Applied Mathematical Sciences, Vol. 6, 2012, no. 12, 579 - 587

New Exact Solutions for the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli Equation

Ying Li and Desheng Li

School of Mathematics and System Science
Shenyang Normal University, Liaoning Province, 110034, P.R. China
liyingpop103@163.com

Abstract

By using the modified Clarkson-Kruskal (CK) direct method, we construct a Bäcklund transformation of (2+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation. Through Bäcklund transformation, we get some new solutions of the BLMP equation, extend the results of previous literature.

Mathematics Subject Classification: 35C05

Keywords: BLMP equation, the modified Clarkson-Kruskal (CK) direct method, Bäcklund transformation, exact solutions

1 Introduction

The symmetry study is one of the most useful methods in each branches of natural science, especially in integrable systems.

In [2], Clarkson and Kruskal (CK) proposed a simple direct method to investigate the symmetry reduction of a nonlinear system without using any Lie group theory and got all the possible similarity reductions. Recently Lou *et al.* modified the CK direct method and searched a more simple method (called the modified CK direct method) to construct Bäcklund transformation[6]. By means of the modified CK direct method, many equations have been discussed, such as (2+1)-dimensional KdV-Burgers equation, Kadomtsev-Petviashvili (KP) equation, Ablowitz-Kaup-Newell-Segur (AKNS) equation, dispersive long-wave equations, Camassa-Holm equation, breaking soliton equation and (1+1)-dimensional shallow water wave system related to the KdV equation[5,6,9,10,13,14]. Literature [8] discusses (1+1)-dimensional Whitham-Broer-Kaup (WBK) equation in shallow water by using the classical Lie approach and the direct method, and gives five types of similarity reductions, including three types of singularity points. It also proves the special similarity reduction containing the general similarity reduction. In [11], they derive the symmetry group theorem of Whitham-Broer-Kaup-Like (WBKL) equations by using the modified CK direct method. Basing on the theorem and the adjoint equations, they obtain the conservation laws of WBKL equations. Literature [3] proposes a simple Bäcklund transformation of potential Boiti-Leon-Manna-Pempinelli (BLMP) system by using the standard truncated Painlevé expansion and symbolic computation, and a solution of the potential BLMP system with three arbitrary functions is given. Then the very rich localized coherent structures and periodical coherent structures are revealed. In [7], Basing on the binary Bell polynomials, the bilinear form for the BLMP equation is obtained. The new exact solutions are presented with an arbitrary function in y, and soliton interaction properties are discussed by the graphical analysis. Literature [4] obtains the symmetry, similarity reductions and new solutions of (2+1)-dimensional BLMP equation. These solutions include rational function solutions, double-twisty function solutions, Jacobi oval function solutions and triangular cycle solutions. Then it finds many conservation laws of BLMP equation. Literature [12] discusses BLMP equation and generalized breaking soliton equations by using the exponential function, and obtains some new exact solutions of the equations.

In this paper, we use the modified CK direct method to find the exact solutions of Boiti-Leon-Manna-Pempinelli (BLMP) equation

$$u_{yt} - 3u_x u_{xy} - 3u_y u_{xx} + u_{xxxy} = 0. \tag{1}$$

which was derived by Boiti M, Leon J, *et al.* during their researched a Korteweg-de Vries (KdV) equation through weak Lax pairs relations[1].

According to the modified CK direct method, some new exact solutions are obtained.

2 Bäcklund Transformation of (2+1)-dimensional BLMP Equation

We assume that Eq.(1) has the Bäcklund transformation of the form

$$u(x, y, t) = W(x, y, t, U(p, q, r)) \tag{2}$$

where $W, p = p(x, y, t), q = q(x, y, t)$ and $r = r(x, y, t)$ are functions to be determined by requiring that $U(p, q, r)$ satisfies the same (2+1)-dimensional

BLMP equation as $u(x,y,t)$ with the transformation $\{u,x,y,t\} \to \{U,p,q,r\}$, i.e.

$$U_{pppq} = 3U_pU_{pq} + 3U_qU_{pp} - U_{qr} \tag{3}$$

It is enough to prove that the transformation(2) has the simple form, says,

$$u(x,y,t) = \alpha(x,y,t) + \beta(x,y,t)U(p,q,r) \tag{4}$$

Substituting Eq.(4) into Eq.(1), we find that the coefficients of U_{pppp} and U_{qqqq} are $\beta p_x^3 p_y$ and $\beta q_x^3 q_y$, respectively. Both coefficients should be zero, and any one of $p_x = 0, q_y = 0$, or $\beta = 0$ will make the transformation(4) meaningless, so the remining choice is

$$p_y = 0, q_x = 0, \quad i.e. \quad p = p(x,t), q = q(y,t) \tag{5}$$

Substituting Eq.(5) into Eq.(1), and eliminating U_{pppq} by Eq.(3), we have

$$-3\beta^2 p_x q_y r_x U_p U_{qr} - 3\beta^2 p_x^2 r_y U_r U_{pp} + F(x,y,t,U,U_p,\cdots) = 0 \tag{6}$$

where $F(x,y,t,U,U_p,\cdots)$ is independent of U_pU_{qr} and U_rU_{pp}, vanishing the coefficients of U_pU_{qr} and U_rU_{pp}, yields

$$r_x = 0, r_y = 0, \quad i.e. \quad r = r(t) \tag{7}$$

Using condition (7), Eq.(6) is reduced to

$$-3\beta\beta_x p_x q_y U U_{pq} - 3\alpha_y \beta p_x^2 U_{pp} + \beta q_y q_t U_{qq}$$

$$+3\beta p_x^2 q_y (p_x - \beta) U_p U_{pq} + 3\beta p_x^2 q_y (p_x - \beta) U_q U_{pp} + F_1(x,y,t,U,U_p,\cdots) = 0 \tag{8}$$

where $F_1(x,y,t,U,U_p,\cdots)$ is independent of $UU_{pq}, U_{pp}, U_{qq}, U_pU_{pq}$ and U_qU_{pp}, so Eq.(8) shows us

$$\beta_x = 0, \alpha_y = 0, q_t = 0, \beta = p_x, \quad i.e. \quad \beta = \beta(t), q = q(y), \alpha = \alpha(x,t) \tag{9}$$

Using condition (9), Eq.(8) is reduced to

$$\beta q_y (r_t - p_x^3) U_{qr} + \beta q_y (p_t - 3\alpha_x p_x) U_{pq} + (\beta_t q_y + \beta q_{yt} - 3\alpha_{xx}\beta q_y) U_q = 0 \tag{10}$$

By vanishing the coefficients of Eq.(10), the equations of functions α, p, q, r read

$$\beta q_y (r_t - p_x^3) = 0, \beta q_y (p_t - 3\alpha_x p_x) = 0, \beta_t q_y + \beta q_{yt} - 3\alpha_{xx}\beta q_y = 0 \tag{11}$$

Solving the determining equations (11), we can get

$$\alpha(x,t)=\frac{x(\frac{d}{dt}F_2(t)x+2\frac{d}{dt}F_3(t))}{6F_2(t)}+F_4(t),\quad \beta(t)=C_1F_2(t)$$

$$p(x,t)=F_2(t)x+F_3(t),\quad q(y)=F_1(y),\quad r(t)=\int F_2^3(t)\,dt+C_2 \tag{12}$$

where $F_1(y)$ is arbitrary function of $y, F_2(t), F_3(t)$ and $F_4(t)$ are arbitrary functions of t, C_1, C_2 are arbitrary constants. Then we conclude the following theorem.

Theorem 2.1 *If $U = U(x,y,t)$ is a solution of the (2+1)-dimensional BLMP equation (1),then so is*

$$u(x,y,t)=\frac{x(\frac{d}{dt}F_2(t)x+2\frac{d}{dt}F_3(t))}{6F_2(t)}+F_4(t)+C_1F_2(t)U(p,q,r) \tag{13}$$

with (12), where $F_1(y)$ is arbitrary function of $y, F_2(t), F_3(t)$ and $F_4(t)$ are arbitrary functions of t, C_1, C_2 are arbitrary constants.

3 New Exact Solutions of (2+1)-dimensional BLMP Equation

Via Theorem 1, many kinds of new exact solutions from the known ones of (2+1)-dimensional BLMP equation (1) are presented. For instance, if we choose the following solutions obtained in Ref. [3] as seed solutions

$$u(x,y,t)=\frac{1}{3}\int\frac{s'(t)+m'''(x)}{m'(x)}\,dx-\frac{2m'(x)}{f} \tag{14}$$

where $f=m(x)+n(y)+s(t)$.

We can get new solutions of Eq.(1) as follows

$$u(x,y,t)=\frac{x(\frac{d}{dt}F_2(t)x+2\frac{d}{dt}F_3(t))}{6F_2(t)}+F_4(t)+C_1F_2(t)[\frac{1}{3}\int\frac{s'(r)+m'''(p)}{m'(p)}\,dp-\frac{2m'(p)}{f}] \tag{15}$$

where $f=m(p)+n(q)+s(r)$, and p,q,r are determined by (12).

Taking different expressions of parameters m,n,s, we can obtain many kinds of solutions to Eq.(1) from the general form solutions (15).

Case 1. $m(p) = e^p, n(q) = e^q, s(r) = 3r^2$, the seed solution is

$$u(p,q,r) = \frac{p}{3} - 2re^{-p} - \frac{2e^p}{e^p + e^q + 3r^2} \tag{16}$$

As a result, the corresponding new exact solution of Eq.(1) can be written as

$$u(x,y,t) = \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) + C_1F_2(t)[\frac{p}{3} - 2re^{-p} - \frac{2e^p}{e^p + e^q + 3r^2}]$$
$$= \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) - 2C_1F_2(t)(\int F_2^3(t)\,dt + C_2)e^{-(F_2(t)x+F_3(t))}$$
$$+\frac{C_1F_2(t)}{3}(F_2(t)x + F_3(t)) - \frac{2C_1F_2(t)e^{(F_2(t)x+F_3(t))}}{e^{(F_2(t)x+F_3(t))} + e^{F_1(y)} + 3(\int F_2^3(t)\,dt + C_2)^2}$$

Case 2. $m(p) = p, n(q) = q^2, s(r) = e^r$, the seed solution is

$$u(p,q,r) = \frac{p}{3}e^r - \frac{2}{p + q^2 + e^r} \tag{17}$$

We can get the new exact solutions of Eq.(1)

$$u(x,y,t) = \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) + C_1F_2(t)[\frac{p}{3}e^r - \frac{2}{p + q^2 + e^r}]$$
$$= \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + \frac{C_1F_2(t)}{3}(F_2(t)x + F_3(t))e^{\int F_2^3(t)\,dt + C_2}$$
$$+F_4(t) - \frac{2C_1F_2(t)}{(F_2(t)x + F_3(t)) + F_1^2(y) + e^{\int F_2^3(t)\,dt + C_2}}$$

Case 3. $m(p) = \sin p, n(q) = \sin q, s(r) = r^2$, the seed solution is

$$u(p,q,r) = \frac{2r}{3}\ln|\csc p - \cot p| - \frac{p}{3} - \frac{2\cos p}{\sin p + \sin q + r^2} \tag{18}$$

We can get the new exact solutions of Eq.(1)

$$u(x,y,t) = \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) + C_1F_2(t)[\frac{2r}{3}\ln|\csc p - \cot p| - \frac{p}{3}$$
$$-\frac{2\cos p}{\sin p + \sin q + r^2}] = \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} - \frac{C_1F_2(t)}{3}(F_2(t)x + F_3(t))$$
$$+F_4(t) + \frac{2C_1F_2(t)}{3}(\int F_2^3(t)\,dt + C_2)\ln|\csc(F_2(t)x + F_3(t)) - \cot(F_2(t)x$$
$$+F_3(t))| - \frac{2C_1F_2(t)\cos(F_2(t)x + F_3(t))}{\sin(F_2(t)x + F_3(t)) + \sin F_1(y) + (\int F_2^3(t)\,dt + C_2)^2}$$

Case 4. $m(p) = e^p, n(q) = q^2, s(r) = \sin r$, the seed solution is

$$u(p, q, r) = \frac{p}{3} - \frac{1}{3}e^{-p}\cos r - \frac{2e^p}{e^p + q^2 + \sin r} \tag{19}$$

We can get the new exact solutions of Eq.(1)

$$\begin{aligned} u(x, y, t) &= \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) + C_1F_2(t)[\frac{p}{3} - \frac{1}{3}e^{-p}\cos r \\ &-\frac{2e^p}{e^p + q^2 + \sin r}] = \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + \frac{C_1F_2(t)}{3}(F_2(t)x + F_3(t)) \\ &+F_4(t) - \frac{C_1F_2(t)}{3}e^{-(F_2(t)x+F_3(t))}\cos(\int F_2^3(t)\,dt + C_2) \\ &-\frac{2C_1F_2(t)e^{F_2(t)x+F_3(t)}}{e^{F_2(t)x+F_3(t)} + F_1^2(y) + \sin(\int F_2^3(t)\,dt + C_2)} \end{aligned}$$

Case 5. $m(p) = e^p, n(q) = e^q + e^{-q}, s(r) = \cos r$, the seed solution is

$$u(p, q, r) = \frac{p}{3} + \frac{1}{3}e^{-p}\sin r - \frac{2e^p}{e^p + e^q + e^{-q} + \cos r} \tag{20}$$

We can get the new exact solutions of Eq.(1)

$$\begin{aligned} u(x, y, t) &= \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) + C_1F_2(t)[\frac{p}{3} + \frac{1}{3}e^{-p}\sin r \\ &-\frac{2e^p}{e^p + e^q + e^{-q} + \cos r}] = \frac{x(\frac{d}{dt}F_2(t)x + 2\frac{d}{dt}F_3(t))}{6F_2(t)} + F_4(t) \\ &+\frac{C_1F_2(t)}{3}(F_2(t)x + F_3(t)) + \frac{C_1F_2(t)}{3}e^{-(F_2(t)x+F_3(t))}\sin(\int F_2^3(t)\,dt \\ &+C_2) - \frac{2C_1F_2(t)e^{F_2(t)x+F_3(t)}}{e^{F_2(t)x+F_3(t)} + e^{F_1(y)} + e^{-F_1(y)} + \cos(\int F_2^3(t)\,dt + C_2)} \end{aligned}$$

We select functions $F_2(t) = e^t, F_3(t) = \sin t, F_4(t) = 0$ parameters $t = 0$ and constants $C_1 = 0.1, C_2 = 0.2$. According to the different selection for the function $F_1(y)$, we draw the figures of case 2.

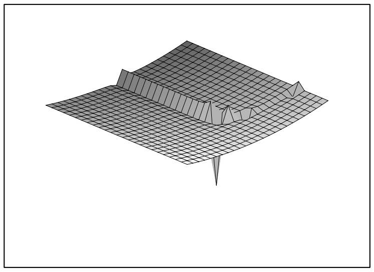

Fig.1 $F_1(y) = (2^y)^{\frac{1}{2}}$

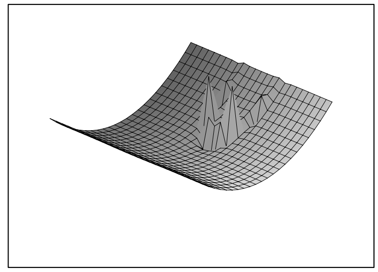

Fig.2 $F_1(y) = (2^{y^2})^{\frac{1}{2}}$

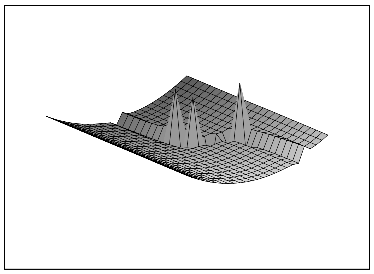

Fig.3 $F_1(y) = (\arctan y)^{\frac{1}{2}}$

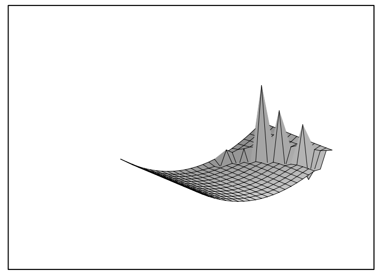

Fig.4 $F_1(y) = (\ln y)^{\frac{1}{2}}$

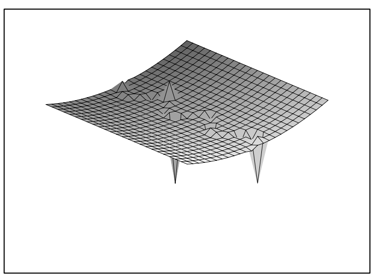

Fig.5 $F_1(y) = (\sin y)^{\frac{1}{2}}$

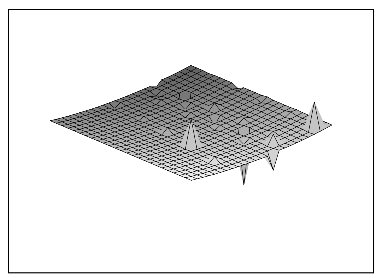

Fig.6 $F_1(y) = (\tan y)^{\frac{1}{2}}$

In fact, if we choose proper functions $F_1(y), F_2(t), F_3(t), F_4(t)$, we can portray more graphs of solutions for Eq.(1). In this paper, we only choose one of new solutions.

4 Main Results

In this paper, by applying the modified CK direct method, we investigate the Bäcklund transformation of the BLMP equation. Then theorem 1 gives the relationship between new solutions and old ones of Eq.(1). Further, many new exact solutions of Eq.(1) are obtained.

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Received: July, 2011