \alpha, \alpha \in [0,1] \} \]

Addition of two Trapezoidal fuzzy numbers can be performed as

\[ (a_1,b_1,c_1,d_1) + (a_2,b_2,c_2,d_2) = (a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2). \]

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3. Non-preemptive systems with many classes
Suppose that customers of the \( k^{th} \) priority (the smaller the number, the higher the priority) arrive at a single channel queue according to a poisson process with \( \lambda_k \) \((k = 1,2,3,...,r)\) and that these customers wait on a first-come, first-served within their respective priorities. Let the service distribution for the \( k^{th} \) priority be exponentially with mean \( 1/\mu_k \) unit that begins service and completes its service before another item is admitted, regardless of priorities. We begin \( \rho_k = \frac{\lambda_k}{\mu_k} (1 \leq k \leq r), \sigma_k = \sum_{i=1}^{r} \rho_{i}, \sigma_{\text{total}} = \rho \) The system is stationary for \( \sigma_{\text{total}} < 1 \). Let \( \mu(x) = \mu(y) \) are membership functions of arrival rate and service rate respectively.

Without loss of generality let us assume that the performance measures of interest for 3-priority queues. From the knowledge of traditional queueing theory under the steady-state conditions \( \rho_k = \frac{\lambda_k}{\mu_k} < 1 \), the expected queue size is

\[ L_q = \sum_{i=1}^{r} \frac{\lambda_i^{(i)}}{(1-\sigma_i)^2} = \sum_{i=1}^{r} \frac{\lambda_i}{\sigma_i} \]

where \( \lambda_i \) and \( \mu_i \) denote the universal set of arrival rate and service rate respectively.
and by using Little’s formula, the waiting time in queue

\[ W_q = \frac{\sum_{i=1}^n \lambda w_q^{(i)}}{\lambda} \]

where

\[ w_q^{(i)} = \frac{\sum_{k=i}^n \lambda_k}{(1-\sigma_{i-1})(1-\sigma_i)} \]

### 3.1 Robust Ranking Technique – Algorithm

For solving the problem we defuzzify the fuzzy numbers into crisp ones by a fuzzy number ranking method. Robust ranking technique [14] which satisfies compensation, linearity, and additive properties and provides results which are consistent with human perceptions. Give a convex fuzzy number \( \tilde{a} \), the Robust ranking index is defined by

\[ R(\tilde{a}) = \int_0^1 0.5(a_{L\alpha} + a_{U\alpha}) d\alpha \]

Where \( (a_{L\alpha}, a_{U\alpha}) \) is the \( \alpha \)–level cut of the fuzzy number \( \tilde{a} \).

In this paper we use this method for ranking the fuzzy numbers. The Robust ranking index \( R(\tilde{a}) \) gives the representative value of the fuzzy number \( \tilde{a} \). It satisfies the linearity and additive property.

### 4. Mathematical formulation

Consider a single server FM/FM/1 queuing system with 3-priority queues. The inter arrival times \( A_i \), \( i=1,2,3 \) of units in the first, second and third priority queues and service time \( S \) are approximately known and are represented by the following fuzzy sets.

\[ A_i = \{(x, \mu_{A_i}(x))/x \in X \}, \]

\[ S = \{(y, \mu_{S}(y))/y \in Y \}, \]

Where X and Y are crisp universal sets of the inter arrival time and inter service time respectively and \( \mu_{A_i}(x) \) and \( \mu_{S}(y) \) are the respective membership functions.

The \( \alpha \)-cut of \( A_i(\alpha) = \{ x \in X/\mu_{A_i}(x) \geq \alpha \}, i=1, 2, 3 \) and \( S(\alpha) = \{ y \in Y/\mu_{S}(y) \geq \alpha \}. \)

Where \( 0<\alpha\leq1 \). All \( A_i(\alpha) \), \( i=1, 2, 3 \) and \( S(\alpha) \) are the crisp sets. Using \( \alpha \)-cut, the inter arrival times and service time can be represented by different levels of confidence intervals \([0,1]\).

Hence a fuzzy queue can be reduced to a family of crisp queues with different \( \alpha \)-cuts \( \{ A_i(\alpha)/0<\alpha\leq1 \}, i=1, 2, 3 \) and \( \{ S(\alpha)/0<\alpha\leq1 \} \).

These two sets represent sets of movable boundaries and they form nested structure [Zimmermann] for expressing the relationship between the crisp sets and fuzzy sets.
Let the confidence intervals of the fuzzy sets $A_i(a), i=1,2,3$ and $S(a)$ be $[A_i(a), u(A_i(a))], i=1,2,3$ and $[S(a), u(S(a))]$ respectively.

Using the concept of $\alpha$-cut the FM/FM/1 queue with 3-priority queues can be reduced to M/M/1 queue with 3-priority customers with equal service rates.

\[ \rho_1 = \frac{\lambda_1}{\mu}, \rho_2 = \frac{\lambda_2}{\mu}, \rho_3 = \frac{\lambda_3}{\mu} \]

Further $\rho_1 = \frac{\lambda_1 + \lambda_2 + \lambda_3}{\mu}, \rho = \lambda / \mu, \lambda = \lambda_1 + \lambda_2 + \lambda_3, \rho = \frac{1}{\rho_1 + \lambda_2 + \lambda_3}, \sigma = \sum_{i=1}^{n} \rho_i, \sigma_0 = 0$

\[ W_q^{(i)} = \frac{(\rho_1 + \rho_2 + \rho_3)}{\mu (1-\sigma_1 - \sigma_2)} \]

From which we can deduce that

\[ W_q^{(1)} = \frac{\lambda}{\mu (\rho - \lambda)} \]
\[ W_q^{(2)} = \frac{\lambda}{(\mu - \lambda_1)(\mu - \lambda_2)} \]
\[ W_q^{(3)} = \frac{\lambda}{(\mu - \lambda)(\mu - \lambda_2)} \]

\[ L_q^{(1)} = \frac{(\rho_1 + \rho_2 + \rho_3)}{\mu (1-\lambda_1)} \]
\[ L_q^{(2)} = \frac{(\rho_1 + \rho_2 + \rho_3)}{\mu (1-\lambda_2)} \]
\[ L_q^{(3)} = \frac{(\rho_1 + \rho_2 + \rho_3)}{\mu (1-\lambda_3)} \]

Where $\lambda_1, \lambda_2, \lambda_3$ are the arrival rates of first, second and third priority units respectively and $\mu$ is the service rate.

5. Numerical example

Expected waiting time and expected number of customer in the queue for FM/FM/1 queue with 3-priority classes.

Example-1 (For Trapezoidal Fuzzy Number)

Suppose that the rates of first, second and third priority with the same service rates are Trapezoidal fuzzy numbers represented by $A_1 = [2,3,5,6], A_2 = [3,4,6,7], A_3 = [4,5,7,8]$ and $S = [22,23,25,26]$ per hour respectively. The $\alpha$-cut of the membership functions $\mu_{A_1}(a), \mu_{A_2}(a), \mu_{A_3}(a)$ and $\mu_S(a)$ are $[2+\alpha, .6-\alpha], [3+\alpha, .7-\alpha], [4+\alpha, .8-\alpha]$ and $[22+\alpha, .26-\alpha]$ respectively.
Now we calculate $R(2,3,5,6)$ by applying Robust ranking method. The membership function of the Trapezoidal fuzzy number $(2, 3, 5, 6)$ is

$$
\mu(x) = \begin{cases} 
\frac{x-2}{1}, & 2 \leq x \leq 3 \\
1, & 3 \leq x \leq 5 \\
\frac{6-x}{1}, & 5 \leq x \leq 6 \\
0, & \text{otherwise}
\end{cases}
$$

The $\alpha$-cut of the fuzzy number $(2, 3, 5, 6)$ is $(\alpha, (\alpha + 2.6 - \alpha))$ for which

$$
R(\alpha) = R(2,3,5,6) = \int_0^1 0.5(\alpha, (\alpha + 2.6 - \alpha)) d\alpha = 4
$$

Proceeding similarly, the Robust ranking indices for the fuzzy numbers $\tilde{A}_2, \tilde{A}_3, \tilde{A}_5$ are calculated as:

$$
R(\tilde{A}_2) = 5, R(\tilde{A}_3) = 6, R(\tilde{A}_5) = 24
$$

From traditional queuing theory formulas

Average waiting time of units of first priority in the queue is

$$
W_q(1) = \frac{\lambda}{\mu} = 0.0312
$$

Average waiting time of units of second priority in the queue is

$$
W_q(2) = \frac{\lambda}{(\mu - \lambda_2)(\mu - (\lambda_1 + \lambda_2))} = 0.05
$$

Average waiting time of units of third priority in the queue is

$$
W_q(3) = \frac{\lambda}{(\mu - \lambda_3)(\mu - (\lambda_1 + \lambda_2))} = 0.1111
$$

Average queue length of first priority is

$$
L_q(1) = \frac{(3+1+3+3) \lambda}{(1+\lambda_1 + \lambda_2)} = 0.125
$$

Average queue length of second priority is

$$
L_q(2) = \frac{(3+1+3+3) \lambda^2}{(1+\lambda_1 + \lambda_2)^2} = 0.25
$$

Average queue length of third priority is

$$
L_q(3) = \frac{(3+1+3+3) \lambda^3}{(1+\lambda_1 + \lambda_2)^3} = 0.666
$$

Example-2 (For Triangular Fuzzy Number)

Suppose that the rates of first, second and third priority with the same service rates are triangular fuzzy numbers represented by $\tilde{A}_1 = [2,5,6], \tilde{A}_2 = [3,6,7], \tilde{A}_3 = [4,7,8]$ and $\tilde{A}_5 = [22,25,26]$ per hour respectively. The $\alpha$-cut of the membership functions

$$
\mu_{\tilde{A}_1}(\alpha), \mu_{\tilde{A}_2}(\alpha), \mu_{\tilde{A}_3}(\alpha), \mu_{\tilde{A}_5}(\alpha)
$$

are $[2+3\alpha, 6-\alpha], [3+3\alpha, 7-\alpha], [4+3\alpha, 8-\alpha]$ and $[22+3\alpha, 26-\alpha]$ respectively.

Now we calculate $R(2,5,6)$ by applying Robust ranking method.

The membership function of the Triangular fuzzy number $(2, 5, 6)$ is
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\[ \mu(x) = \begin{cases} 
\frac{x-2}{3}, & 2 \leq x \leq 5 \\
1, & x = 5 \\
\frac{6-x}{5}, & 5 \leq x \leq 6 \\
0, & \text{otherwise} 
\end{cases} \]

The \( \alpha \)–cut of the fuzzy number \((2, 5, 6,\alpha, \lambda, \mu)\) is \((\alpha_{a}, \alpha_{a}^{L}, \alpha_{a}^{U})\) for which

\[ R(\bar{A}_1) = R(2, 5, 6) = \int_{0}^{1} 0.5(\alpha_{a}^{L}, \alpha_{a}^{U}) \, d\alpha = \int_{0}^{1} 0.5(3\alpha + 2 + 6 - \alpha) \, d\alpha = 0.5 \left[ 8\alpha + \alpha^2 \right]_{0}^{1} = 4.5 \]

Proceeding similarly, the Robust ranking indices for the fuzzy numbers \( \bar{A}_2, \bar{A}_3, \bar{S} \) are calculated as:

\[ R(\bar{A}_2) = 5.5, \quad R(\bar{A}_3) = 6.5, \quad R(\bar{S}) = 24.5 \]

Hence \( \lambda_1 = 4.5, \lambda_2 = 5.5, \lambda_3 = 6.5, \mu = 24.5, \lambda = 16.5 \)

From traditional queuing theory formulas

Average waiting time of units of first priority in the queue is

\[ w_q^{(1)} = \frac{\lambda_1}{\mu(\mu - \lambda_2)} = 0.0336 \]

Average waiting time of units of second priority in the queue is

\[ w_q^{(2)} = \frac{\lambda_2}{\mu - \lambda_1(\mu - (\lambda_1 + \lambda_2))} = 0.0824 \]

Average waiting time of units of third priority in the queue is

\[ w_q^{(3)} = \frac{\lambda_3}{\mu - (\mu - (\lambda_1 + \lambda_2))} = 0.1422 \]

Average queue length of first priority is

\[ L_q^{(1)} = \frac{\lambda_1}{\mu(1 - \lambda_2)} = 0.1514 \]

Average queue length of second priority is

\[ L_q^{(2)} = \frac{\lambda_2}{\mu - (1 - \lambda_1)(1 - \lambda_2)} = 0.4535 \]

Average queue length of third priority is

\[ L_q^{(3)} = \frac{\lambda_3}{\mu - (1 - \lambda_1)(1 - \lambda_2)} = 0.9244 \]

6. Conclusion

In this paper, Fuzzy set theory has been applied to 3-priority queues. The arrival rate and service rates are described by fuzzy numbers which are more realistic and general in nature. Moreover, the fuzzy problem has been converted into crisp problem using
Robust ranking indices. Since the performance measures such as the system length, the waiting time are crisp values, the management can take the best and optimum decisions. One can conclude that the solution of fuzzy problems can be obtained by Robust ranking method effectively. The approach proposed in this paper provides practical information for system manager and practitioners.

References


Received: July 9, 2013