Applied Mathematical Sciences, Vol. 6, 2012, no. 12, 589 - 596

Derivation of Polynomial Equations for Eight-Point Prismatic Arrays

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*Los Alamos National Laboratory is operated by the Los Alamos National Security, LLC for the National Nuclear Security Administration of the U.S. Department of Energy contract DE-AC52-06NA25396.

Abstract

Described are symbolic methods for obtaining quadratic and cubic equations for the eight-point rectangular prism. They apply coefficients taken from the trilinear equation to generate equations of the second and third degrees. The methods are illustrated by examples.

Mathematics subject classification: 65D05, 65D07, 65D17

Keywords: three-dimensional interpolation, polynomial interpolation, prismatic data array

1. Introduction

The estimation of a datum missing from an eight- or a nine-point rectangular prismatic array, as in Fig. 1, has scholarly as well as economic interest. Several equations for this purpose have been illustrated [2]. One of them is presently useful: Eq. (11) in Ref. [2]. It is reproduced as Eq. (1) below. Except for E, which represents a datum at the center point of Fig. 1, the terms in Eq. (1) are taken from the trilinear equation, Eq. (2). The letters x,y,z in Eq. (2) refer to coordinate numbers in the –1 .. 1 coordinate system in which x is from left to right, y is into the page, and z is from the bottom to the top of the prism. The suffix letter c signifies a coefficient so that xyc represents the coefficient of the coordinate product xy. The arithmetic mean of the data A .. D and F .. I is represented by (avg) [2].

$$2E(yzc)(xzc)(xyc) - 2(avg)(yzc)(xzc)(xyc) + (xyc)^2(xzc)^2 + (xyc)^2(yzc)^2 + (xzc)^2(yzc)^2 = 0 \quad (1)$$

$$R = (avg) + (xc)x + (yc)y + (zc)z + (xyc)xy + (xzc)xz + (yzc)yz + (xyzc)xyz \quad (2)$$

The suggestive property of Eq. (1) is its accuracy on first-, second-, and third-power relationships among trilinear numbers. See Table 5 in [2]. This property can be expressed in other ways: coefficients in the trilinear equation can be combined so that they apply to trilinear numbers in relationships of degrees higher than unity. Cross-product coefficients in the cited equation can be applied to nine numbers A .. I as $A=1^N$, $B=2^N$, $C=3^N$ .. $I=9^N$ where N=1, 2, or 3 [2]. There may be other such relationships, without the prism center point E, that have useful properties with respect to the eight-point rectangular prism in Fig. 1.

2. Quadratic equation for the eight-point cube

A linear expression among three coordinate parameters, x,y,z, is (P+Qx+Ry+Sz). This expression can be squared and expanded as in Eq. (3). The last three terms on its right hand side are the quadratic terms. If quadratic terms can be introduced into Eq. (2), it becomes a basis for an interpolation equation for the eight-point rectangular prism in Fig. 1. Let the coefficients of the x^2, y^2, and z^2 terms in Eq. (3) be denoted x2c, y2c, and z2c, respectively.

$$R = (P + Qx + Ry + Sz)^2 =$$
$$P^2 + 2PQx + 2PRy + 2PSz + 2QRxy + 2QSxz + 2RSyz + Q^2x^2 + R^2y^2 + S^2z \quad (3)$$

A method for finding the quadratic terms has been illustrated in [3] and the complete equation appears in [4]. However, a more general approach is desirable. The expression on the left hand side of Eq. (1), and the forms of x2c, y2c, and z2c in [4], suggest the possibility of a general expression. A simple one appears in the line for Eq. (4). FNC defines it by the means of the sign of equality. Applying the symbols in Eq. (3) permits Eq. (4) to be rewritten as Eq. (5). The process is symbolic so a constant (W) can be introduced into Eqs. (4) and (5).

$$FNC = (W)(xyc)^J(xzc)^K(yzc)^L \quad (4)$$
$$FNC = (W)(QR)^J(QS)^K(RS)^L \quad (5)$$

Only xyc and xzc in Eq. (5) contain Q. The power of Q in x2c in Eq. (3) is 2. That coefficient does not contain R and S so their powers are zero. Three new equations can be written based on Eq. (5). They are Eqs. (6), (7), (8).

J + K = 2 (power of Q in x2c of Eq. (3)) (6)
J + L = 0 (power of R in x2c of Eq. (3)) (7)
K + L = 0 (power of S in x2c of Eq. (3)) (8)

Equations (6)-(8) can be solved as a simultaneous set. The results are J=(1), K=(1), L=(–1). These exponents can be entered into the right hand side of Eq. (4) with the result that x2c is symbolically defined as in Eq. (9). The same form, excluding the undetermined constant (W), appears in Eq. (11) in [4].

x2c = (W)(xyc)(xzc)/(yzc) (9)

The symbolic form of the coefficient y2c is determined in the same way that applies Eq. (5) and three new simultaneous equations. Only xyc and yzc contain R. The power of R in y2c in Eq. (3) is 2 but it lacks the terms Q and S. Their powers are zero. Three new simultaneous equations can be written based on Eq. (5). They are equations (10), (11), (12). They are based on Eqs. (3) and (5).

J + K = 0 (power of Q in y2c of Eq. (3)) (10)
J + L = 2 (power of R in y2c of Eq. (3)) (11)
K + L = 0 (power of S in y2c of Eq. (3)) (12)

Equations (10)-(12) can be solved as a simultaneous set. The solution is J=(1), K=(–1), L=(1). This set can be entered into Eq. (4). The result for y2c is Eq. (13). A similar process establishes z2c as in Eq. (14).

$$y2c = (W)(xyc)(yzc)/(xzc) \quad (13)$$
$$z2c = (W)(xzc)(yzc)/(xyc) \quad (14)$$

The literal forms of xyc, xzc, and yzc are Eqs. (15)-(17), respectively. Let A .. D be 1^2 .. 4^2, and F .. I be 6^2 .. 9^2, respectively, as in Fig. 1. Then Eqs. (15), (16), and (17) have the numerical values 1, 5/2, and 5, respectively.

$$xyc = (A–B–C+D+F–G–H+I)/8 \quad (15)$$
$$xzc = (A–B+C–D–F+G–H+I)/8 \quad (16)$$
$$yzc = (A+B–C–D–F–G+H+I)/8 \quad (17)$$

When the preceding numbers are substituted into Eqs. (9), (13), and (14), the symbolic results are W/2, 2W, 25W/2, for x2c, y2c, and z2c, respectively. Expansion of the expression $(5+x/2+y+5z/2)^2$ yields Eq. (18). The true values of x2c, y2c, and z2c are 1/4, 1, and 25/4, respectively. A numerical comparison between symbolic x2c, y2c, z2c and true values of x2c, y2c, and z2c, respectively, indicates that W=1/2. The symbolic approach establishes the forms of x2c, y2c, z2c and a numerical comparison establishes the value of W. The constant term in the interpolation equation is the mean of the eight data minus the sum (x2c+y2c+z2c) [4]. The numerical, quadratic equation that interpolates Fig. 1 is Eq. (18). It reproduces the original data at vertices A .. I.

$$R = 25 + 5x + 10y + 25z + xy + 5xz/2 + 5yz + x^2/4 + y^2 + 25z^2/4 \quad (18)$$

In summary, the cross-product terms in Eq. (2) contain information supporting relationships among the first, second, and third powers of linear numbers. That is illustrated by Eq. (1) [2]. A new relationship has been posited: Eq. (4). Symbolically it embodies the properties of Eq. (1) and thereby renders the forms of x2c, y2c, and z2c in the quadratic equation for Fig. 1.

3. Cubic equation for the eight-point cube

A cubic equation for the eight-point prism in Fig. 1 was illustrated several years ago [5]. Its general, symbolic form is Eq. (1) therein. It is long and tedious so it is not reproduced here. The derivation of the cubic equation for the eight-point prism proceeds as for the quadratic equation in section 2.

The expanded form of the of $(P+Qx+Ry+Sz)^3$ appears as Eq. (19). It is long and tedious so it easy to make mistakes when transcribing it. The coefficients in Eq. (19) are to be expressed as powers of xyc, xzc, yzc, and xyzc in Eq. (1). A new expression FNC analogous to Eq. (4) is now posited. It is Eq. (20).

$$R = Q^3x^3 + R^3y^3 + S^3z^3 + 3Q^2x^2Ry + 3Q^2x^2Sz + 3QxR^2y^2 + 3QxS^2z^2 + 3R^2y^2Sz + 3RyS^2z^2 + 3PR^2y^2 + 3P^2Sz + 3P^2Qx + 3PQ^2x^2 + 3P^2Ry + 3PS^2z^2 + 6PQRxy + 6PQSxz + 6PRSyz + 6QRSxyz + P^3 \qquad (19)$$

$$FNC = (W)(xyc)^J(xzc)^K(yzc)^L(xyzc)^M \qquad (20)$$

The symbolic coefficient of the term x^2y in Eq. (19) is Q^2R. It does not contain P or S. Their powers are zero. Four simultaneous linear equations are constructed using J, K, L, M as in Eqs. (21)-(24).

$J + K + L = 0$ (powers of P in coefficient of x^2y) (21)
$J + K + M = 2$ (powers of Q in coefficient of x^2y) (22)
$J + L + M = 1$ (powers of R in coefficient of x^2y) (23)
$K + L + M = 0$ (powers of S in coefficient of x^2y) (24)

Equations (21)-(24) render J=(1), K=(0), L=(–1), M=(1). If these numbers are substituted into Eq. (20), the symbolic form of the coefficient of x^2y in Eq. (19) is (W)(xyc)(xyzc)/(yzc). The constant (W) is determined by a numerical comparison as in section 2. See Eq. (16) in [5].

What is the symbolic form of the coefficient of y^3 in Eq. (19)? According to that equation, the coefficient of y^3 is R^3. Form a second series of four simultaneous equations as in Eqs. (25)-(28).

$J + K + L = 0$ (powers of P in coefficient of y^3) (25)
$J + K + M = 0$ (powers of Q in coefficient of y^3) (26)
$J + L + M = 3$ (powers of R in coefficient of y^3) (27)
$K + L + M = 0$ (powers of S in coefficient of y^3) (28)

The preceding equations render J=(1), K=(–2), L=(1), M=(1). The symbolic form of the coefficient of y^3 is therefore $(W)(xyc)(yzc)(xyzc)/(xzc)^2$. The constant term (W)=(1/6) is determined as in section 2. See Eq. (23) in [5].

Express the coefficient of xz^2 in terms of xyc, xzc, yzc, xyzc. The symbolic form of this coefficient in Eq. (19) is QS^2. Form four new simultaneous equations as in Eqs. (29)-(32).

J + K + L = 0 (powers of P in coefficient of xz^2) (29)
J + K + M = 1 (powers of Q in coefficient of xz^2) (30)
J + L + M = 0 (powers of R in coefficient of xz^2) (31)
K + L + M = 2 (powers of S in coefficient of xz^2) (32)

Equations (29)-(32) render J=(–1), K=(1), L=(0), M=(1). Substitute these numbers into the right hand side of Eq. (20). It yields the symbolic form of the coefficient of xz^2 as (W)(xzc)(xyzc)/(xyc). The constant term is found as outlined in section 2. It is (W)=(1/2). See Eq. (20) in [5]. Other third-power terms for the interpolating equation are also constructed by the symbolic method.

The forms of some of the other coefficients, such as those for x^2, y^2, and z^2, are illustrated in section 2 and in [5]. Coefficients like xyzc are available in the trilinear equation. The linear-term coefficients, those for x, y, and z, remain unknown. They are separately determined from the general, symbolic form of the equation for the eight-point prism, Eq. (1) in [5].

A set of eight simultaneous equations is developed from the symbolic form of the general equation (Eq. (1) in [5]), the data, and their respective coordinates as in Fig. 1. Into this equation are substituted the coordinates of one prism point, say point A in Fig. 1. Its (x,y,z) coordinates are (–1,–1,–1), respectively. Then the datum at A is subtracted from the substituted equation. Eight such equations are prepared in this manner. The unknowns in the set of eight simultaneous equations are the three linear-term coefficients. The set is solved for the linear-term coefficients by the method of least squares [5]. The linear-term coefficients then have simple forms: Eqs. (1)-(3) in [6] or Eqs. (33)-(35) below. They are preferred to the tedious forms, Eqs. (6)-(8) in [5]. The constant term in the cubic equation is the mean of the eight data minus the three quadratic-term coefficients. The cubic equation is further described in [5-7].

nxc = xc – y2xc – z2xc – x3c (33)
nyc = yc – x2yc – z2yc – y3c (34)
nzc = zc – x2zc – y2zc – z3c (35)

4. Discussion

A ratio like $(xyc)(yzc)(xyzc)/(xzc)^2$ has another interpretation if the suffix letter c is removed from all of its factors. The algebraic cancellation of letters results in y^3 as in Eq. (36). This is a check on symbolic coefficients. It is not a perfect check because y^3 can result from other combinations of such factors.

$$(xy)(yz)(xyz)/(xz)^2 = y^3 \quad (36)$$

The forms of the quadratic-term coefficients in Eq. (10) of [4], and the properties of Eq. (1), are suggestive. They lead to a conjecture that the cited forms, and analogous ones, can be alternatively expressed. The conjecture is expressed in two ways: Eq. (4) and Eq. (20). Section 2 is based on Eq. (2), an equation limited to first-degree coordinate parameters: x, y, and z. By means of the conjecture, the coefficients in Eq. (2) are used to generate an equation with second-degree parameters: x^2, y^2, and z^2 as in Eq. (18). Section 3 is also based on Eq. (2) and the conjecture rewritten as Eq. (20). Together, they render an equation containing third-degree coordinate parameters: x^3, y^3, and z^3 [5,7]. The illustrated process also generates an equation for the four-point rectangle, Eq. (6) in [3].

The symbolic approach illustrated here is suggested by the operational method of data analysis [8]. It uses the shifting operator and symbolic methods to illustrate relationships among points in space [2-8]. However, the preceding applications are not operational methods because they do not invoke the shifting operator [8]. Symbolic methods, as illustrated here and in [8], tend to be misunderstood and maligned. That impression derives from E.T. Bell's interpretation of Heaviside's experience with analogous methodology [1].

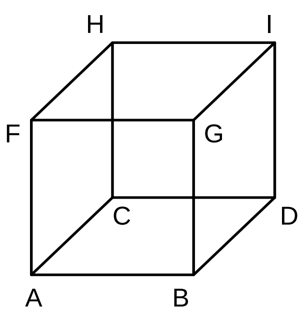

Fig. 1. The eight-point rectangular prism or cube. Center point E is not shown in the diagram.

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Received: September, 2011