Restarted Adomian’s Decomposition Method for the Bratu-Type Problem

A. R. Vahidi \(^a\) and M. Hasanzade \(^b\)

\(^a\) Department of Mathematics, Shahr-e-Rey Branch
Islamic Azad University, Tehran, Iran
alrevahidi@yahoo.com

\(^b\) Department of Mathematics, Science and Research Branch
Islamic Azad University, Tehran, Iran
mehdi_hasanzade@yahoo.com

Abstract

In this paper, the restarted Adomian decomposition method (RADM) is applied to solve the Bratu-type problem. The results are compared with those of the Adomian decomposition method (ADM). The obtained results indicate that the RADM in identical conditions give more suitable and accurate solutions for the Bratu-type problem in comparison with those for the ADM.

Keywords: Adomian’s decomposition method; Restarted Adomian decomposition method; Bratu-type problem

1 Introduction

The Adomian decomposition method is a solution method with a wide range of applications including the solution of algebraic, differential, integral and integro-differential equations or system of equations. This method was first introduced by Adomian [1, 2] in the beginning of the 1980’s. In this method, the solution is considered as an rapidly converging, infinite series.

It is well known that Bratu boundary value problem in one-dimensional planar coordinates is of the form [6-11]

\[ u''(x) + \lambda e^{u(x)} = 0, \quad u(0) = u(1) = 0, \quad 0 < x < 1. \]  

(1)

The analytical solution to Eq. (1) is given by

\[ u(x) = -2 \ln \left[ \frac{\cosh(0.5(x - 0.5)\theta)}{\cosh(\theta/4)} \right], \]  

(2)
where $\theta$ is the solution of $\theta = \sqrt{2\lambda} \cosh(\theta/4)$. The problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$, and $\lambda < \lambda_c$, respectively, where $\lambda_c = 3.513830719$ [7,9,10,12].

The standard Bratu problem (1) was used to model a combustion problem in a numerical slab. The Bratu model appears in a number of applications such as the fuel ignition of the thermal combustion theory [9] and in the Chandrasekhar model of the expansion of the universe. It simulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction [11].

Several authors have proposed a variety of approaches to obtain the approximate solution of the Bratu-type problem. Wazwaz [13] applied the ADM to solve the Bratu-type problem. Y. Aregbesola suggested the weighted residual method [11] and S. A.khuri in [14] used the Laplace transform decomposition method to obtain the approximate solutions of the Bratu-type problem.

In present study, we apply the RADM based on the ADM introduced by E. Babolian et al [3, 4, 5] to solve the initial value problem of the Bratu-type and show that the RADM, in identical conditions, results in more suitable solutions for the Bratu-type problem than those for the ADM.

The present paper is organized as follows: In section 2, the ADM for initial value problem of the Bratu-type is explained in detail. The RADM is presented for initial value problem of the Bratu-type in section 3. In section 4, the conclusion is briefly discussed.

2 The ADM for initial value problem of the Bratu-type

In this section, we apply the ADM for initial value problem of the Bratu-type in two cases. In the first case, as the standard ADM, we consider the nonlinear term as $e^{u(x)}$ and in the second case we use Taylor series of $e^{u(x)}$.

Case 1. Wazwaz in [13] considered the initial value problem of the Bratu-type

$$u''(x) - 2e^{u(x)} = 0, \quad 0 < x < 1,$$

(3)

with the initial conditions

$$u(0) = u'(0) = 0,$$

(4)

and applied the ADM to solve it. We describe this procedure briefly. To do this, consider Eq. (3) in an operator form

$$Lu(x) = 2e^{u(x)},$$

(5)

Denoting $\frac{d^2}{dx^2}(.)$ by $L$, we have $L^{-1}$ as a two-fold integration from 0 to $x$. Applying the inverse operator $L^{-1}$ to both sides of Eq. (5) and using the
initial conditions $u(0) = u'(0) = 0$ yield
$$u(x) = 2L^{-1}e^{u(x)}. \quad (6)$$

To apply the ADM to Eq. (6), let $u(x) = \sum_{m=0}^{\infty} u_m(x)$ and $e^{u(x)} = \sum_{m=0}^{\infty} A_m(x)$ where the $A_m$’s are the Adomian polynomials that represent the nonlinear term $e^{u(x)}$ and are given by
$$A_0(x) = e^{u_0(x)}, \quad (7)$$
$$A_1(x) = u_1(x)e^{u_0(x)}, \quad (8)$$
$$A_2(x) = (u_2(x) + \frac{1}{2}u_1^2(x))e^{u_0(x)}, \quad (9)$$
$$A_3(x) = (u_3(x) + u_1(x)u_2(x) + \frac{1}{6}u_1^3(x))e^{u_0(x)}, \quad (10)$$
$$A_4(x) = (u_4(x) + u_1(x)u_3(x) + \frac{1}{2}u_2^2(x) + \frac{1}{2}u_1^2u_2(x) + \frac{1}{24}u_1^4(x))e^{u_0(x)}, \quad (11)$$

By identifying the zeroth component $u_0(x)$ by 0, the remaining components $u_m(x)$ for $m \geq 0$ can be obtained recurrently by using the relation
$$u_0(x) = 0, \quad (12)$$
$$u_{m+1}(x) = 2 \int_0^x \int_0^x A_m(x), \quad m \geq 0 \quad (13)$$

The solution components $u_m(x)$ from (13) can be calculated as
$$u_1(x) = x^2, \quad (14)$$
$$u_2(x) = \frac{x^4}{6}, \quad (15)$$
$$u_3(x) = \frac{2x^6}{45}, \quad (16)$$
$$u_4(x) = \frac{17x^8}{1260}. \quad (17)$$

By calculating five terms of the series solution, we obtain
$$u_{ADMI}(x) \cong \varphi_5(x) = \sum_{m=0}^{4} u_m(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260}. \quad (18)$$
Case 2. To accelerate the convergence of the ADM, when used for nonlinear differential equations, we replace nonlinear terms by their Taylor expansion. Toward this end, we consider $e^u$ as

$$e^u \simeq 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!}.$$  

(19)

Therefore, Eq. (3) can be written as

$$u'' = 2(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!}).$$

(20)

Now, we apply the ADM for Eq. (20). In fact, by substituting $u(x) = \sum_{i=0}^{\infty} u_i(x)$ into Eq. (20), we get

$$\sum_{i=0}^{\infty} u_i(x) = 2 \int_0^x \int_0^x (1+\sum_{i=0}^{\infty} u_i(x))dx dx + \int_0^x \int_0^x \sum_{i=0}^{\infty} A_i(x)dx dx + \frac{1}{3} \int_0^x \int_0^x \sum_{i=0}^{\infty} B_i(x)dx dx,$$

(21)

where

$$A_i = \frac{1}{i!} \frac{d^i}{d\lambda^i} \left[ (\sum_{n=0}^{\infty} \lambda^n u_n(x))^2 \right]_{\lambda=0}, \quad i = 0, 1, 2, \ldots$$

(22)

and

$$B_i = \frac{1}{i!} \frac{d^i}{d\lambda^i} \left[ (\sum_{n=0}^{\infty} \lambda^n u_n(x))^3 \right]_{\lambda=0}, \quad i = 0, 1, 2, \ldots$$

(23)

Each term of the series in Eq. (21) is given by the recurrence relation

$$u_0(x) = x^2,$$

(24)

$$u_{n+1}(x) = 2 \int_0^x \int_0^x u_n(x)dx dx + \int_0^x \int_0^x A_n(x)dx dx + \frac{1}{3} \int_0^x \int_0^x B_n(x)dx dx.$$ (25)

The solution components $u_n(x)$ from Eq. (25) can be calculated as

$$u_0(x) = x^2,$$

$$u_1(x) = \frac{1}{6} x^4 + \frac{1}{30} x^6 + \frac{1}{168} x^8,$$

$$u_2(x) = \frac{1}{168} x^8 + \frac{1}{1350} x^{10} + \frac{1}{11088} x^{12} + \cdots,$$

$$u_3(x) = \frac{1}{1800} x^{10} + \frac{1}{10395} x^{12} + \frac{323}{687960} x^{14} + \cdots,$$

$$\vdots$$

By calculating five terms of the series solution, we obtain

$$u_{ADM2}(x) \cong \varphi_5(x) = \sum_{m=0}^{4} u_m(x) = x^2 + \frac{1}{6} x^4 + \frac{2}{45} x^6 + \frac{17}{1260} x^8 + \cdots.$$ (27)
3 The RADM for initial value problem of the Bratu-type

In this section, we extend the RADM for solving initial value problem of the Bratu-type. Basically, the RADM has the same structure as that of the ADM but the ADM is used more than once. Note that we apply the RADM for the translated Eq. (20). In the following, the algorithm RADM for Eq. (20) is introduced.

3.1 The algorithm

Step 1. Choose positive natural numbers $m, n, m'$. 

Step 2. Use the ADM to solve Eq. (20) and obtain $\phi_m(t)$, then let $P(x) = \phi_m(x)$. 

Step 3. Add and subtract $P(x)$ to right side of Eq. (21) to obtain

$$\sum_{i=0}^{\infty} u_i(x) = P(x) + 2 \int_0^x \int_0^x (1 + \sum_{i=0}^{\infty} u_i(x)) dx dx + \int_0^x \int_0^x A_i(x) dx dx$$

$$+ \frac{1}{3} \int_0^x \int_0^x B_i(x) dx dx - P(x),$$

(28)

(29)

For $k = 1$ to $n$, do,

Step 4. Let

$$u_{k,0}^{res}(x) = P(x),$$

$$u_{k,1}^{res}(x) = x^2 + 2 \int_0^x \int_0^x (u_{k,0}^{res}(x)) dx dx + \int_0^x \int_0^x A_i(x) dx dx$$

$$+ \frac{1}{3} \int_0^x \int_0^x B_i(x) dx dx - P(x),$$

(30)

$$u_{k,i}^{res}(x) = 2 \int_0^x \int_0^x u_{k,i-1}^{res}(x) dx dx + \int_0^x \int_0^x A_i(x) dx dx$$

$$+ \frac{1}{3} \int_0^x \int_0^x B_i(x) dx dx, \ i = 2, 3, \ldots, m'$$

(31)

(32)

Step 5. Let

$$u^{res}(x) = \sum_{i=0}^{m'} u_{k,i}^{res}(x),$$

(33)

$$P(x) = u^{res}(x),$$

(34)
End of For.
Step 6. Consider the approximate solution of the problem as \( \varphi(t) = u^{res}(x) \).

To apply the RADM to obtain the approximate solution of Eq. (20), considering algorithm 3.1, we choose \( m = 2, k = 1, m' = 3 \) and hence

\[
\varphi_2(x) = u_0(x) + u_1(x) = x^2 + \frac{1}{6}x^4 + \frac{1}{30}x^6 + \frac{1}{168}x^8, \tag{35}
\]

then we let \( P(x) = \varphi_2(x) \) and the RADM as follows

\[
u_{res}^{1,0}(x) = P(x), \tag{36}
\]

\[
u_{res}^{1,1}(x) = x^2 + 2 \int_0^x \int_0^x (u_{res}^{1,0}(y)) dy dx + \int_0^x \int_0^x A_0'(y) dy dx + \frac{1}{3} \int_0^x \int_0^x B_0'(y) dy dx - P(x) = \frac{x^4}{6} + \frac{17x^{10}}{16200} + \frac{29x^{12}}{166320} + \ldots, \tag{37}
\]

\[
u_{res}^{1,2}(x) = 2 \int_0^x \int_0^x u_{res}^{1,1}(y) dy dx + \int_0^x A_1'(y) dy dx + \frac{1}{3} \int_0^x \int_0^x B_1'(y) dy dx + \frac{x^{10}}{4050} + \frac{17x^{12}}{124740} + \frac{521x^{14}}{10319400} + \ldots, \tag{38}
\]

\[
u_{res}^{1,3}(x) = 2 \int_0^x \int_0^x u_{res}^{1,2}(y) dy dx + \int_0^x A_2'(y) dy dx + \frac{1}{3} \int_0^x \int_0^x B_2'(y) dy dx = \frac{x^{12}}{166320} + \frac{121x^{14}}{20638800} + \frac{3193x^{16}}{898128000} + \ldots, \tag{39}
\]

Finally, the approximate solution of Eq. (20) using the RADM is obtained by calculating

\[
u^{res}(x) \simeq P(x) + \nu_{res}^{1,0}(x) + \nu_{res}^{1,1}(x) + \nu_{res}^{1,2}(x) + \nu_{res}^{1,3}(x), \tag{40}
\]

that is

\[
u^{res}(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \ldots. \tag{41}
\]

It is important to note that using more terms of Taylor expansion of \( e^{u(x)} \) in case 2 would cause more accuracy in solutions. In order to verify the efficiency of the RADM in comparison with that of the ADM, we report the absolute errors of solutions obtained by these methods for \( x \in [0, 1] \) relative to the exact solution in Table 1.
Restarted Adomian’s decomposition method

<table>
<thead>
<tr>
<th>Time</th>
<th>Error of ADM</th>
<th>Error of RADM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>4.3876E-13</td>
<td>9.3045E-14</td>
</tr>
<tr>
<td>0.2</td>
<td>4.5402E-10</td>
<td>9.7158E-11</td>
</tr>
<tr>
<td>0.3</td>
<td>2.6638E-8</td>
<td>5.7790E-9</td>
</tr>
<tr>
<td>0.4</td>
<td>4.8488E-7</td>
<td>1.0727E-7</td>
</tr>
<tr>
<td>0.5</td>
<td>4.6664E-6</td>
<td>1.0595E-6</td>
</tr>
<tr>
<td>0.6</td>
<td>3.0124E-5</td>
<td>7.0675E-6</td>
</tr>
<tr>
<td>0.7</td>
<td>1.4821E-4</td>
<td>3.6212E-5</td>
</tr>
<tr>
<td>0.8</td>
<td>6.0039E-4</td>
<td>1.5413E-4</td>
</tr>
<tr>
<td>0.9</td>
<td>2.1074E-3</td>
<td>5.7446E-4</td>
</tr>
<tr>
<td>1</td>
<td>6.6498E-3</td>
<td>1.9496E-4</td>
</tr>
</tbody>
</table>

Table 1. Errors of the ADM and the RADM for initial value problem of the Bratu-type

4 Conclusion

In this study, we applied the RADM to obtain a more accurate approximation of analytical solution for initial value problem of the Bratu-type. The results were compared with those of the ADM. The numerical results showed that the RADM, in identical conditions, result in more suitable solutions for initial value problem of the Bratu-type than those for the ADM.

References


Received: June, 2011