

Structure Properties of M-Fuzzy Groups

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Abstract. The theory of fuzzy sets has developed in many directions and is finding application in a wide variety of fields. Rosenfeld (7) in 1971 used this concept to develop the theory of fuzzy groups, In this paper we have given independent proof of several theorems on M- fuzzy groups. we discuss about M- fuzzy groups and investigate some of their structures on the concept of M- fuzzy group family.

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1 Introduction

After the introduction of the concept of fuzzy sets by zadeh (10) several researches were conducted on the generalizations of the notion of fuzzy sets. The study of fuzzy group was started by Rosenfeld (7) and it was extended by Roventa (8) who have introduced the concept of fuzzy groups operating on fuzzy sets. Wu (9) studied the fuzzy normal subgroups. Gu (6) put forward the notion of fuzzy groups with operators. In this paper, we introduce the concept of M- fuzzy groups with operators and obtain some related results. For the sake of convenience, we set out the former concepts.

2 Preliminaries

Let X be a nonempty set. A fuzzy set A is a map $A: X \rightarrow [0,1]$, and $F(x)$ will denote the fuzzy power set of X . we first recall some basic definitions and results that will be needed in the sequel.

Definition 2.1: Let G be a group and $A \in F(G)$. If

- (i) $A(xy) \geq \min \{ A(x), A(y) \}$
- (ii) $A(x^{-1}) = A(x)$, $x \in G$, then A is called fuzzy group on G . If the supplementary condition $A(e) = 1$ called standardized fuzzy group.

Definition 2.2: A group with operators is an algebraic system consisting of a group G , a set M and a function defined in the product set $M \times G$ and having the values in G such that if mx denotes the element in G determined by the element x of G and the element m of M , and $m \in M$, then G is called M- group with operators.

Definition 2.3: A subgroup A of an M- group G is said to be the fuzzy subgroup if $mx \in A$ for all $m \in M$ and $x \in A$.

Definition 2.4: Let A be a fuzzy set in U and $\bullet : G \times G \rightarrow G$ be a composition law, such that (G, \bullet) forms M- group. If two conditions

(FG1) $A(m(xy)) \geq \min \{ A(mx), A(my) \}$ and (FG2) $A(mx^{-1}) = A(mx)$ for all x, y in A . If the supplementary conditions $A(me_G) = 1$ is also satisfied, then this M- fuzzy group is called a standardized M- fuzzy group, where e_G is an identity of M- group (G, \bullet) .

3 Some properties of M- fuzzy groups

Proposition 3.1: Let A be M- fuzzy group and S be a fuzzy subset of A then S is a M-fuzzy subgroup of A iff $A_S(m(xy)) \geq \min \{ A_S(mx) , A_S(my) \}$ for all x,y in S.

Proof: case-1 : Let S be a M- fuzzy subgroup of A. S itself is a M- fuzzy group. Thus (FG1) and (FG2) are satisfied in S. Therefore we obtain

$$A_S(m(xy^{-1})) \geq \min \{ A_S(mx) , A_S(my^{-1}) \} = \min \{ A_S(mx) , A_S(my) \}$$

Case-2: Let $A_S(m(xy^{-1})) \geq \min \{ A_S(mx) , A_S(my) \}$ for all $x,y \in S$ ----- (1)

If $y = x$ in (1) , we have $A_S(m(xy^{-1})) = A_S(me) \geq \min \{ A_S(mx) , A_S(my) \} = A_S(mx)$. If $x=e$ in (1) , $A_S(my^{-1}) = A_S(m(ey^{-1})) \geq \min \{ A_S(me) , A_S(my^{-1}) \} = A_S(my)$, for all $x,y \in S$. Implies that $A_S(my) \geq A_S(my^{-1})$ for all x, y in S.(since substituting $y=y^{-1}$ in the above equation) Therefore $A_S(my) = A_S(my^{-1})$,FG2 is satisfied in S. further $A_S(m(x(y^{-1}))) \geq \min \{ A_S(mx) , A_S(m(y^{-1})) \} = \min \{ A_S(mx) , A_S(my) \}$, FG1 is satisfied in S.

Lemma 3.2: for all $a,b \in [0,1]$, $m \in M$ and p is any positive integer , verify that (i) If $ma \leq mb$ then $(ma)^p \leq (mb)^p$ and (ii) $\min \{ ma,mb \}^p = \min \{ (ma)^p , (mb)^p \}$

Proof: It is obvious.

Proposition 3.3: If A is a M- fuzzy group, then $A^p = \{ (mx, (A(mx))^p) / mx \in A \}$ is M-fuzzy group.

Proof: Let A is M- fuzzy group, where (A, \bullet) is M- group. Thus (A^p, \bullet) is M- fuzzy group for all the positive integer p. let $x,y \in A^p$ and p is any positive integer.

(FG1) $A^p(m(xy)) = (A(m(xy))^p) \geq \min \{ A(mx) , A(my) \}^p = \min \{ A(mx)^p , A(my)^p \}$
 FG1 is satisfied in A^p . (FG2) $A^p(mx) = (A(mx))^p = (A(mx^{-1}))^p = A^p(mx^{-1})$. (A^p, \bullet) is M-fuzzy group by (2.4) $A^p(me) = (A(me))^p = (1)^p$ therefore (A^p, \bullet) is standardized M- fuzzy group.

Corollary 3.4: The M- fuzzy group A^q is a M- fuzzy subgroup A^p , if $q \leq p$.

Proof: clearly A^q and A^p are M- fuzzy groups by (3.3). For all $x \in [0,1]$, $x^q \geq x^p$ implies that $A^q \subset A^p$ (since $A^q(x) \leq A^p(x)$ for all $x \in A$).

Proposition 3.5: If A^i and A^j are M- fuzzy groups, then $A^i \cup A^j$ is also M- fuzzy group for all natural numbers i and j.

Proof: suppose $i < j$

$$\begin{aligned} \text{(FG1) } A^i \cup A^j (m(xy)) &= \max \{ A^i(m(xy)) , A^j(m(xy)) \} \\ &= \max \{ (A(m(xy)))^i , (A(m(xy)))^j \} \end{aligned}$$

$$\begin{aligned}
&= (A(m(xy)))^i \\
&= A^i(m(xy)) \\
&\geq \min \{ A^i(mx), A^j(my) \} \text{ by (3.3)} \\
&= \min \{ \max \{ (A^i(mx), A^j(mx)) \}, \max \{ (A^i(my), A^j(my)) \} \}. \\
&\geq \min \{ A^i UA^j(mx), A^i UA^j(my) \}. \text{ Therefore } A^i UA^j \text{ is M-}
\end{aligned}$$

fuzzy group by 2.4.

$$\begin{aligned}
(\text{FG2}) \ A^i UA^j (mx) &= \max \{ A^i(mx), A^j(mx) \} \\
&= \max \{ (A(mx))^i, (A(mx))^j \} \\
&= \max \{ (A(mx^{-1}))^i, (A(mx^{-1}))^j \} \\
&= A^i UA^j (mx^{-1}).
\end{aligned}$$

Proposition 3.6 : If A^i and A^j are M- fuzzy groups, then $A^i \cap A^j$ is also M- fuzzy groups, where i and j are natural numbers.

Proof: It is obvious.

Lemma 3.7: Prove that $A^p \subset A$ for all p .

Proof: Let $x \in A$ therefore $A^p(mx) \leq A(mx)$ for all natural number P since $(A(mx))^p \leq A(mx)$.

Definition 3.8: Let $f: G \rightarrow G^1$ be a homomorphism's of M- groups. For any fuzzy set $A \in G^1$ we define a new fuzzy set $A^f \in G$ by $A^f(mx) = Af(mx)$ for all $x \in G$.

Proposition 3.9: Let G and G^1 be M – groups and ‘ f ’ an M- homomorphism from G onto G^1 , (i) if A is M – fuzzy group of G^1 then A^f is M- fuzzy group of G . (ii) if A^f is M- fuzzy group of G then A is M- fuzzy group of G^1 .

Proof: (i) Let $x, y \in G$ and $m \in M$, we have

$$\begin{aligned}
(\text{FG1}) \ A^f(m(xy)) &= A(f(mx), f(my)) \\
&\geq \min \{ Af(mx), Af(my) \} \\
&= \min \{ A^f(mx), A^f(my) \}
\end{aligned}$$

$$\begin{aligned}
(\text{FG2}) \ A^f(mx) &= A(f(mx)) \\
&= A(f(mx)^{-1}) \\
&= Af(mx^{-1}) = A^f(mx^{-1}). \text{ Therefore } A^f \text{ is M- fuzzy group of } G.
\end{aligned}$$

(ii) for any $x, y \in G$ and $m \in M$, there exists $a, b \in G$ such that $f(ma) = x$ and $f(mb) = y$.

$$\begin{aligned}
(\text{FG1}) \ A(m(xy)) &= A(f(ma), f(mb)), \\
&= A^f(m(ab)) \\
&\geq \min \{ A^f(ma), A^f(mb) \}
\end{aligned}$$

$$\begin{aligned}
 &= \min \{ Af(ma) , Af(mb) \} \\
 &= \min \{ A(mx) , A(my) \} \\
 \text{(FG2) } A(mx^{-1}) &= A(f(ma))^{-1} = Af(ma^{-1}) = A^f(ma^{-1}) = A^f(ma) = A(mx) .
 \end{aligned}$$

Proposition 3.10: Let $\langle A \rangle = \{ A , A^1 , A^2 , \dots , A^p , \dots , E \}$ then $\bigcup_{p=1}^{\infty} A^p = A$

and $\bigcap_{p=1}^{\infty} A^p = E$

Proof: Let $x \in A$ and $m \in M$. clearly $A \subset \bigcup_{p=1}^{\infty} A^p$ (i)

(since $A(mx) \leq \bigcup_{p=1}^{\infty} A^p(mx)$)
 $\bigcup_{p=1}^{\infty} A^p(mx) = \max \{ A(mx) , A^2(mx) , \dots \}$
 $= A(mx)$ (since $A(mx) \leq (A(mx))^p$ for all natural number p .)

$\bigcup_{p=1}^{\infty} A^p \subset A$ (ii) from (i) and (ii) $\bigcup_{p=1}^{\infty} A^p = A$. Also we have to

Show that $E = \bigcap_{p=1}^{\infty} A^p$. Let $me \in A^p$ implies that $(me, 1) \in A^p$ for all p .

Implies that $(me, 1) \in \bigcap_{p=1}^{\infty} A^p$

Therefore $E \subset \bigcap_{p=1}^{\infty} A^p$ (iii)

Let $mx \in \bigcap_{p=1}^{\infty} A^p$ implies that $mx \in A^p$ for all p .

$$\begin{aligned}
 \bigcap_{p=1}^{\infty} A^p(mx) &= \min \{ A(mx) , A^2(mx) , \dots \} \\
 &= 0 \quad \text{if } mx \neq me \\
 &= 1 \quad \text{if } mx = me
 \end{aligned}$$

$x \in \bigcap_{p=1}^{\infty} A^p$ implies that $mx = me$ thus $\bigcap_{p=1}^{\infty} A^p \subset E$ (iv) hence

From (iii) and (iv) $E = \bigcap_{p=1}^p A^p$

Proposition 3.11: Let G and G^1 be M - groups and f is homomorphism from G onto G^1 . If A^f is M - fuzzy group of G , then A is M - fuzzy group of G^1 .

Proof: By proposition (3.9), A is M - fuzzy group of G^1 . for any $y \in G^1$ and $m \in M$.

$$\begin{aligned}
 \text{(FG1)} \quad A^f(my) &= \sup_{z \in f^{-1}(my)} A(z) \geq \sup_{mf(x) = my} A(mx) \geq \sup_{f(x) = y} A(x) = A^f(y) \\
 \text{(FG2)} \quad A^f(mx^{-1}) &= \sup_{z \in f^{-1}(mx^{-1})} A(z) = \sup_{mx \in f^{-1}(y)} A(mx) \quad \text{where } x \in f^{-1}(y) \\
 &= \sup_{f(mx) = mx^{-1}} A(mx) = \sup_{mf(x) = mx^{-1}} A(mx) \\
 &\geq \sup_{f(x) = x^{-1}} A(x^{-1}) \\
 &= A^f(x^{-1}). \quad A \text{ is } M\text{- fuzzy group of } G^1.
 \end{aligned}$$

Definition 3.12: Let A be a M - fuzzy group, then the following set of M - fuzzy groups $\{A, A^1, A^2, \dots, A^p, \dots, E\}$ is called M - fuzzy group family generated by A . It will be denoted by $\langle A \rangle$.

Proposition 3.13: Let A be a M - fuzzy group, then $A \supset A^2 \supset A^3 \dots \supset A^p \dots \supset E$.

Proof: It is known that $A(ma) \in [0,1]$, hence $A(ma) \geq A(ma)^2$, $A(ma^2) \geq (A(ma^2))^2, \dots, A(ma^n) \geq (A(ma^n))^2$ by using the definition of fuzzy subsets, this gives that $A \supset A^2$. By generalizing it for any natural numbers i and j with $i \leq j$. we obtain $(A_i(a))^i \geq (A_i(a))^j$, $(A_i(a^2))^i \geq (A(ma^2))^j, \dots, (A_i(ma^n))^i \geq A_j(ma^n)^j$.

So $A^i \supset A^j$ for any natural numbers i and j with $i \leq j$ which means that $A \supset A^2 \supset A^3 \dots \supset A^{pnv} \dots$ finally we get $E = \bigcap_{p=1}^p A^p$ which is immediate from proposition (3.10). Since

$$\begin{aligned}
 \text{Lt } A(ma)^p &= 1 \quad \text{if } ma = me. \\
 n \rightarrow \infty &= 0 \quad \text{if } ma \neq me \quad \text{we then obtain the required relations.}
 \end{aligned}$$

Conclusion: W.M . WU(9) and A. Rosenfeld (7) introduced the concept of fuzzy normal subgroups and fuzzy groups. In this paper we investigate the concept of M - fuzzy groups and obtain some Results.

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