

Wealth Distribution in an Asset Pricing Model: the Role of the Switching Mechanism

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Abstract

We present an asset pricing model with heterogeneous agents trading in a Walrasian scenario. Fractions of agents are updated at any time and their wealth is updated not only as a consequence of portfolio growth of agents adopting the same strategy, but also due to the flow of new agents. After introducing the belief types, rational and chartists, the model is built to investigate the role of the switching mechanism in the long-run wealth evolution. In particular, it shows complexity in the wealth distribution and positive strong correlation between the evolution of the fraction of agents using the same predictor and its relative wealth. The former result is explained by the switching mechanism introduced into the wealth dynamics, the latter is proved by computing the statistical distributions of the observed correlation coefficients.

Keywords: asset pricing model, wealth dynamics, strange attractor, correlation coefficient

1 Introduction

In the last years an increasing number of analytical and numerical researches on financial markets have focused on the study of the market equilibrium price within "agent-based" models. Several contributions present analytically tractable models which describe markets as complex systems of boundedly rational and heterogeneous agents (see Hommes (2006) and LeBaron (2006))

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for an extensive survey). This kind of models is able to explain many "stylized facts" observable in real markets, for which the Efficient Market Hypothesis is inadequate.

An important branch of studies within Heterogeneous Agent Models has focused on the evolution of agents' wealth and its effect on price dynamics by assuming constant relative risk aversion (CRRA) utility function. In fact, some simulation-based works lean towards a framework where investors' optimal decisions depend on their wealth, which is in agreement with the assumption of CRRA utility function, see e.g. Levy et al. (1994, 1995, 2000), and Campbell and Viceira (2002). Examples of Heterogeneous Agent Models inspired by the desire to explore the CRRA framework can be found in Chiarella and He (2001, 2002), Chiarella et al. (2006), Anufriev et al. (2006), Anufriev and Bottazzi (2006), Anufriev (2008), Anufriev and Dindo (2010), Brianzoni et al. (2010a) and Brianzoni et al. (2010b).

Most of the contributions to the development and analysis of financial models with heterogeneous agents and CRRA utility are built under the assumption of constant proportions of agents. Differently, Chiarella and He (2002) allow agents to switch following an adaptive belief system. More in detail, the authors assume that traders agree to accept the average wealth level of agents using the adopted strategy.

In order to overcome this unconvincing element, Brianzoni et al. (2010b) introduce a new switching mechanism according to which the wealth of the selected group takes into account the wealth coming from the group of origin. In other words, the wealth of each group is updated from period t to $(t + 1)$ not only as a consequence of portfolio growth of agents adopting the relative strategy, but also due to the flow of agents coming from the other group. The authors introduce two typical traders, fundamentalists and chartists. Fundamentalists believe that the price of an asset is determined by its fundamental value, they sell (buy) assets when their prices are above (below) the market fundamental value. Chartists do not take the fundamental value into account, rather they use simple technical rules. Usually, the latter pushes prices away from the fundamental value, acting as a destabilizing force. In line with the *evolutionary finance* literature (see e.g. Evstigheev et al. (2009), and Blume and Easley (2009)) the authors analyze the survival of agents in a financial market and prove that both types of agents can survive in the long run. This study is based on the assumption that the dividend process is i.i.d. and it has been developed under a market maker scenario. Several numerical simulations show complexity in the wealth dynamics, which is mainly due to the new assumption about wealth redistribution.

In order to better investigate the role of the switching mechanism in determining the long-run evolution of the system, we consider the same switching mechanism as in Brianzoni et al. (2010b) but we allow for a growing dividend

process and assume that agents trade in a Walrasian equilibrium framework. Related papers that consider growing dividends are Brock and Hommes (1997), Hommes (2002) and Chiarella et. al. (2006). Moreover, the market is populated by rational traders and chartists and, in line with existing agent-based models, we will see that both types of agents can survive in the market in the long run. More in detail, the analysis will mainly focus on the wealth evolution in the long run and its strong correlation with the fraction of agents using the same strategy.

The paper is organized as follows. We first present the general framework describing an asset pricing model where agents use different beliefs about future price. As in Briazoni et al. (2010b), we obtain a three-dimensional piecewise-smooth system able to characterize the evolution of the distribution of wealth when traders switch from one group to the other.

As a result we show that the system admits a continuum of steady states located on the border of the phase space. As the final system is quite difficult to be studied analytically, the analysis is developed from the numerical point of view. Differently from the Heterogeneous Agent Models, our diagrams show that the wealth distribution can fluctuate for intermediate values of the intensity of choice, measuring how fast agents choose between different predictors. This result is due to the switching mechanism introduced, which involves complexity in the long run wealth distribution. We also observe strong positive correlation between the evolution of the fraction of agents using a certain strategy and its relative wealth, that can be considered as success indicators for a given strategy. Hence we present the frequency distribution of the observed correlation coefficients, calculated from a great number of numerically generated data series, which support our obtained results.

2 The model

Consider an economy composed of one risky asset paying a random dividend y_t at time t and one risk free asset with constant risk free rate $r = R - 1 > 0$ and denote by p_t the price (ex dividend) per share of the risky asset at time t . In order to describe the wealth's dynamics we assume that all agents belonging to the same group agree to share their wealth whenever an agent joins the group (or leaves it). According to such an assumption, the wealth of agent type h at time t , denoted by $\bar{w}_{h,t}$, is given by the total wealth of group h in the fraction of agents belonging to this group. Hence, the dynamics of the wealth of investor h is described by the following equation:

$$W_{h,t+1} = (1 - z_{h,t})\bar{w}_{h,t}R + z_{h,t}\bar{w}_{h,t}(1 + \rho_{t+1}) = \bar{w}_{h,t}[R + z_{h,t}(\rho_{t+1} - r)] \quad (1)$$

where $z_{h,t}$ is the fraction of wealth that agent-type h invests in the risky asset and $\rho_{t+1} - r = \frac{p_{t+1} + y_{t+1} - (1+r)p_t}{p_t}$ is the excess return in period $t + 1$.

The individual demand function $z_{h,t}$ derives from the maximization problem of the expected utility of $W_{h,t+1}$, i.e. $z_{h,t} = \max_{z_{h,t}} E_{h,t}[u_h(W_{h,t+1})]$, where $E_{h,t}$ is the belief of investor-type h about the conditional expectation, based on the available information set of past prices and dividends. Since each agent is assumed to have a CRRA utility function, investors' optimal decisions depend on their wealth. Following Chiarella and He (2001), the optimal (approximated) solution is given by:

$$z_{h,t} = \frac{E_{h,t}[\rho_{t+1} - r]}{\lambda_h \sigma_h^2} \quad (2)$$

where λ_h is the relative risk aversion coefficient and $\sigma_h^2 = \text{Var}_{h,t}[\rho_{t+1} - r]$ is the belief of investor h about the conditional variance of excess returns.

We consider a market populated by two groups of agents where the fraction $n_{h,t}$ of traders using strategy h at time t will be updated according to a performance measure $\phi_{h,t}$ (as in Brianzoni et. al. 2010b). Hence, the adaptation of beliefs, i.e. the dynamics of the fractions $n_{h,t}$ of different trader types, is given by:

$$n_{h,t+1} = \frac{\exp[\beta(\phi_{h,t} - C_h)]}{Z_{t+1}}, \quad Z_{t+1} = \sum_h \exp[\beta(\phi_{h,t} - C_h)] \quad (3)$$

where the parameter β is the *intensity of choice* measuring how fast agents choose between different predictors and $C_h \geq 0$ are the costs for strategy h . When β increases, more and more agents use the predictor with the highest fitness. In the extreme case $\beta = +\infty$ all agents choose the strategy with the highest fitness, in the other extreme case $\beta = 0$ no switching at all takes place and both fractions are equal to $\frac{1}{2}$.

Let us define the performance measure $\phi_{h,t}$. We assume that at time t the fitness measure used for strategy selection is given by:

$$\phi_{h,t} = z_{h,t} E_{h,t}[\rho_{t+1} - r]. \quad (4)$$

Agents revise their beliefs in a boundedly rational way in the sense that, at any time, most agents choose the predictor which generated the best performance. In other words, the fraction $n_{h,t+1}$ of traders using strategy h at time $t + 1$ will be updated according to $\phi_{h,t}$.

Finally, as in Brock and Hommes (1998), we define the difference in fractions at time t , i.e. $m_t = n_{1,t} - n_{2,t}$, so that $n_{1,t} = \frac{1+m_t}{2}$ and $n_{2,t} = \frac{1-m_t}{2}$. As a consequence:

$$m_{t+1} = \tanh \left[\frac{\beta}{2} (\phi_{1,t} - \phi_{2,t} - C_1 + C_2) \right]. \quad (5)$$

The wealth dynamics

We consider a market populated by two groups of agents, where agents can move from group i to group j while both movements are not simultaneously possible. This assumption can be better understood while considering the net flow of moving agents. Define the difference in the fraction of agents of type h from time t to time $t + 1$ as $\Delta n_{h,t+1} = n_{h,t+1} - n_{h,t}$, $h = 1, 2$, hence $\Delta n_{1,t+1} = -\Delta n_{2,t+1}$. Observe that $\Delta n_{1,t+1} \geq 0$ (i.e. $m_{t+1} \geq m_t$), iff $\Delta n_{1,t+1}$ fraction of agents moves from group 2 to group 1 at time $t + 1$.

In order to describe the wealth dynamics of each group, we define the *wealth of group h* as: $\tilde{W}_{h,t} = n_{h,t}\bar{w}_{h,t}$, $h = 1, 2$.

As in Brianzoni et al. (2010b) we obtain

$$\tilde{W}_{1,t+1} = \begin{cases} \Delta n_{1,t+1}W_{2,t+1} + n_{1,t}W_{1,t+1} = \\ = n_{1,t}(W_{1,t+1} - W_{2,t+1}) + n_{1,t+1}W_{2,t+1}, & \text{if } m_{t+1} \geq m_t \\ n_{1,t+1}W_{1,t+1}, & \text{if } m_{t+1} < m_t \end{cases}, \quad (6)$$

and:

$$\tilde{W}_{2,t+1} = \begin{cases} n_{2,t+1}W_{2,t+1} & \text{if } m_{t+1} \geq m_t \\ n_{2,t}(W_{2,t+1} - W_{1,t+1}) + n_{2,t+1}W_{1,t+1} & \text{if } m_{t+1} < m_t \end{cases}. \quad (7)$$

Finally, we define the relative wealth of each group as the wealth of group h in the total wealth:

$$w_{h,t} = \tilde{W}_{h,t} / \sum_h \tilde{W}_{h,t}$$

where $\tilde{W}_{h,t} = n_{h,t}\bar{w}_{h,t}$ and $h = 1, 2$.

In the following, we will consider the dynamics of the difference in the relative wealths:

$$w_t := w_{1,t} - w_{2,t}.$$

To this end, we recall (6) and (7) and analyze both the cases $m_{t+1} \geq m_t$ and $m_{t+1} < m_t$.

Recalling Brianzoni et al. (2010b) the dynamics of the state variable w_t is described by the following system:

$$w_{t+1} = \begin{cases} \frac{F_1}{G} + 1 & \text{if } m_{t+1} \geq m_t \\ \frac{F_2}{G} - 1 & \text{if } m_{t+1} < m_t \end{cases} \quad (8)$$

where: $F_1 = -2\frac{1-m_{t+1}}{1-m_t}(1-w_t)[R+z_{2,t}(\rho_{t+1}-r)]$, $F_2 = 2\frac{1+m_{t+1}}{1+m_t}(1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]$ and $G = (1-w_t)[R+z_{2,t}(\rho_{t+1}-r)] + (1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]$.

The switching mechanism leads to a continuous piecewise smooth function.

Equilibrium price

The number of shares at price p_t that investor h wishes to hold is given by $N_{h,t}^D = \frac{z_{h,t}\bar{w}_{h,t}}{p_t}$. Summing the demands of all investors gives the aggregate demand:

$$N_t^D = \frac{n_{1,t}z_{1,t}\bar{w}_{1,t} + n_{2,t}z_{2,t}\bar{w}_{2,t}}{p_t}.$$

The supply of shares is assumed to be fixed $N_t^S = N$. Hence the Market Clearing Equation (MCE) at time t , $N_t^D = N_t^S$, is given by:

$$\frac{n_{1,t}z_{1,t}\bar{w}_{1,t} + n_{2,t}z_{2,t}\bar{w}_{2,t}}{p_t} = N.$$

It is well-known that the CRRA framework with heterogeneous agents is difficult to be explored. In order to obtain an analytically-tractable model we focus on the case of zero net supply of shares, as in Brock and Hommes (1998) and Chiarella and He (2001).

In the case of zero net supply of shares the MCE becomes:

$$n_{1,t}z_{1,t}\bar{w}_{1,t} + n_{2,t}z_{2,t}\bar{w}_{2,t} = 0$$

after dividing for $\tilde{W}_{1,t} + \tilde{W}_{2,t}$ we obtain the following equation: $z_{1,t}w_{1,t} + z_{2,t}w_{2,t} = 0$, which can be rewritten in terms of the state variable w_t :

$$z_{1,t}(1 + w_t) + z_{2,t}(1 - w_t) = 0. \quad (9)$$

Specializing the MCE to the case of one type of agents, it is possible to get a bench mark notion of the rational expectations "fundamental solution" p_t^* , i.e. a long-run market clearing price path which would be obtained under homogeneous beliefs on expected excess return (see Brock and Hommes (1998), Chiarella et al.(2006)).

Differently from Brianzoni et al. (2010b), in which a market-maker scenario is investigated under the assumption of an i.i.d. dividend process, we allow for a growing dividend process described by $E_t(y_{t+1}) = (1 + g)y_t$, with $0 \leq g < r$. In this case, the fundamental solution is:

$$p_t^* = \frac{(1 + g)y_t}{r - g}$$

and the fundamental evolves over time according to $E_t(p_{t+1}^*) = (1 + g)p_t^*$. Notice that *along the fundamental path* the expected yield and the capital gain are given by $E_t[y_{t+1}/p_t^*] = r - g$ and $E_t[(p_{t+1}^* - p_t^*)/p_t^*] = g$, hence $E_t[\rho_{t+1} - r] = 0$.

Trading strategies

In order to obtain the final system, we focus on the case of a market populated by two groups of agents, where type 1 are rational agents and type 2 are chartists. Define the excess return as $x_{t+1} = \rho_{t+1} - r$, then rational agents have perfect foresight and they are able to compute x_{t+1} correctly. Hence, the rational predictor is defined as $E_{1,t}[x_{t+1}] = x_{t+1}$. Differently, type-2 agents are chartists, who use simple linear trading rules. More precisely, we assume that $E_{2,t}[x_{t+1}] = ax_t$ with $a \in \mathbb{R}$. Furthermore we assume that rational agents pay a cost $C > 0$, while chartists' beliefs are freely available.

Considering our expectation schemes, the MCE (9) becomes:

$$\frac{x_{t+1}}{\lambda_1 \sigma_1^2} (1 + w_t) + \frac{ax_t}{\lambda_2 \sigma_2^2} (1 - w_t) = 0. \quad (10)$$

1. **Case** $w_t \neq -1$.

Formula (10) implies that the state variable x_{t+1} evolves according to the following equation:

$$x_{t+1} = -\frac{\lambda_1 \sigma_1^2}{\lambda_2 \sigma_2^2} \frac{1 - w_t}{1 + w_t} ax_t. \quad (11)$$

Consider now the wealth dynamics in the case $m_{t+1} \geq m_t$. After substituting (11) into the first equation of System (8) we obtain:

$$w_{t+1} = -\frac{1}{R} \frac{1 - m_{t+1}}{1 - m_t} (1 - w_t) \left[R + \frac{ax_t}{\lambda_2 \sigma_2^2} \left(-\frac{\lambda_1 \sigma_1^2}{\lambda_2 \sigma_2^2} \frac{1 - w_t}{1 + w_t} ax_t \right) \right] + 1.$$

Also, using relation $1 - \tanh(x) = \frac{2}{e^{2x} + 1}$ and Equation (5),

$$w_{t+1} = \frac{2(1-w_t)}{R(1-m_t)\{\exp[\beta(\phi_{1,t}-\phi_{2,t}-C)]+1\}} \left[\left(\frac{a}{\lambda_2 \sigma_2^2} \right)^2 \lambda_1 \sigma_1^2 \frac{1-w_t}{1+w_t} x_t^2 - R \right] + 1.$$

Consider now the case $m_{t+1} < m_t$ of System (8):

$$w_{t+1} = \frac{1}{R} \frac{1 + m_{t+1}}{1 + m_t} (1 + w_t) \left[R + \frac{x_{t+1}^2}{\lambda_1 \sigma_1^2} \right] - 1.$$

Using of (11) and simplification of the corresponding expression lead to equation:

$$w_{t+1} = \frac{1}{R} \frac{1 + m_{t+1}}{1 + m_t} (1 + w_t) \left[R + \frac{\lambda_1 \sigma_1^2}{(\lambda_2 \sigma_2^2)^2} a^2 \left(\frac{1 - w_t}{1 + w_t} \right)^2 x_t^2 \right] - 1.$$

Finally, using relation $1 + \tanh(x) = \frac{2}{e^{-2x} + 1}$ and Equation (5), one obtains the following equation for the wealth w_{t+1} :

$$w_{t+1} = \frac{2(1+w_t)}{R(1+m_t)\{exp[-\beta(\phi_{1,t}-\phi_{2,t}-C)]+1\}} \left[R + \left(\frac{a}{\lambda_2\sigma_2^2} \right)^2 \lambda_1\sigma_1^2 \left(\frac{1-w_t}{1+w_t} \right)^2 x_t^2 \right] - 1.$$

2. **Case** $w_t = -1$.

From Equation (10) it immediately follows $x_t = 0$, as a consequence $z_{2,t} = 0$ and the demand of the second group is zero. On the other hand, the wealth of the first group at time t is zero. This implies that the demand of the first group is zero as well. Being both the aggregate demand and the supply of shares zero at time t , it follows: $x_{t+1} = x_t$, and from Equation (5) we obtain $m_{t+1} = \tanh \left\{ -\frac{\beta}{2}C \right\}$.

Consider now the wealth dynamics. Remembering that $w_t = -1$ and $x_{t+1} = x_t = 0$, from System (8) we immediately obtain:

$$w_{t+1} = \begin{cases} -2\frac{1-m_{t+1}}{1-m_t} + 1 & \text{if } m_{t+1} \geq m_t \\ -1 & \text{if } m_{t+1} < m_t \end{cases}.$$

2.1 The final system

Our model is given by a noisy nonlinear system, since dividends and fundamental price are stochastic processes. Netherless, in this work we focus on the dynamics of the deterministic skeleton of the model by assuming that dividends evolve according to their expected value. Notice that under the assumption of a growing dividend process y_t such that $E_t(y_{t+1}) = (1 + g)y_t = (r - g)p_t^*$, the excess return in period $t + 1$ can be rewritten as:

$$x_{t+1} = \frac{p_{t+1} + y_{t+1} - (1 + r)p_t}{p_t} = \frac{p_{t+1} - (1 + r)p_t}{p_t} + \frac{(r - g)p_t^*}{p_t}.$$

Our adaptive asset pricing and wealth dynamics model is written in terms of the state variables x_t , m_t and w_t as follows:

1. if $w_t \neq -1$

$$T_1 : \begin{cases} x_{t+1} = f(x_t, w_t) = -\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2} \frac{1-w_t}{1+w_t} ax_t \\ m_{t+1} = g(x_t, w_t) = \tanh \left[\frac{\beta}{2} (\Delta\phi_t - C) \right] \\ w_{t+1} = h(x_t, m_t, w_t) = \begin{cases} h_1(x_t, m_t, w_t), & \forall m_{t+1} \geq m_t \\ h_2(x_t, m_t, w_t), & \forall m_{t+1} < m_t \end{cases} \end{cases} \quad (12)$$

where:

$$h_1(x_t, m_t, w_t) = \frac{2(1-w_t)}{R(1-m_t)\{\exp[\beta(\Delta\phi_t-C)]+1\}} \left[\left(\frac{a}{\lambda_2\sigma_2^2} \right)^2 \lambda_1\sigma_1^2 \frac{1-w_t}{1+w_t} x_t^2 - R \right] + 1$$

$$h_2(x_t, m_t, w_t) = \frac{2(1+w_t)}{R(1+m_t)\{\exp[-\beta(\Delta\phi_t-C)]+1\}} \left[\left(\frac{a}{\lambda_2\sigma_2^2} \right)^2 \lambda_1\sigma_1^2 \left(\frac{1-w_t}{1+w_t} \right)^2 x_t^2 + R \right] - 1$$

and $\Delta\phi_t = \phi_{1,t} - \phi_{2,t} = \frac{x_{t+1}^2}{\lambda_1\sigma_1^2} - \frac{a^2x_t^2}{\lambda_2\sigma_2^2} = \left[\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2} \left(\frac{1-w_t}{1+w_t} \right)^2 - 1 \right] \frac{a^2x_t^2}{\lambda_2\sigma_2^2}$

2. if $w_t = -1$

$$T_2 : \begin{cases} x_{t+1} = x_t = 0 \\ m_{t+1} = \tanh \left\{ -\frac{\beta}{2}C \right\} \\ w_{t+1} = \begin{cases} -2\frac{1-m_{t+1}}{1-m_t} + 1, & \forall m_{t+1} \geq m_t \\ -1, & \forall m_{t+1} < m_t \end{cases} \end{cases} \quad (13)$$

Hence the final dynamical system is given by $T = T_1 \cup T_2$.

3 Wealth dynamics and correlation between variables

3.1 Fixed points

About the fixed points owned by the final system the following lemma holds.

Lemma 1. *For all $a \in \mathbb{R}$ every point $E(x^* = 0, m^* = \tanh \left\{ -\frac{\beta}{2}C \right\}, w = w^*)$ is a fixed point in which the long-run wealth distribution is given by any constant $w^* \in [-1, 1]$. In other words, the system T admits a continuum of steady states which are located in a one-dimensional subset (a straight line) of the phase space.*

Proof. If $a < 0$, then from the MCE (10) we obtain: $\{x = 0\} \cup \{w = \frac{-A-B}{A-B}\}$, with $A = \frac{1}{\lambda_1\sigma_1^2}$ and $B = \frac{a}{\lambda_2\sigma_2^2}$. Considering System (12), $x = 0$ implies $m = \tanh \left\{ -\frac{\beta}{2}C \right\}$ and $w = \bar{w}$, $\forall \bar{w} \in (-1, 1]$, while $w = \frac{-A-B}{A-B}$ implies $x = 0$ and $m = \tanh \left\{ -\frac{\beta}{2}C \right\}$; considering System (13) we obtain $x = 0$, $m = \tanh \left\{ -\frac{\beta}{2}C \right\}$ and $w = -1$. If $a > 0$, then the MCE admits a unique admissible solution given by $x = 0$. As in the previous case, from System (12) we obtain $m = \tanh \left\{ -\frac{\beta}{2}C \right\}$ and $w = \bar{w}$, $\forall \bar{w} \in (-1, 1]$, while System (13) implies $m = \tanh \left\{ -\frac{\beta}{2}C \right\}$ and $w = -1$. Finally, if $a = 0$, then the MCE implies: $\{x = 0\} \cup \{w = -1\}$. Again, for $x = 0$ we obtain $m = \tanh \left\{ -\frac{\beta}{2}C \right\}$ and

$w = w^*$, $\forall w^* \in (-1, 1]$ (by System (12)), and $m = \tanh\{-\frac{\beta}{2}C\}$ and $w = -1$ (by System (13)). For $w = -1$ we immediately have $m = \tanh\{-\frac{\beta}{2}C\}$ and $x = 0$.

□

Observe that any steady state E is characterized by $x^* = 0$, i.e. $(1+r)p_{t-1} - (r-g)p_{t-1}^* = p_t$, for all t , hence $(1+r)p_{t-1} - (1+r)p_{t-1}^* + (1+g)p_{t-1}^* = p_t$ and, being $p_t^* = (1+g)p_{t-1}^*$ we obtain: $p_t - p_t^* = (1+r)(p_{t-1} - p_{t-1}^*) \forall t$. As a consequence $p_t = p_t^* \forall t$ implies $x_t = 0 \forall t$; this means that some (infinitely many) steady states defined by Lemma 1 are fundamental equilibria. The presence of a continuum of steady states depends on the fact that the expectation schemes are equivalent in equilibrium. Moreover, both assets are equivalent in terms of return and all the agents earn the same return independent of their investment shares.

Observe that all the equilibria are on the boundary of the phase space and by investigating the eigenvalues of the system, it is possible to see that they are non-hyperbolic. Hence, as our system is quite difficult to be studied analytically, in what follows we present some numerical simulations useful to show the existence of complicated dynamics in the evolution of the wealth distribution.

3.2 Numerical evidences

The main purpose of our study is to show that the switching mechanism introduced in the wealth dynamics is a source of complexity in the evolution of the wealth distribution. In order to show such a result, as our final system is quite difficult to be studied analytically, we firstly present some numerical simulations.

In what follows we assume $\lambda_1 = \lambda_2 = \sigma_1^2 = \sigma_2^2 = 1$, $R = 1.02$ and $C = 0.5$ while we let parameters β and a vary. In Figure 1 we present the one-dimensional bifurcation diagram of the state variables m_t and w_t with respect to a while in Figure 2 we present the one-dimensional bifurcation diagram of the state variables m_t and w_t with respect to β .

Besides observing the well-known period doubling bifurcations, we found border collision bifurcations, i.e. non-canonical bifurcations which occur in piecewise smooth maps.² In fact, successions of periodic windows occur. As in Brianzoni et al. (2010b), this type of bifurcation is involved by the wealth dynamics which is described by a piecewise smooth function.

²These bifurcations have mainly been studied in the context of piecewise linear maps. Hommes and Nusse (1991) showed, for instance, that a “period three to period two” bifurcation occurs for a class of piecewise linear maps. More recent interesting contributions on this topic are from Jain and Banerjee (2003), Avrutin et al. (2008a, 2008b, 2009).

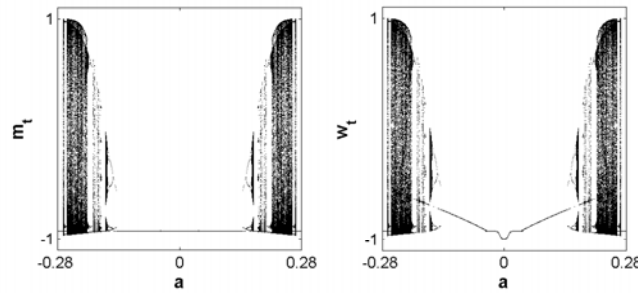


Figure 1: One-dimensional bifurcation diagrams of the state variables m_t and w_t with respect to a for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and $\beta = 6.5$.

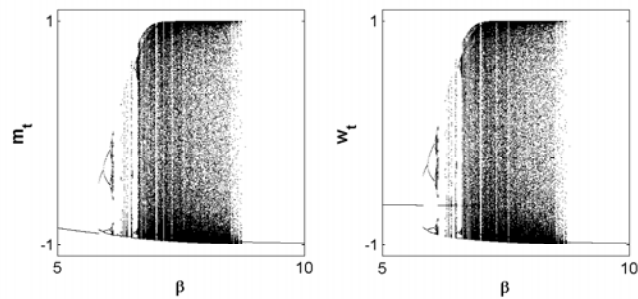


Figure 2: One-dimensional bifurcation diagrams of the state variables m_t and w_t with respect to β for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and parameter value $a = 0.2$.

While observing the diagrams presented in Figure 1, an interesting property emerges, i.e. the same trajectories are associated to opposite values of parameter a , for a given initial condition and once fixed the values of the other parameters of the model. In order to better understand the role of the parameter a , define $\alpha = -|a|$ and $\xi_0 = |x_0|$ so that $T^h(\xi_0, m_0, w_0; \alpha) = (\xi_h, m_h, w_h; \alpha)$ is the h -iterate of $(\xi_0, m_0, w_0; \alpha)$ under the application of T . Then the following proposition holds.

Proposition 2. (a) If $a < 0$ and $x_0 > 0$ then $T^h(x_0, m_0, w_0; a) = (\xi_h, m_h, w_h; \alpha)$;

(b) If $a < 0$ and $x_0 < 0$ then $T^h(x_0, m_0, w_0; a) = (-\xi_h, m_h, w_h; \alpha)$;

(c) If $a > 0$ and $x_0 > 0$ then $T^h(x_0, m_0, w_0; a) = ((-1)^h \xi_h, m_h, w_h; \alpha)$;

(d) If $a > 0$ and $x_0 < 0$ then $T^h(x_0, m_0, w_0; a) = ((-1)^{h+1}\xi_h, m_h, w_h; \alpha)$.

Proof. Consider systems (12) and (13). Then it is easy to observe that $g(x_t, m_t, w_t, a) = g(\xi_t, m_t, w_t, \alpha)$ and $h(x_t, m_t, w_t, a) = h(\xi_t, m_t, w_t, \alpha)$. Furthermore the following property holds:

- (a) If $a < 0$ and $x_0 > 0$ then $f(x_t, m_t, w_t, a) = f(\xi_t, m_t, w_t, \alpha)$;
- (b) If $a < 0$ and $x_0 < 0$ then $f(x_t, m_t, w_t, a) = f(-\xi_t, m_t, w_t, \alpha) = -f(\xi_t, m_t, w_t, \alpha)$;
- (c) If $a > 0$ and $x_0 > 0$ then $f(x_t, m_t, w_t, a) = f(\xi_t, m_t, w_t, -\alpha) = f(\xi_t, m_t, w_t, \alpha)$ if $t = 2k$ while $f(x_t, m_t, w_t, a) = f(\xi_t, m_t, w_t, -\alpha) = -f(\xi_t, m_t, w_t, \alpha)$ if $t = 2k + 1, \forall k \in \mathbb{N}$;
- (d) If $a > 0$ and $x_0 < 0$ then $f(x_t, m_t, w_t, a) = f(-\xi_t, m_t, w_t, -\alpha) = -f(\xi_t, m_t, w_t, \alpha)$ if t is pair while $f(x_t, m_t, w_t, a) = f(-\xi_t, m_t, w_t, -\alpha) = f(\xi_t, m_t, w_t, \alpha)$ if $t = 2k + 1, \forall k \in \mathbb{N}$.

□

Observe that $a = 0$ or $x_0 = 0$ implies $T^h(x_0, m_0, w_0; a) = (0, m_h, w_h; a), (\forall h = 1, 2, \dots)$.

According to the previous proposition, given any initial condition, the evolution of both the fraction of type- h agents and their wealth is not affected by the signs of x_0 and a . More precisely, given the parameter values and other conditions, the evolutions of m_t and w_t only depend on the the distance between x_0 and its equilibrium value $x = 0$. Furthermore, opposite values of the parameter a only affect the sign of the elements of the sequence of the iterates $\{x_0, x_1, x_2, \dots\}$.

In order to better explain the meaning of the previous proposition, we present in Figure 3 the attractor owned by system T in the plain (x_t, m_t) for $a = 0.2$. Observe that such an attractor is symmetric w.r.t. the straightline $x_t = 0$. We numerically obtain two different values of β , that is $\beta_1 \simeq 7.0959$ and $\beta_2 \simeq 8.495$ such that, given the same initial condition of Figure 2, the attractor is strange or characterized by very high period cycles for $\beta \in (\beta_1, \beta_2)$. In Figure 3 we consider a β -value belonging to such an interval.

Differently, for $\beta > \beta_2$ or for $\beta < \beta_1$ and small enough, the system converges to a fixed point (fundamental steady state).

In Figure 4 the attractor is depicted for the opposite value of a : by comparing the two figures the meaning of Proposition 2 is clarified.

We now focus on the long-run evolution of the state variables m_t and w_t . More in detail, the model of Brianzoni et al. (2010b) leads to different success indicators: a certain strategy can be successful in terms of the number of agents using it or in terms of the wealth of the respective group, but not necessarily in both senses as these two state variables perform in an *independent* way (for

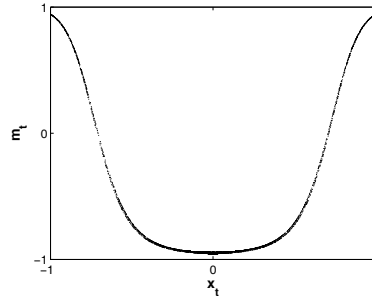


Figure 3: Attractor in the (x_t, m_t) plain for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and parameter values $a = 0.2$ and $\beta = 7.09595$.

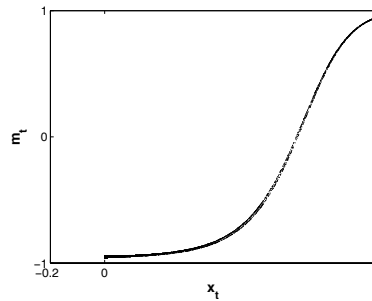


Figure 4: Attractor in the (x_t, m_t) plain for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and parameter values $a = -0.2$ and $\beta = 7.09595$.

instance the case in which most of the traders belongs to group i while most of the wealth is owned by type- j ($\neq i$) traders can be observed). On the contrary, in our framework the state variables w_t and m_t are characterized by strong correlation.

In Figures 5 and 6 the evolution of the two state variables versus time is depicted for two different initial conditions: it is quite immediate to observe that the two state variables show a similar feature and the same evidence has been numerically confirmed in a great number of simulations. First of all both variables seem to increase (or decrease) from a period to another (see again Figures 5 and 6); secondly the final attractor of both variables seem to have the same qualitative structure (as it can be observed while looking at Figures 1 and 2). An important question is whether this relation emerges also from the quantitative point of view.

In order to give an answer to such a question in Figure 7 we plot in the plane (m_t, w_t) the i -iterate $(m(i), w(i))$, for $i = 1, 2, \dots, 150$, numerically obtained under the application of System T , using the same parameter values and initial

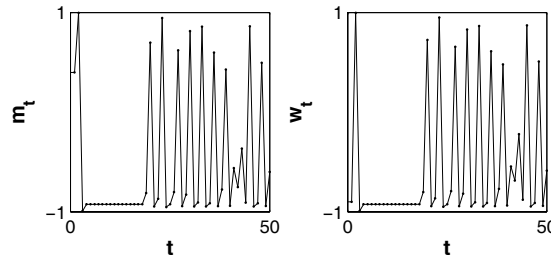


Figure 5: State variables m_t and w_t versus time for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and parameter values $a = 0.25$ and $\beta = 6.5$.

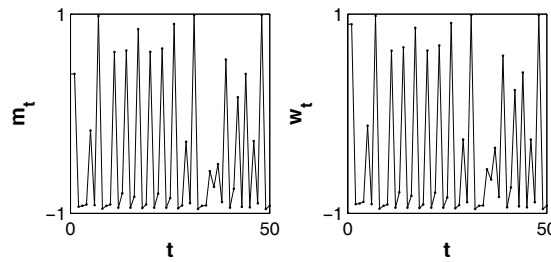


Figure 6: State variables m_t and w_t versus time for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = 0.9$ and parameter values $a = 0.25$ and $\beta = 6.5$.

conditions of Figures 5 and 6. Observe that we are not depicting the attractor of the system since we are interested in the location of the two state variables in the plane over time (not only in the long run). It is immediate to observe that most of the points is located *near* the diagonal, proving that, at any time t , the fraction of agents using strategy i is very close to the fraction of wealth owned by group i .

Define \underline{m} the vector of the first K calculated values for the state variable $m(t)$ and \underline{w} the vector of the first K calculated values for the state variable $w(t)$ (being K sufficiently high). Then, given the same initial condition and parameter values, the correlation coefficient $R(\underline{m}, \underline{w})$ measures the strength and the sign of the linear relationship between the two state variables generated by the iteration of System T . In the cases plotted in Figure 7 the correlation coefficients are respectively 0.8467 in case (a) and 0.9319 in case (b). Hence in the two studied cases a strong positive correlation between the two state variables emerge.

In order to conclude in a more general case, we can repeat the previous

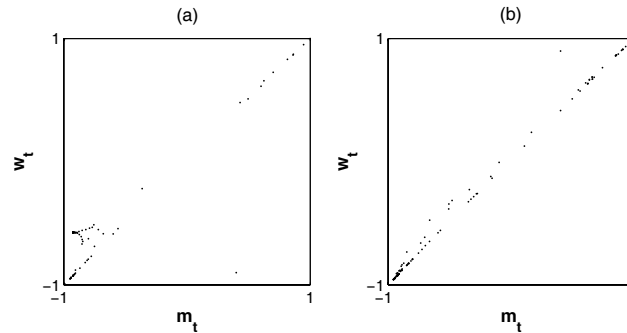


Figure 7: Points $(m(i), w(i))$ for $i = 1, 2, \dots, 150$ obtained by the iteration of T for parameter values $a = 0.25$ and $\beta = 6.5$ and the initial conditions: (a) $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and (b) $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = 0.9$.

procedure. For instance, we now fix parameters a and β and also the initial level of the state variable x_t at the following values: $a = 0.25$, $\beta = 6.5$, $x_0 = 0.5$. We let the initial conditions for the other state variables m_0 and w_0 vary in the following way: m_0 and w_0 can assume 21 different values being the elements of a linearly spaced vector between -1 and 1 , hence m_0 and w_0 can be equal to $-1, -0.9, \dots, 0, \dots, 0.9, 1$. By considering each possible initial value for both state variables we obtain a set of 441 initial conditions. At any initial condition, by the iteration of System T , the vectors \underline{m} and \underline{w} are obtained and the correlation coefficient is calculated (in such a case we calculate the first 1500 iterations of the map). Finally the distribution of the correlation coefficients can be obtained.

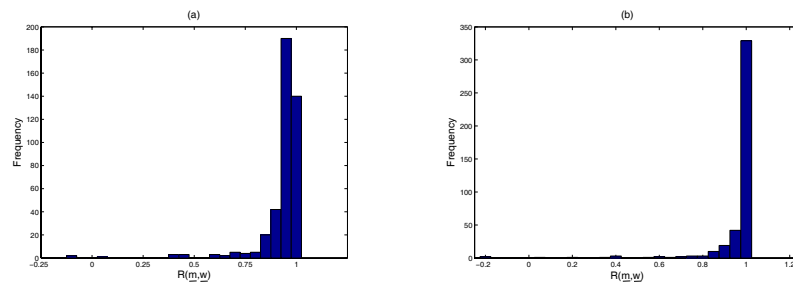


Figure 8: (a) Frequency distribution of the correlation coefficient for different values of m_0 and w_0 and $\beta = 6.5$. (b) Frequency distribution of the correlation coefficient for different values of m_0 and w_0 and $\beta = 7$.

In Figure 8 panel (a) the observed frequency distribution of coefficient $R(\underline{m}, \underline{w})$ for different choices of the initial condition for m_0 and w_0 is plotted.

The mean of such a distribution is 0.9297 while the variance is 0.0145 proving the strong correlation between the dynamics of the two success indicators: the fraction of agents using a given strategy and its correspondent wealth.

In Figure 8 panel (b) the same procedure has been used to obtain the frequency distribution of the correlation coefficients for $\beta = 7$. Also in this case a strong correlation between the two state variables is obtained; the distribution has mean 0.9621 and variance 0.0165.

This procedure has been repeated for other parameter selections and for different grids of initial conditions: the evidence of the existence of a strong positive correlation is confirmed. This feature is in agreement with the concept of *survivor*, as the agent with positive wealth share (see Amir et al. (2005) and Anufriev and Dindo (2010) among others).

3.3 The role of the switching mechanism

By performing a lot of simulations another important result which emerge in our model is that the wealth dynamics is able to produce complex features for intermediate values of the intensity of choice β . In order to prove that the switching mechanism is responsible for complicated dynamics in the wealth evolution, we investigate what happens in the limiting case, when the intensity of choice β goes to infinity.

For $\beta \rightarrow +\infty$, we obtain:

1. if $\Delta\phi_0 - C = \left[\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2} \left(\frac{1-w_0}{1+w_0} \right)^2 - 1 \right] \frac{a^2x_0^2}{\lambda_2\sigma_2^2} - C > 0$ then $x_{t+1} = -\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2} \frac{1-w_t}{1+w_t} ax_t$,
 $m_{t+1} = 1$, $w_{t+1} = 1$. In this case the system will converge to the globally stable fixed point $E_1(x^* = 0, m^* = 1, w^* = 1)$,
2. if $\Delta\phi_0 - C = \left[\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2} \left(\frac{1-w_0}{1+w_0} \right)^2 - 1 \right] \frac{a^2x_0^2}{\lambda_2\sigma_2^2} - C < 0$ then $x_{t+1} = -\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2} \frac{1-w_t}{1+w_t} ax_t$,
 $m_{t+1} = -1$, $w_{t+1} = -1$. In this case the system will converge to the globally stable fixed point $E_2(x^* = 0, m^* = -1, w^* = -1)$.

In other words, depending on the initial condition, both the fractions of agents and the relative wealths converge in the long run, i.e. some classes survive, some classes do not. The surviving group is defined by the trading strategy which performs better at the initial time. In fact, when β goes to infinity, all agents use the predictor with the highest fitness and they accumulate the total market wealth, the two assets become equivalent in terms of return.

Such a result enable us to conclude that the switching mechanism introduced in our model is a source of complexity for intermediate values of β differently from what happens in previous models in which the wealth dynamics is not studied and complexity is observed when β is high enough.

4 Conclusions

By overcoming models based on fixed proportions of agents, we have established an adaptive model able to characterize the evolution of the distribution of wealth when agents switch between different trading strategies.

The model is developed in the discrete time setting of standard portfolio theory, in that agents are allowed to revise their portfolios over any time interval. Both expectation feedback and adaptiveness are common features of recent heterogeneous asset pricing models. In addition, our model is able to characterize the evolution of the distribution of wealth when agents switch from an old strategy to a new strategy, according to their past performances.

Our main assumption is that all agents belonging to a group agree to share their wealth whenever an agent joins the group (or leaves it). In such a way we can characterize equilibrium price and wealth evolution among heterogeneous agents. Moreover, it leads to different *success indicators* of each strategy, the difference in the fraction of agents and the difference in the relative wealths of the groups. In other words, a certain strategy can be successful in terms of the number of agents using it or in terms of the wealth of the respective group.

It is found that the presence of heterogeneous agents and switching mechanism leads the stationary model to have multiple steady-states for wealth distribution. As far as stability is concerned, the intensity of choice (β) plays an important role: the dynamics tend to be more complicated for intermediate values of this parameter. This result is unexpected and interesting since, in heterogeneous agent models, complexity increases as the intensity of choice increases. In our framework, a different outcome depends on the new assumption on the wealth redistribution, as proved by studying the limiting case (β goes to infinity), showing simple dynamics and convergence of relative wealths in the long-run.

In line with existing heterogeneous agent models, our framework is able to explain some economic issues such as: the survival of *irrational* agents, the role of market forces (wealth-driven selection) and some stylized facts (fluctuations, excess of volatility, bubbles and crashes).

An interesting further contribution would be to consider the possibility of a mutual switch between strategies.

The same wealth dynamics has been implemented by Brianzoni et al. (2010b) in a market maker scenario and under an i.i.d. dividend process. The new contribution of our model comes from the use of the market clearing condition to determine the asset price. Moreover, we have allowed for a growing dividend process. Differently from Brianzoni et al. (2010b), in our model a strong positive correlation characterizes the evolution of the difference in the relative wealths and the evolution of the difference in the fractions of agents. This result has been obtained by performing several frequency distributions of

the correlation coefficients.

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Received: July, 2011