Mathematical Model of Electromagnetic Response from a Positively Skewed Bulge Ground Structure

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Abstract

The purpose of this paper is to study the structure of the earth by constructing a mathematical model of electromagnetic response of a half-space earth with a positively skewed bulge conductivity. Taylor-series expansion and approximation are used to find the solution of the homogeneous differential equation. The Hankel transform is introduced to the problems and analytical result is achieved. The responses of electric field from the ground surface show some subject matter to the variations of conductivity and are compelled by the conductivity ground profile.

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1 Introduction

There are many methods that can be used in the exploration geophysics to obtain the information about the structure of the earth beneath the ground surface. The direct samplings is inadequate because it is quite costly and invasive such as drilling boreholes and trenching. The electromagnetic methods have been found in mining geophysics to search for conductive ore body and occasionally used in geophysical exploration because it is far less expensive than most other investigation methods and the depth of exploration can probe the near surface to as deep as thousand kilometers depending on frequency and also on the purpose of the exploration. Lee and Ignetik [7] considered the forward problem of the transient electromagnetic response of a half-space with an exponentially varying conductivity profiles. They pointed out that the conductivity variation of the ground may sometimes be reasonably approximated by an exponential variation since soil salinity profiles frequently show monotonically increasing or decreasing salt conductivity of the ground. Yooyuanyong and Siew [10] gave the mathematical model of electromagnetic response of a disk beneath an exponentially varying conductive overburden. Ketchanwit [5] used Lee and Ignetik’s assumption to model the transient electromagnetic response of a two-layered earth.

In this paper, we present a mathematical model and techniques for studying the structure of the earth’s surface layers. We consider the ground having the conductivity which is given by

\[ \sigma(z) = (\sigma_0 + z)\exp(-\frac{bz}{2}), \]

where \( \sigma_0 \) is a positive constant, \( b \) is constant. The conductivity ground profile in this paper is different from the models used by Ketchanwit [5] and Lee and Ignetik [7].

2 Formulation and Solution of the Problem

We start off by considering a cylindrical polar coordinates \((r, \theta, z)\) which is introduced with \( z > 0 \) and is taken vertically positive downwards. In addition, the origin under the center of the horizontal circular loop is used. The azimuthally symmetry yields merely electric field \( E_\theta \), and magnetic fields \( H_r \) and \( H_z \) components. Following Morrison et al., [8], these field quantities are established to satisfy Maxwell’s equations (Hohmann and Raiche, [3]) in the form of equations

\[ i\omega \mu H_r = -\frac{\partial E}{\partial z}, \]

\[ i\omega \mu H_z = \frac{1}{r}\frac{\partial (rE)}{\partial r}, \]
Mathematical model of electromagnetic response

and

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = (\sigma(z) - i\omega\varepsilon)E + J_s, \quad (3)$$

where $i = \sqrt{-1}$ is an imaginary number, $J_s = aI(\omega)\delta(r - a)\delta(z + h)/r$ is the source current density, $\omega$ is the angular frequency, $\delta$ is the delta function, $\sigma(z)$ is the electrical conductivity of medium depending on depth only, $\varepsilon$ is the electric permittivity of medium, $\mu$ is the magnetic permeability of medium, and $I(\omega)$ is the current in a coil of a small radius $a$. Eliminating $H_r$ and $H_z$ from above equations lead to get the differential equation for electric fields as.

$$i\omega\mu J_s = -\frac{\partial^2 E}{\partial z^2} - \frac{\partial^2 E}{\partial r^2} - \frac{1}{r} \frac{\partial E}{\partial r} + \frac{E}{r^2} - (i\omega\mu\sigma(z) + \omega^2\mu\varepsilon)E. \quad (4)$$

By applying Hankel transform which is defined as

$$\tilde{E}(\lambda, z, \omega) = \int_0^\infty r J_1(\lambda r) E(r, z, \omega) dr,$$

where $J_1$ is Bessel function of the first kind of order 1, and equation (4) becomes

$$-\frac{\partial^2 \tilde{E}}{\partial z^2} + [\lambda^2 - (i\omega\mu\sigma(z) + \omega^2\mu\varepsilon)]\tilde{E} = i\omega\mu aI(\omega)\delta(z + h)J_1(\lambda a). \quad (5)$$

We next consider the ground having a positively skewed bulge conductivity profile with depth denoted by $\sigma(z) = (\sigma_0 + z)\exp(-\frac{b^2}{2})$, where $\sigma_0$ is a positive constant, $b$ is constant. We consider a primary alternating source current carried by a coil of radius $a$, at $z = -h$ above the surface of the earth $z = 0$, (see figure 1).

Figure 1: Illustration of a half-space model with a positively skewed bulge conductivity ground profile.
The electric field in air can be denoted by \( \widetilde{E}_{\text{air}}(\lambda, z, \omega) \) and expressed as the sum of two components,

\[
\widetilde{E}_{\text{air}}(\lambda, z, \omega) = \widetilde{E}_{\text{air}}^p(\lambda, z, \omega) + \widetilde{E}_{\text{air}}^s(\lambda, z, \omega),
\]

where \( \widetilde{E}_{\text{air}}^p(\lambda, z, \omega) \) is the primary field, and \( \widetilde{E}_{\text{air}}^s(\lambda, z, \omega) \) is the secondary field. Both electric fields can be obtained from equation (5). That is,

\[
\widetilde{E}_{\text{air}}^p(\lambda, z, \omega) = B_1 \exp(-\lambda z) + B_2 \exp(\lambda z) - \frac{\beta_a}{2\lambda} \int_{-\infty}^{0} G(z, x) dx,
\]

where \( \beta_a = -i\omega\mu_0aI(\omega)J_1(\lambda a) \)

\[
G(z, x) = \exp(-\lambda|z - x|)\delta(x + h) = \begin{cases} 
\exp(-\lambda(z - x)) & ; z \geq x \\
\exp(\lambda(z - x)) & ; z < x
\end{cases}
\]

When \( z \to -\infty \), \( \widetilde{E}_{\text{air}}^p \to 0 \), therefore, we require \( B_1 = 0 \). In addition, we obtain the secondary field as

\[
\widetilde{E}_{\text{air}}^s(\lambda, z, \omega) = B_3 \exp(-\lambda z) + B_4 \exp(\lambda z),
\]

In order to obtain \( \widetilde{E}_{\text{air}}^s \to 0 \), we force \( B_3 = 0 \) as \( z \to -\infty \). In air, \( \sigma_{\text{air}}(z) \cong 0 \), and the electric field is given by

\[
\widetilde{E}_{\text{air}}(\lambda, z, \omega) = \frac{i\omega\mu_0aI(\omega)J_1(\lambda a)\exp(-\lambda|z + h|)}{2\lambda} + C \exp(\lambda z), \quad z \leq 0 \quad (6)
\]

which remains bounded as \( z \to -\infty \) where \( C = B_2 + B_4 \) is arbitrary constant to be determined.

From equation (4), we can derive the partial differential equation for the electric field in ground as

\[
\frac{\partial^2}{\partial z^2} E_g(r, z, \omega) + \frac{\partial^2}{\partial r^2} E_g(r, z, \omega) + \frac{1}{r} \frac{\partial}{\partial r} E_g(r, z, \omega)
- \frac{1}{r^2} E_g(r, z, \omega) + k_g^2 E_g(r, z, \omega) = 0. \quad (7)
\]

where \( E_g(r, z, \omega) \) is the electric field in ground, \( k_g^2 = i\omega\mu_g\sigma(z) + \omega^2\mu_g\varepsilon_g \), \( \mu_g \) is the magnetic permeability of ground, \( \varepsilon_g \) is the electric permittivity of ground and \( \sigma(z) = (\sigma_0 + z)\exp(-\frac{h}{z}) \) is the conductivity of ground. Taking Hankel transform to equation (7), we obtain

\[
\frac{\partial^2}{\partial z^2} \tilde{E}_g(\lambda, z, \omega) - (\lambda^2 - k_g^2) \tilde{E}_g(\lambda, z, \omega) = 0, \quad (8)
\]
Equation (8) is a homogeneous differential equation, it can be solved by first using Taylor-series expansion to write

\[
\exp\left(-\frac{bz}{2}\right) = 1 - \frac{bz}{2} + \frac{b^2 z^2}{4 \cdot 2!} - \frac{b^3 z^3}{8 \cdot 3!} + \frac{b^4 z^4}{16 \cdot 4!} - \ldots, \tag{9}
\]

And by Taylor-series approximation, we obtain the solution of equation (8) as

\[
\tilde{E}_g(\lambda, z, \omega) = C_1 \exp\left(-\left(\lambda^2 - i\omega \mu_g \sigma_0 + b\right) z\right) + C_2 \left[\left(\lambda^2 - i\omega \mu_g \sigma_0 + b + 1\right) z + \left(\lambda^2 - i\omega \mu_g \sigma_0 - (\lambda^2 - i\omega \mu_g \sigma_0 + b)\right) \frac{z^2}{2}\right], \tag{10}
\]

where \(C_1\) and \(C_2\) are arbitrary constants to be determined from the boundary conditions. Under the condition \(z \to \infty\), we have \(\tilde{E}_g(\lambda, z, \omega) \to 0\). Hence, we require \(C_2 = 0\). Therefore, equation (10) becomes

\[
\tilde{E}_g(\lambda, z, \omega) = C_1 \exp\left(-\left(\lambda^2 - i\omega \mu_g \sigma_0 + b\right) z\right), \tag{11}
\]

We can find constants \(C\) from equation (6) and \(C_1\) from equation (11) by imposing the continuity of \(\tilde{E}(\lambda, z, \omega)\) and \(\frac{\partial}{\partial z} \tilde{E}(\lambda, z, \omega)\) at air-earth interface by setting \(z = 0\). Consequently, we obtain the electric field in air as

\[
\tilde{E}_{air}(\lambda, z, \omega) = \frac{i \omega \mu_0 I(\omega) a J_1(\lambda a) \exp(-\lambda|z + h|)}{2\lambda} + \left[\frac{2\lambda}{(\lambda + \lambda^2 - i\omega \mu_g \sigma_0 + b) - 1}\right] \eta_a \exp(\lambda z), \quad z \leq 0, \tag{12}
\]

The electric field in ground as

\[
\tilde{E}_g(\lambda, z, \omega) = \frac{2\lambda \eta_a}{(\lambda + \lambda^2 - i\omega \mu_g \sigma_0 + b)} \exp\left(-\left(\lambda^2 - i\omega \mu_g \sigma_0 + b\right) z\right), \quad z \geq 0, \tag{13}
\]

where \(\eta_a = \frac{i \omega \mu_0 I(\omega) a J_1(\lambda a) \exp(-\lambda|h|)}{2\lambda}\).

Applying the inverse Hankel transform to equations (12) and (13), we obtain
the electric field in the air as
\[
E_{\text{air}}(r, z, \omega) = \int_0^\infty \lambda \tilde{E}_{\text{air}}(\lambda, z, \omega) J_1(\lambda r) d\lambda,
\]
\[
\frac{i \omega \mu_0 a I(\omega)}{2} \int_0^\infty [\exp(-\lambda|z + h|) - \exp(\lambda(z - |h|))] J_1(\lambda a) J_1(\lambda r) d\lambda
\]
\[
+ i \omega \mu_0 a I(\omega) \int_0^\infty \left[ \frac{\lambda}{(\lambda + \lambda^2 - i \omega \mu_g \sigma_0 + b)} \right] J_1(\lambda a) J_1(\lambda r) \exp(\lambda(z - |h|)) d\lambda,
\] (14)

and the electric field in ground as
\[
E_g(r, z, \omega) = \int_0^\infty \lambda \tilde{E}_g(\lambda, z, \omega) J_1(\lambda r) d\lambda,
\]
\[
= i \omega \mu_0 a I(\omega) \int_0^\infty \left[ \frac{\lambda}{(\lambda + \lambda^2 - i \omega \mu_g \sigma_0 + b)} \right] J_1(\lambda a)
\]
\[
\times J_1(\lambda r) \exp(-\lambda^2 - i \omega \mu_g \sigma_0 + b) z - \lambda |h|) d\lambda,
\] (15)

The electric field on the ground surface can be determined from equation (14) or equation (15). Hence, we have
\[
E_{\text{sur}}(r, z, \omega) = i \omega \mu_0 a I(\omega) \int_0^\infty \left[ \frac{\lambda}{(\lambda + \lambda^2 - i \omega \mu_g \sigma_0 + b)} \right] J_1(\lambda a)
\]
\[
\times J_1(\lambda r) \exp(-\lambda^2 - i \omega \mu_g \sigma_0 + b) z - \lambda |h|) d\lambda.
\] (16)

By considering \(z = 0\) to equation (16), we derive the electric field \(E_{\text{sur}}(r, 0, \omega)\) on the ground surface, and Chave’s algorithm [2] is used for numerical calculating the inverse Hankel transform of the electric field solutions.

3 Conclusion and Future Work

The objective of this paper is to present a mathematical model and techniques needed for studying the structures of the Earth. We formulated the problem to get the electric fields which could be used to find the electric fields on the ground surface. We consider a half-space model having a positively skewed
bulge conductivity profile which is denoted by \( \sigma(z) = (\sigma_0 + z)\exp\left(-\frac{b z^2}{2}\right) \), where \( \sigma_0 \) is a positive constant, \( b \) is constant. Taylor-series expansion and approximation are applied to find the solution of a homogeneous differential equation. The graphs are shown the behavior of the electric fields against source-receiver spacing \( r \) at different depths. In the experiments, we set the frequency 50 Hz and fix the value of \( \sigma_0 \) while values of \( b \) are varied. In addition, fixing value of \( b \) while values of \( \sigma_0 \) are varied. We increase the frequency up to 100 Hz. The curves of electric fields are plotted separately in real and imaginary parts. The curves from both parts are oscillated first (see figures 2 - 7), then tend to zero. The result is caused by the conductivity and by setting the parameters. We also see that as the distance \( z \) increases, the electric fields get smaller and approach to zero which is resemble to a reality. The deeper the distance from ground surface, the smaller the electric field. The inversion process will be studied in the near future work to find out the optimal values for all the parameters. Besides, more complicated layered structures can be conducted and new problems should be studied by means of formulated equations.

![Graph of real part of electric field](image)

Figure 2: Graph of real part of electric field \( E \) versus \( r \) for a half-space model with a positively skewed bulge conductivity ground profile, \( \sigma_0 = 0.01 \text{ S m}^{-1}, \)
\( b_1 = 0.1 \text{ m}^{-1}, b_2 = 0.2 \text{ m}^{-1}, b_3 = 0.3 \text{ m}^{-1}, \) frequency = 50 Hz.
Figure 3: Graph of imaginary part of electric field $E$ versus $r$ for a half-space model with a positively skewed bulge conductivity ground profile, $\sigma_0 = 0.01 \, S\, m^{-1}$, $b_1 = 0.1 \, m^{-1}$, $b_2 = 0.2 \, m^{-1}$, $b_3 = 0.3 \, m^{-1}$, frequency = 50 Hz.

Figure 4: Graph of real part of electric field $E$ versus $r$ for a half-space model with a positively skewed bulge conductivity ground profile, $b = 0.1 \, m^{-1}$, $\sigma_0 = 0.01 \, S\, m^{-1}$, $\sigma_0 = 0.02 \, S\, m^{-1}$, $\sigma_0 = 0.03 \, S\, m^{-1}$, frequency = 50 Hz.
Figure 5: Graph of imaginary part of electric field $E$ versus $r$ for a half-space model with a positively skewed bulge conductivity ground profile, $b = 0.1 \text{ m}^{-1}$, $\sigma_0 = 0.01 \text{ S m}^{-1}$, $\sigma_0 = 0.02 \text{ S m}^{-1}$, $\sigma_0 = 0.03 \text{ S m}^{-1}$, frequency = 50 Hz.

Figure 6: Graph of real part of electric field $E$ versus $r$ for a half-space model with a positively skewed bulge conductivity ground profile, $\sigma_0 = 0.01 \text{ S m}^{-1}$, $b_1 = 0.1 \text{ m}^{-1}$, $b_2 = 0.2 \text{ m}^{-1}$, $b_3 = 0.3 \text{ m}^{-1}$, frequency = 100 Hz.
Figure 7: Graph of imaginary part of electric field $E$ versus $r$ for a half-space model with a positively skewed bulge conductivity ground profile, $\sigma_0 = 0.01 \text{ S m}^{-1}$, $b_1 = 0.1 \text{ m}^{-1}$, $b_2 = 0.2 \text{ m}^{-1}$, $b_3 = 0.3 \text{ m}^{-1}$, frequency = 100 Hz.

References


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