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Solving System of Nonlinear Equations by Restarted Adomian's method

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Abstract

In this paper, we apply the restarted Adomian decomposition method, based on the standard Adomian decomposition method (ADM), for solving the system of nonlinear equations. Illustrative examples have been presented to demonstrate the method and the obtained results are compared with those derived from the standard Adomian decomposition method.

Keywords: Adomian decomposition method; Restarted Adomian method; System of nonlinear equations

1 Introduction

Since the beginning of the 1980s, the Adomian decomposition method has been applied to a wide class of functional equations [1, 2, 3]. Adomian gives the solution as an infinite series usually converging to an accurate solution. K. Abboui and Y. Cherrualt applied the ADM to solve the equation $f(x) = 0$, where $f(x)$ is a nonlinear function [4]. E. Babolian et al [5] applied the ADM to solve the system of nonlinear equations. The restarted Adomian decomposition method (RADM), based on the standard Adomian introduced by E. Babolian et al [6] for algebraic equations. H. Sadeghi et al [7] applied the RADM for solving the system of nonlinear Volterra integral equations.

The purpose of this paper is to introduce the RADM for solving the system of nonlinear equations. We will show by some examples that the convergence rate of this method is more accelerate than that of the ADM.

The present paper has been organized as follows. Section 2 deals with the analysis of the ADM applied to a system of nonlinear equations. In section 3,

we introduce the RADM for the system of nonlinear equations. The RADM and the ADM with illustrative examples have been compared in section 4. Conclusions are summarized in section 5.

2 The ADM for system of nonlinear equations

Consider the following system of nonlinear equations

$$f_i(x_1, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, n, \tag{1}$$

where $f_i : \mathbf{R}^n \to \mathbf{R}$. Eqs. (1) can be written in the following form

$$x_i = c_i + N_i(x_1, \ldots, x_n), \quad i = 1, 2, \ldots, n, \tag{2}$$

where c_is are constants and N_is are in general nonlinear functions of their arguments. The standard ADM [2] yields the solution x_i in terms of the series

$$x_i = \sum_{j=0}^{\infty} x_{i,j} \ , \quad i = 1, 2, \ldots, n, \tag{3}$$

and nonlinear functions N_is are expressed in terms of an infinite series of Adomian's polynomials

$$N_i(x_1, \ldots, x_n) = \sum_{j=0}^{\infty} A_{i,j} \ , \quad i = 1, 2, \ldots, n, \tag{4}$$

where $A_{i,j}$s depend upon $x_{1,0}, x_{1,1}, \ldots, x_{1,j}, x_{2,0}, x_{2,1}, \ldots, x_{2,j}, \ldots, x_{n,1}, \ldots, x_{n,j}$. In view of Eqs. (3) and (4)

$$N_i(\sum_{j=0}^{\infty} x_{1,j}\lambda^j, \ldots, \sum_{j=0}^{\infty} x_{n,j}\lambda^j) = \sum_{j=0}^{\infty} A_{i,j}\lambda^j \ , \quad i = 1, 2, \ldots, n, \tag{5}$$

which yields

$$A_{ij} = \frac{1}{j!}[\frac{d^j}{d\lambda^j} N_i(\sum_{j=0}^{\infty} x_{1,j}\lambda^j, \ldots, \sum_{j=0}^{\infty} x_{n,j}\lambda^j)]_{\lambda=0} \ , \quad i = 1, 2, \ldots, n, \tag{6}$$

where λ is a parameter introduced for convenience. Hence Eqs. (2) can be written as

$$\sum_{j=0}^{\infty} x_{i,j} = c_i + \sum_{j=0}^{\infty} A_{i,j} \ , \quad i = 1, 2, \ldots, n. \tag{7}$$

The ADM defines the components $x_{i,j}$, for $i = 1, 2, \ldots, n$ and $j \geq 0$, by the following recursion relation

$$\begin{aligned} &x_{i,0} = c_i, \\ &x_{i,j+1} = A_{i,j}, \quad j = 0, 1, \ldots. \end{aligned} \tag{8}$$

We approximate the solution x_i by the truncated series

$$\varphi_{i,k} = \sum_{j=0}^{k-1} x_{i,j}, \quad \text{with} \quad \lim_{k\to\infty} \varphi_{i,k} = x_i \ , \quad i = 1, 2, \ldots \tag{9}$$

In computing x_i, as n increases, the number of terms in the expression for $A_{i,n}$ increases and this causes the propagation of round off errors, on the other hand, the factor $\frac{1}{n!}$ in the formula of $A_{i,n}$ makes it very small. Considering this, we introduce a new algorithm based on the Adomian method to improve the accuracy dramatically.

3 The RADM for system of nonlinear equations

In this section, the RADM will be applied to the system of nonlinear equations. Consider the system of nonlinear (2) with the exact solution $X^* = (x_1^*, \ldots, x_n^*)$. Let s_i be a point close to x_i^* such that

$$|x_i^* - s_i| < |x_i^* - c_i|, \quad i = 1, 2, \ldots, n. \tag{10}$$

We add s_i to both sides of (2). Then

$$x_i - N_i + s_i = c_i + s_i, \quad i = 1, 2, \ldots, n. \tag{11}$$

Now we can solve Eq. (11) with the ADM instead of Eq. (2).

$$\begin{aligned}
&x_{i,0} = s_i \ , && i = 1, 2, \ldots, n,\\
&x_{i,1} = c_i - s_i + A_{i,0}(x_{1,0}, \ldots, x_{n,0}), && i = 1, 2, \ldots, n,\\
&x_{i,2} = A_{i,1}(x_{1,0}, \ldots, x_{n,0}, x_{1,1}, \ldots, x_{1,1}, \ldots, x_{n,1}), && i = 1, 2, \ldots, n,\\
&\vdots\\
&x_{i,k+1} = A_{i,k}(x_{1,0}, \ldots, x_{n,0}, x_{1,1}, \ldots, x_{n,1}, \ldots, x_{n,1}, \ldots, x_{n,k}), && i = 1, 2, \ldots, n,
\end{aligned} \tag{12}$$

and we introduce the following algorithm.

3.1 RADM algorithm

Step 1. Choose small natural numbers k, m.
Step 2. Apply the ADM on Eq. (2) and calculate $x_{i,0}, x_{i,1}, \ldots, x_{i,k}$ for $i = 1, 2, \ldots, n$.
Step 3. Set $\varphi_i^{(1)} = x_{i,0} + x_{i,1} + \cdots + x_{i,k}$.
Step 4. For $j = 2 : m$ do

$$
\begin{aligned}
& s_i = \varphi_i^{j-1}, && i = 1, 2, \ldots, n \\
& x_{i,0} = s_i, && i = 1, 2, \ldots, n \\
& x_{i,1} = c_i - s_i + A_{i,0}(x_{1,0}, \ldots, x_{n,0}), && i = 1, 2, \ldots, n \\
& x_{i,2} = A_{i,1}(x_{1,0}, \ldots, x_{n,0}, x_{1,1}, \ldots, x_{n,1}), && i = 1, 2, \ldots, n \\
& \vdots \\
& x_{i,k+1} = A_{i,k}(x_{1,0}, \ldots, x_{n,0}, x_{1,1}, \ldots, x_{n,1}, \ldots, x_{n,1}, \ldots, x_{n,k}), && i = 1, 2, \ldots, n \\
& \text{Set } \varphi_i^{(j)} = x_{i,0} + x_{i,1} + \cdots + x_{i,k}\ , \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

end of For.

When k increases, the number of terms in the expression for $A_{i,k}$ increases and this causes the propagation of round of errors. In this algorithm, we update $x_{i,0},\ i = 1, 2, \ldots, n$ in each step. Therefore, we choose m, k small i.e consider $2 \leq m, k \leq 5$.

3.2 Numerical examples

In this section, a example is solved by applying the RADM and the results are compared with those of the ADM. Mathematica 5 is used to carry computations.

Example 1. Consider the following system of nonlinear equations

$$
\begin{cases}
3x_1 - cos(x_2x_3) - \frac{1}{2} = 0, \\
x_1^2 - 81(x_2 + 0.1)^2 + sinx_3 + 1.06 = 0, \\
e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.
\end{cases}
$$

The exact solutions are $X^* = (x_1^*, x_2^*, x_3^*)^t = (0.5, 0, -\frac{\pi}{6})^t$. For applying the ADM, we get

$$
\begin{aligned}
x_1 &= \frac{1}{6} + \frac{1}{3}cos(x_2x_3) \\
x_2 &= \frac{0.25}{16.2} + \frac{1}{16.2}x_1^2 - \frac{81}{16.2}x_2^2 + \frac{1}{16.2}sinx_3 \\
x_3 &= -\frac{10\pi - 3}{60} - \frac{1}{20}e^{-x_1x_2}
\end{aligned}
$$

By using Eq. (3), we have

$$
\begin{aligned}
\sum_{j=0}^{\infty} x_{1,j} &= \frac{1}{6} + \frac{1}{3}\sum_{j=0}^{\infty} A_{1,j}(cos(x_2x_3)) \\
\sum_{j=0}^{\infty} x_{2,j} &= \frac{0.25}{16.2} + \frac{1}{16.2}\sum_{j=0}^{\infty} A_{2,j}(x_1^2) - \frac{81}{16.2}\sum_{j=0}^{\infty} A_{2,j}(x_2^2) + \frac{1}{16.2}\sum_{j=0}^{\infty} A_{2,j}(sin(x_3))
\end{aligned}
$$

$$\sum_{j=0}^{\infty} x_{3,j} = -\frac{10\pi - 3}{60} - \frac{1}{20}\sum_{j=0}^{\infty} A_{3,j}(e^{(-x_1x_2)})$$

Let $x_{1,0} = \frac{1}{6}$, $x_{2,0} = \frac{0.25}{16.2}$ and $x_{3,0} = -\frac{10\pi-3}{60}$. The Adomian polynomials for the nonlinear term x_i^n are given by (6) easily. The first few Adomian polynomials for the nonlinear term $sinx_3$ are formulated as

$$A_0 = sinx_{3,0}$$

$$A_1 = x_{3,1}cosx_{3,0}$$

$$A_2 = -\frac{1}{2!}x_{3,1}^2 sinx_{3,0} + x_{3,2}cosx_{3,0}$$

$$A_3 = -\frac{1}{3!}x_{3,1}^3 cosx_{3,0} - x_{3,2}x_{3,1}sinx_{3,0} + x_{3,3}cosx_{3,0}$$

and the first few Adomian polynomials for the nonlinear term $cos(x_2x_3)$ are formulated as

$$A_0 = cos(x_{2,0}x_{3,0})$$

$$A_1 = -(x_{3,0}x_{2,1} + x_{2,0}x_{3,1})sin(x_{2,0}x_{3,0})$$

$$A_2 = (\frac{-1}{2!}x_{2,1}^2x_{3,0}^2 - \frac{1}{2!}x_{3,1}^2x_{2,0}^2 + x_{2,0}x_{3,0}x_{2,1}x_{3,1})cos(x_{2,0}x_{3,0})$$

$$-(x_{2,1}x_{3,1} + x_{2,2}x_{3,0} + x_{3,2}x_{2,0})sin(x_{2,0}x_{3,0}),$$

$$A_3 = (\frac{1}{3!}x_{3,0}^2x_{2,1}^3 + \frac{1}{3!}x_{2,0}^3x_{3,1}^3 + \frac{1}{2!}x_{1,1}x_{2,1}^2x_{3,0}^2x_{2,0} + \frac{1}{2!}x_{2,1}^2x_{3,1}x_{2,0}x_{3,0}^2$$

$$+\frac{1}{2!}x_{1,1}x_{3,1}^2x_{2,0}^2x_{3,0} - x_{3,1}x_{2,2} - x_{2,1}x_{3,2} - x_{2,3}x_{3,0} - x_{3,3}x_{2,0})sin(x_{2,0}x_{3,0})$$

$$-(x_{1,1}x_{2,1}^2x_{3,0} + x_{2,1}^2x_{3,1}x_{3,0} + x_{1,1}x_{3,1}^2x_{2,0} + x_{3,1}x_{3,2}x_{2,0}^2$$

$$-x_{3,1}x_{2,2}x_{3,0}x_{2,0} + x_{2,1}x_{3,2}x_{2,0}x_{3,0})cos(x_{2,0}x_{3,0}),$$

and the first few Adomian polynomials for the nonlinear term $e^{-x_1x_2}$ are formulated as

$$A_0 = e^{-x_{1,0}x_{2,0}}$$

$$A_1 = -(x_{2,0}x_{1,1} + x_{2,1}x_{1,3})e^{-x_{1,0}x_{2,0}}$$

$$A_2 = (\frac{1}{2!}x_{1,1}^2x_{2,0}^2 + \frac{1}{2!}x_{2,1}^2x_{1,0}^2 - x_{1,1}x_{2,1} + x_{1,1}x_{2,1}x_{1,0}x_{2,0}$$

$$-x_{1,2}x_{2,0} - x_{2,2}x_{1,0})e^{-x_{1,0}x_{2,0}}$$

$$A_3 = (-\frac{1}{3!}x_{1,1}^3x_{2,0}^3 - \frac{1}{3!}x_{2,1}^3x_{1,0}^3 + x_{1,1}^2x_{2,1}x_{2,0} - \frac{1}{2!}x_{1,1}^2x_{2,1}x_{1,0}x_{2,0}x_{2,0}^2$$

$$+x_{1,1}x_{1,2}x_{2,0}^2 + x_{2,2}x_{1,0}^2x_{2,1} - x_{2,1}x_{1,2} + x_{2,1}x_{1,2}x_{1,0}x_{2,0} - x_{1,1}x_{2,2}$$

$$+x_{1,1}x_{2,2}x_{1,0}x_{2,0} - x_{1,3}x_{2,0} - x_{2,3}x_{1,0}))e^{-x_{1,0}x_{2,0}}$$

By applying the ADM and calculating seven terms of series, we have

$$x_1 \simeq \varphi_{1,7} = x_{1,0} + x_{1,1} + \ldots + x_{1,6} = 0.500002$$

$$x_2 \simeq \varphi_{2,7} = x_{2,0} + x_{2,1} + \ldots + x_{2,6} = 0.000264$$

$$x_3 \simeq \varphi_{3,7} = x_{3,0} + x_{3,1} + \ldots + x_{3,6} = -0.52360$$

Therefore,

$$ESA_1 = |x_1^* - \varphi_{1,7}| = 1.69904 \times 10^{-6}$$

$$ESA_2 = |x_2^* - \varphi_{2,7}| = 2.64923 \times 10^{-4}$$

$$ESA_3 = |x_3^* - \varphi_{3,7}| = 1.64335 \times 10^{-6}$$

Now applying the RADM by calculating seven terms, but in two steps, the results are

$$\varphi_1^{(1)} = x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} = 0.49999$$

$$\varphi_1^{(2)} = x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} = 0.49999$$

and

$$\varphi_2^{(1)} = x_{2,0} + x_{2,1} + x_{2,2} + x_{2,3} = -0.00203028$$

$$\varphi_2^{(2)} = \varphi_2^{(1)} + x_{2,4} + x_{2,5} + x_{2,6} = 9.96392 \times 10^{-7}$$

and

$$\varphi_3^{(1)} = x_{3,0} + x_{3,1} + x_{3,2} + x_{3,3} = -0.523833$$

$$\varphi_3^{(2)} = \varphi_3^{(1)} + x_{3,4} + x_{3,5} + x_{3,6} = -0.523598$$

Hence we have

$$ERA_1 = |x_1^* - \varphi_1^{(2)}| = 9.32325 \times 10^{-9}$$

$$ERA_2 = |x_2^* - \varphi_2^{(2)}| = 9.96392 \times 10^{-7}$$

$$ERA_3 = |x_3^* - \varphi_3^{(2)}| = 4.25556 \times 10^{-7}$$

Comparing the absolute errors of results by two methods (ADM and RADM), relative to the exact solution, shows that RADM in two steps gives more accurate results than those of the ADM.

4 Conclusion

In this paper, we applied the RADM and the ADM to approximate the system of solution of non-linear equations and showed that the RADM gave better approximate solutions in comparison with those of the ADM.

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