Comparison of Numerical Method for Forward and Backward Time Centered Space for Long - Term Simulation of Shoreline Evolution

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Abstract

Mathematical modeling of shoreline evolution becomes a useful engineering technique for investigating and predicting the evolution of the plan view of the sandy beach. In this paper two numerical schemes for the shoreline evolution in the long-term scale are presented, and comparisons to analytical solution for some cases are presented for a satisfactory level of suitable numerical scheme. Analytical solutions of shoreline evolution for simple configuration are presented under idealized wave condition. Based on the result that backward time centered space scheme more suitable than forward time centered space scheme to simulate shoreline evolution in the long-term scale.

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Keyword: mathematical modeling, shoreline evolution, long-term scale, backward time centered space, forward time centered space
Introduction

Three time scales of shoreline evolution which can be distinguished are geological evolution over hundreds and thousands of years, long-term evolution from year to year or decades and short-term or seasonal evolution during a major storm. For the problem under consideration, long-term evolution is the primary importance because the long-term evolution is water wave or wave-generated currents. Three phenomena intervene in the action which wave has on shoreline evolution are erosion of beach material by short period seas versus accretion by longer period swells, effect of water level changes on erosion and effect of coastal structures [10].

In order to investigate of beach behavior is needed qualitative understanding of idealized shoreline response to the governing process. Analytical solution originating from a mathematical model which describes the basic physics is the one tool to understanding it. The analytical solutions are often valuable for giving qualitative insight and understanding the properties of shoreline change in the long-term scale. The analytical solution cannot be expected to provide quantitatively accurate solutions to problems involving complex boundary conditions and wave inputs. In the real situation, a numerical model of shoreline evolution would be more appropriate.

Many authors obtained an analytical solution to shoreline evolution by using a simple mathematical formula. The one-line theory was introduced by many authors, several contributors in the analytical solution of shoreline evolution include Grijm[5, 6], Bakker and Edelman [3], Bakker [4], Le Mahute and Soldate [9], Walton and Chiu [13], and Larson et al. [8]. Two numerical schemes of shoreline evolution for simplified configuration beach are examined and presented in this paper.

Fundamental assumptions of the model

In the one-line model, the beach profile is assumed to move landward and seaward while retaining the same shape, implying that all bottom contours are parallel. Consequently, under this assumption it is sufficient to specify the horizontal location of the profile with respect to baseline, and one contour line can be used to describe changes in the beach plan shape and volume as the beach erodes and accretes. The major assumption of the model is the sand is transported alongshore between two well-defined limiting elevations on the profile. One contribution to the volume change results if there is a difference in the alongshore sand transport rate at the lateral sides of the section and the associated sand continuity. The principles of mass conservation must apply to the system at all times. By considering above definitions, the following differential equation for shoreline evolution is obtained:

$$\frac{\partial y}{\partial t} = \frac{1}{(D_b + D_c)} \left( - \frac{\partial Q}{\partial x} \right)$$

(1)
where $x$ is the alongshore coordinate (m); $y$ is the shoreline positions (m) and perpendicular to $x$-axis; $t$ is time (s); $Q$ is the long-shore sand transport rate (m$^3$/s); $D_B$ is the average berm height (m) and $D_C$ is the closure depth (m).

In order to solve the equation (1), necessary to specify an expression for the long-shore sand transport rate, $Q$. This quantity is considered to be generated by the wave obliquely incident to the shoreline. A general expression for the long-shore sand transport rate was developed by the US Army Corp [12]:

$$ Q = Q_o \sin(2\alpha_b) $$

(2)

where $Q_o$ is the amplitude of the long-shore sand transport rate. The empirical predictive formula for the amplitude of the long-shore sand transport rate is[7]:

$$ Q_o = \frac{\rho}{16} (H^2 c_{sw}) \frac{K}{(\rho_s - \rho)(1 - n)} $$

(3)

where the subscript $b$ denotes value at the point of breaking, $c_s$ is the wave group velocity, $H$ is the wave height, $\rho_s$ is the density of the sediment (kg/m$^3$), $\rho$ is the density of the sea water, $n$ is the porosity and $K$ is the dimensionless coefficient which is a function of particle size. The quantity $\alpha_b$ is the angle between breaking wave crest and local shoreline, and may be written as:

$$ \alpha_b = \alpha_o - \tan^{-1}\left(\frac{\partial y}{\partial x}\right) $$

(4)

where $\alpha_o$ is the angle between breaking wave crests and the $x$-axis.

For beaches with mild slope, it can be assumed that breaking wave angle to the shoreline is small. In this case, $\sin(2\alpha_b) = 2\alpha_b$ and $\tan^{-1}\left(\frac{\partial y}{\partial x}\right)$. Substituting equation (4) into the equation (2), and assuming the beach with mild slope, yields:

$$ Q = Q_o \left(2\alpha_b - 2\frac{\partial y}{\partial x}\right) $$

(5)

Substituting equation (5) into the equation (1) and neglecting the sources or sinks along the coast gives:

$$ \frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} $$

(6)

where $D = \frac{2Q_o}{D_B + D_C}$. Equation (6) is analogous to the one-dimensional heat diffusion equation, it can be solved analytically for various initial and boundary conditions.

**Numerical Scheme**

For practical problem, the equation of shoreline evolution and boundary condition cannot normally be simplified sufficiently for the analytical solution to be valid. In that case, the equation of shoreline evolution must be solved numerically.
Numerical schemes that use in this paper are Forward Time Centered Space (FTCS) and Backward Time Centered Space (BTCS). FTCS is the numerical scheme uses finite difference technique and is stepped forward in time using increments of time interval [1, 14]. The information used in forming the finite difference quotient in FTCS comes from above of grid point \((i, j)\); that is, it uses \(y_{i, j+1}\) as well as \(y_{i, j}\). No information to the bottom of \((i, j)\) is used (see Figure 1a). While BTCS is the numerical scheme uses finite difference technique and is stepped backward in time using increments of time interval [1, 2]. The information used in forming the finite difference quotient in FTCS comes from bottom of grid point \((i, j)\); that is, it uses \(y_{i, j-1}\) as well as \(y_{i, j}\). No information for the above of \((i, j)\) is used (see Figure 1b). Both of the FTCS and BTCS use the finite difference quotient of space comes from both sides of the grid point located at \((i, j)\); that is, it uses \(y_{i, j+1}\) as well as \(y_{i, j-1}\). Grid point \((i, j)\) falls between the two adjacent grid points [1].

Finite difference expression for the FTCS equation (6) can be written as:

\[
y_{i, j+1} = y_{i, j} + \frac{\Delta t}{\Delta x^2} \left( y_{i+1, j} - 2y_{i, j} + y_{i-1, j} \right)
\]  

(7)

Finite difference expression for the BTCS equation (6) can be written as:

\[
y_{i, j} = \frac{\Delta t}{\Delta x^2} \left( y_{i+1, j} - 2y_{i, j} + y_{i-1, j} \right) = y_{i, j-1}
\]  

(8)

To calculate the solution of \(y\) is needed initial and boundary condition. The initial condition is the value at all grid points at time level \(j\). For FTCS scheme to obtain values \(y\) at time level \(j + 1\) are calculated from the known value at the time level \(j\). When these calculations are finished, the values \(y\) at time level \(j + 1\) are obtained. This calculation will repeat until the final time level of simulation. While For BTCS scheme, the initial condition is the value at all grid points at time level \(j - 1\). Equation (8) represents one equation with three unknown values \(y\) at...
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Time level \( j \), namely, \( y_{i+1,j} \), \( y_{i,j} \) and \( y_{i-1,j} \). Hence, equation (8) applied at a given grid point \( i \) does not stand alone; it cannot by itself result in a solution \( y_{i,j} \). Rather equation (8) must be written at all interior grid points, resulting in system of algebraic equation from which unknowns \( y_{i,j} \) for all \( i \) can be solved simultaneously. This method is usually involved with the manipulation algebra of large matrices.

Result and discussion

In order to investigate the shoreline evolution in the long-term scale. The numerical results of the different beach situation are considered and the solution the idealized problem is presented. During all these simulations, the value of the constant \( D = 500.000 \text{ m/year} \). Parameter. \( D \) depends on the wave climate and beach material and has the role of diffusion coefficient [11].

Problem 1. Straight Impermeable groin

For this problem, the initials of beach is parallel to the \( x \)-axis with the same breaking wave angle \( \alpha_0 \) existing everywhere (see Figure 2), thus leading to uniform sand transport rate along the beach. At time \( t = 0 \) a thin groin is instantaneously place at \( x = 0 \) and block all transport. Mathematically, this boundary condition can be formulated as follows[8]:

\[
\frac{\partial y}{\partial x} = \tan \alpha_0 \quad \text{at} \quad x = 0
\]  

This equation states that the shoreline at the groin is instant parallel to the wave crests. A groin interrupts the transport of the sand alongshore, cause an accumulation on the up-drift side and erosion on the down-drift side.

The analytical solution describing the accumulation part on up-drift side of the groin is:

\[
y(x,t) = \tan \alpha_0 \sqrt{\frac{4Dt}{\pi}} \left( e^{-\frac{x^2}{4Dt}} - \frac{x\sqrt{\pi}}{2\sqrt{Dt}} \text{erfc} \left( \frac{x}{2\sqrt{Dt}} \right) \right)
\]  

(10)
In this case, $\alpha_0$ was set to $0.02^\circ$, since the boundary condition at the groin which is totally blocking the transport of sand alongshore so that long-shore sediment transport rate taken to be $Q = 0$. The shoreline changes calculated by using the analytical solution equation (10) and numerical solutions by using FTCS scheme and BTCS scheme are shown in Figure 3. The comparison between the analytical and numerical solutions is only investigated on the up-drift side, since the analytical solution on the down-drift side has not been considered in this paper.

In case of time durations are equal to 1 year and 5 years, the FTCS scheme and BTCS scheme produce an almost identical shoreline to the analytical (see Figure 3a and Figure 3b). This implies that these numerical scheme can handle these situations. When the time duration is increased to be 15 years, the FTCS scheme cannot handle the solution. It is becoming unstable. The solution becomes unstable and periodic (see Figure 3c). While the BTCS scheme still can handle simulation until 100 years (see Figure 3c and Figure 3d).

![Figure 3. Shoreline evolution of interruption from straight impermeable groin.](image)

Problem 2. Rectangular cut in beach
This problem represents an excavation or natural employment of rectangular shape. The initial conditions for rectangular cut in a beach are formulated as\([13]\):

$$\begin{align*}
y(x, t) &= \begin{cases} 
y_0, & |x| \geq a \\
0, & |x| < a
\end{cases} \quad (11)
\end{align*}$$

The analytical solution is

$$y(x, t) = \frac{y_0}{2} \left[ \text{erfc} \left( \frac{a - x}{2\sqrt{Dt}} \right) + \text{erfc} \left( \frac{a + x}{2\sqrt{Dt}} \right) \right] \quad (12)$$
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for \( t > 0 \) and \(-\infty < x < \infty\).

This situation is the inverse problem of the rectangular beach fill [8], so that this situation can be used to evaluate the rate of infilling of certain volumetric percentage of sand. In this problem, \( y_0 \) was set to 30 m and \( a \) was set to 2500 m, since this problem doesn’t block the transport of sand alongshore so that the boundary condition is specified on the initial and final \( x \)-axis that depend from Eq. 12 (see Figure 4).

The FTCS scheme and BTCS scheme produce an almost identical shoreline to the analytical during simulated until 4 years (see Figure 5a and Figure 5b). When the time duration is increased to be 15 years, the FTCS scheme cannot handle the solution. It is becoming unstable and grow lead into oscillation (see Figure 5c). While the BTCS scheme still can handle simulation until 100 years (see Figure 5c and Figure 5d).

Figure 4. Initial Shoreline with configuration a rectangular cut in an infinite beach.

Figure 5. Shoreline evolution of a rectangular cut in an infinite beach.
Conclusion

The comparison between the analytical solution and numerical solution for two different shoreline situations under idealized wave condition are discussed. The obtained result for the first case show that the FTCS and the BTCS scheme can handle numerical solution for straight impermeable groin fill finely until 14 years. When the time duration increased for the FTCS scheme cannot handle numerical solution for this problem, while the BTCS scheme still can handle numerical solution until long-term duration. The obtained result for the second case shows that the FTCS and the BTCS scheme can handle numerical solution for rectangular cut in beach finely until 14 years. Similarly the first problem, when time duration increased for the FTCS scheme cannot handle numerical solution for this problem. While the BTCS scheme still can handle numerical solution until long-term duration. These results imply that the BTCS scheme more suitable than FTCS to simulate shoreline evolution in the long-term scale.

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