

On the Coupling of Homotopy Perturbation and Laplace Transformation for System of Partial Differential Equations

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Abstract

The aim of this article is to propose a new algorithm, namely homotopy perturbation transform algorithm (HPTA). This new algorithm provides us with a convenient way to find exact solution with less computation as compared with standard homotopy perturbation algorithm (HPA). The proposed algorithm is used to handle linear and nonlinear partial differential equations. Explicatory examples are incorporated to reveal the high accuracy and fast convergence of proposed new algorithm.

Keywords: Homotopy perturbation transform algorithm, Homotopy perturbation method, Linear and nonlinear partial differential equations

1 Introduction

Systems of partial differential equations have attracted much attention in a variety of applied sciences. The general ideas and the essential features of these systems are of wide applicability. These systems were formally derived to describe wave propagation, to control the shallow water waves, and to examine the chemical reaction-diffusion model of Brusselator. The method of characteristics, the Riemann invariants, Adomian decomposition method [1], Homotopy perturbation method [2 – 5] Homotopy analysis method [6 – 7] and Laplace decomposition method [8 – 14]

were the commonly used methods. In this work, we will use Homotopy perturbation transform algorithm introduced by Yasir et al [15 – 16]. Majid et al [17] solved exponential stretching sheet equation with the help of this method. This new algorithm basically illustrates how two powerful algorithms, homotopy perturbation method and Laplace decomposition method can be combined and used to approximate the solutions of the nonlinear partial differential equations by manipulating the homotopy perturbation method.

2 Homotopy perturbation transform algorithm

In this section, we present a Homotopy perturbation transform algorithm for solving of partial differential equations written in an operator form

$$\begin{cases} L_t u + L_x v + N_1(u, v) = h_1, \\ L_t v + L_x u + N_2(u, v) = h_2, \end{cases} \quad (2.1)$$

with the initial conditions

$$\begin{cases} u(x, 0) = k_1(x), \\ v(x, 0) = k_2(x). \end{cases} \quad (2.2)$$

where L_t and L_x are considered as a first-order partial differential operators and N_1 and N_2 are nonlinear operators and h_1 and h_2 are forcing terms. The method consists of first applying the Laplace transform to both sides of equations in system (2.1) and then by using initial conditions (2.2), we have:

$$\mathcal{L} [L_t u] + \mathcal{L} [L_x v] + \mathcal{L} [N_1(u, v)] = \mathcal{L} [h_1], \quad (2.3)$$

$$\mathcal{L} [L_t v] + \mathcal{L} [L_x u] + \mathcal{L} [N_2(u, v)] = \mathcal{L} [h_2]. \quad (2.4)$$

Using the differential property of Laplace transform and initial conditions, we have

$$s\mathcal{L} [u(x, t)] - u(x, 0) + \mathcal{L} [L_x v] + \mathcal{L} [N_1(u, v)] = \mathcal{L} [h_1(x, t)], \quad (2.5)$$

$$s\mathcal{L} [v(x, t)] - v(x, 0) + \mathcal{L} [L_x u] + \mathcal{L} [N_2(u, v)] = \mathcal{L} [h_2(x, t)]. \quad (2.6)$$

Or

$$\mathcal{L}[u] = \frac{k_1(x)}{s} + \frac{1}{s}\mathcal{L}[g_{h1}] - \frac{1}{s}\mathcal{L}[L_x v] - \frac{1}{s}\mathcal{L}[N_1(u, v)], \quad (2.7)$$

$$\mathcal{L}[v] = \frac{k_2(x)}{s} + \frac{1}{s}\mathcal{L}[h_2] - \frac{1}{s}\mathcal{L}[L_x u] - \frac{1}{s}\mathcal{L}[N_2(u, v)]. \quad (2.8)$$

Applying inverse Laplace transform on both sides of Eqs. (2.7) – (2.8), we get

$$u = G_1(x) - \mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L}[N_1(u, v)] + \mathcal{L}[L_x v]] \right], \quad (2.9)$$

$$v = G_2(x) - \mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L}[N_2(u, v)] + \mathcal{L}[L_x u]] \right], \quad (2.10)$$

where $G_1(x)$ and $G_2(x)$ represents the terms arising from source term and prescribe initial conditions. According to standard homotopy perturbation method the solution u and v can be expanded into infinite series as

$$u = \sum_{m=0}^{\infty} p^m u_m, \quad v = \sum_{m=0}^{\infty} p^m v_m. \quad (2.11)$$

where $p \in [0, 1]$ is an embedding parameter. Also the nonlinear term N_1 and N_2 can be written as

$$N_1(u, v) = \sum_{m=0}^{\infty} p^m H_{1m}(u, v), \quad N_2(u, v) = \sum_{m=0}^{\infty} p^m H_{2m}(u, v). \quad (2.12)$$

where H_{1m} and H_{2m} are the He's polynomials [11]. By substituting Eqs. (2.11) and (2.12) in Eqs. (2.9) – (2.10), the solution can be written as

$$\sum_{m=0}^{\infty} p^m u_m = G_1(x) - p \left(\mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L}[H_{1m}] + \mathcal{L}[L_x v]] \right] \right), \quad (2.13)$$

$$\sum_{m=0}^{\infty} p^m v_m = G_2(x) - p \left(\mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L}[H_{2m}] + \mathcal{L}[L_x u]] \right] \right). \quad (2.14)$$

In Eqs. (2.13) – (2.14), H_{1m} and H_{2m} are He's polynomials can be generated by several means. Here we used the following recursive formulation

$$H_m(u_0, \dots, u_m) = \frac{1}{m!} \frac{\partial^m}{\partial p^m} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad m = 0, 1, 2, \dots \quad (2.15)$$

Equating the terms with identical powers in p in Eqs. (2.13) – (2.14), we obtained the following approximations

$$\begin{aligned}
 p^0 & : u_0 = G_1(x), \\
 p^1 & : u_1 = -\mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L} [H_{10}] + \mathcal{L} [L_x v_0]] \right], \\
 p^2 & : u_2 = -\mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L} [H_{11}] + \mathcal{L} [L_x v_1]] \right], \\
 & \vdots
 \end{aligned} \tag{2.16}$$

$$\begin{aligned}
 p^0 & : v_0 = G_2(x), \\
 p^1 & : v_1 = -\mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L} [H_{20}] + \mathcal{L} [L_x u_0]] \right], \\
 p^2 & : v_2 = -\mathcal{L}^{-1} \left[\frac{1}{s} [\mathcal{L} [H_{21}] + \mathcal{L} [L_x u_1]] \right], \\
 & \vdots
 \end{aligned} \tag{2.17}$$

The best approximation for the solutions are

$$u = \lim_{p \rightarrow 1} u_m = u_0 + u_1 + u_2 + \dots \tag{2.18}$$

$$v = \lim_{p \rightarrow 1} v_m = v_0 + v_1 + v_2 + \dots \tag{2.19}$$

3 Numerical Experiments

In this part we present three examples. The first and second examples are considered to illustrate the method for homogeneous and inhomogeneous linear systems of partial differential equations. While in third example we solve a inhomogeneous nonlinear partial differential equation.

3.1 The homogeneous linear system

Let us consider the homogeneous linear system of PDEs:

$$u_t + v_x - (u + v) = 0, \tag{3.1}$$

$$v_t + u_x - (u + v) = 0, \tag{3.2}$$

with initial conditions

$$u(x, 0) = \sinh(x), \quad v(x, 0) = \cosh(x). \quad (3.3)$$

Applying Laplace transform algorithm we have

$$su(x, s) - u(x, 0) = -\mathcal{L}[v_x] + \mathcal{L}[(u + v)], \quad (3.4)$$

$$sv(x, s) - v(x, 0) = -\mathcal{L}[u_x] + \mathcal{L}[(u + v)]. \quad (3.5)$$

$$u(x, s) = \frac{u(x, 0)}{s} - \frac{1}{s} \mathcal{L}[v_x - (u + v)], \quad (3.6)$$

$$v(x, s) = \frac{v(x, 0)}{s} - \frac{1}{s} \mathcal{L}[u_x - (u + v)]. \quad (3.7)$$

Using given initial condition Eqs. (3.6) – (3.7), becomes

$$u(x, s) = \frac{\sinh(x)}{s} - \frac{1}{s} \mathcal{L}[v_x - (u + v)], \quad (3.8)$$

$$v(x, s) = \frac{\cosh(x)}{s} - \frac{1}{s} \mathcal{L}[u_x - (u + v)]. \quad (3.9)$$

Applying inverse Laplace transform to Eqs. (3.8) – (3.9), we get

$$u(x, t) = \sinh(x) - \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L}[v_x - (u + v)] \right], \quad (3.10)$$

$$v(x, t) = \cosh(x) - \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L}[u_x - (u + v)] \right]. \quad (3.11)$$

The Homotopy perturbation transform algorithm (HPTA) assumes a series solutions of the functions $u(x, t)$ and $v(x, t)$ is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x, t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x, t). \quad (3.12)$$

Using Eq. (3.12) into Eqs. (3.10) – (3.11), we get:

$$\sum_{m=0}^{\infty} p^m u_m(x, t) = \sinh(x) - p \left(\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[\begin{array}{c} (\sum_{m=0}^{\infty} p^m v_m(x, t))_x - \\ \left(\sum_{m=0}^{\infty} p^m v_m(x, t) + \right) \\ \sum_{m=0}^{\infty} p^m v_m(x, t) \end{array} \right] \right] \right), \quad (3.13)$$

$$\sum_{m=0}^{\infty} p^m v_m(x, t) = \cosh(x) - p \left(\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[\begin{array}{c} (\sum_{m=0}^{\infty} p^m u_m(x, t))_x - \\ \left(\sum_{m=0}^{\infty} p^m v_m(x, t) + \right) \\ \sum_{m=0}^{\infty} p^m v_m(x, t) \end{array} \right] \right] \right). \quad (3.14)$$

From Eqs. (3.9) – (3.12), comparing like powers of p we get:

$$p^0 : \begin{cases} u_0(x, t) = \sinh(x), \\ v_0(x, t) = \cosh(x), \end{cases} \quad (3.15)$$

$$p^1 : \begin{cases} u_1(x, t) = -t \sinh(x), \\ v_1(x, t) = -t \cosh(x), \end{cases} \quad (3.16)$$

$$p^2 : \begin{cases} u_2(x, t) = \frac{t^2}{2!} \sinh(x), \\ v_2(x, t) = \frac{t^2}{2!} \cosh(x), \end{cases} \quad (3.17)$$

⋮

and so on for other components. Using Eqs. (2.18) – (2.19), the series solutions are therefore given by

$$\begin{cases} u(x, t) = \sinh(x) \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \right) + \cosh(x) \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right), \\ v(x, t) = \cosh(x) \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \right) + \sinh(x) \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right), \end{cases} \quad (3.18)$$

using the Taylor expansion for $\sinh t$ and $\cosh t$, we can find the exact solutions

$$\begin{cases} u(x, t) = \sinh(x + t), \\ v(x, t) = \cosh(x + t). \end{cases} \quad (3.19)$$

3.2 The inhomogeneous linear system

Consider the inhomogeneous linear system

$$u_t - v_x - (u - v) = -2, \quad (3.20)$$

$$v_t + u_x - (u - v) = -2, \quad (3.21)$$

with initial conditions

$$u(x, 0) = 1 + e^x, \quad v(x, 0) = -1 + e^x. \quad (3.22)$$

Taking the Laplace transform on both sides of Eqs. (3.20) – (3.21) then, by using the differentiation property of Laplace transform and initial conditions (3.22) gives

$$u(x, s) = \frac{1}{s} + \frac{e^x}{s} - \frac{2}{s^2} + \frac{1}{s} \mathcal{L} [v_x + (u - v)], \quad (3.23)$$

$$v(x, s) = -\frac{1}{s} + \frac{e^x}{s} - \frac{2}{s^2} + \frac{1}{s} \mathcal{L} [(u - v) - u_x]. \quad (3.24)$$

Taking the inverse Laplace transform of both sides of the (3.23) and (3.24), we have

$$u(x, t) = 1 + e^x - 2t + \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} [v_x + (u - v)] \right], \quad (3.25)$$

$$v(x, t) = -1 + e^x - 2t + \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} [(u - v) - u_x] \right]. \quad (3.26)$$

By using homotopy perturbation transform algorithm (HPTA) the solutions functions $u(x, t)$ and $v(x, t)$ is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x, t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x, t). \quad (3.27)$$

Invoking Eq. (3.27) in Eqs. (3.25) – (3.26), we have

$$\sum_{m=0}^{\infty} p^m u_m(x, t) = 1 + e^x - 2t + p \left(\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[\left(\begin{array}{c} (\sum_{m=0}^{\infty} p^m v_m(x, t))_x + \\ \left(\sum_{m=0}^{\infty} p^m v_m(x, t) - \right) \\ \sum_{m=0}^{\infty} p^m v_m(x, t) \end{array} \right) \right] \right] \right), \quad (3.28)$$

$$\sum_{m=0}^{\infty} p^m v_m(x, t) = -1 + e^x - 2t + p \left(\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[\left(\begin{array}{c} \left(\sum_{m=0}^{\infty} p^m v_m(x, t) - \right) \\ \sum_{m=0}^{\infty} p^m v_m(x, t) \\ - (\sum_{m=0}^{\infty} p^m u_m(x, t))_x \end{array} \right) \right] \right] \right) \quad (3.29)$$

On comparing the coefficients of like powers of p we get require solution components:

$$p^0 : \begin{cases} u_0(x, t) = 1 + e^x - 2t, \\ v_0(x, t) = -1 + e^x - 2t, \end{cases} \quad (3.30)$$

$$p^1 : \begin{cases} u_1(x, t) = te^x + 2t, \\ v_1(x, t) = -te^x + 2t, \end{cases} \quad (3.31)$$

$$p^2 : \begin{cases} u_2(x, t) = \frac{t^2}{2!}e^x, \\ v_2(x, t) = \frac{t^2}{2!}e^x, \end{cases} \quad (3.32)$$

and so on for other components. Using (2.18) – (2.19), the series solutions are therefore given by

$$\begin{cases} u(x, t) = 1 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right), \\ v(x, t) = -1 + e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right), \end{cases} \quad (3.33)$$

that converges to the exact solutions

$$\begin{cases} u(x, t) = 1 + e^{x+t}, \\ v(x, t) = -1 + e^{x-t}. \end{cases} \quad (3.34)$$

3.3 The inhomogeneous nonlinear system

Consider the system of inhomogeneous partial differential equations

$$u_t + u_x v + u = 1, \quad (3.35)$$

$$v_t - uv_x - v = 1, \quad (3.36)$$

with initial conditions

$$u(x, 0) = e^x, \quad v(x, 0) = e^{-x}. \quad (3.37)$$

Taking the Laplace transform on both sides of Eqs. (3.35) – (3.36) then, by using the differentiation property of Laplace transform and initial conditions (3.37) gives

$$u(x, s) = \frac{e^x}{s} + \frac{1}{s^2} - \frac{1}{s} \mathcal{L} [u_x v + u], \quad (3.38)$$

$$v(x, s) = \frac{e^{-x}}{s} + \frac{1}{s^2} + \frac{1}{s} \mathcal{L} [uv_x + v]. \quad (3.39)$$

Applying inverse Laplace transform of both sides of the (3.38) and (3.39), we have

$$u(x, t) = e^x + t - \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[u_x v + \sum_{m=0}^{\infty} p^m u_m(x, t) \right] \right], \quad (3.40)$$

$$v(x, t) = e^{-x} + t + \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[uv_x + \sum_{m=0}^{\infty} p^m v_m(x, t) \right] \right]. \quad (3.41)$$

We represent $u(x, t)$ and $v(x, t)$ by the infinite series (2.11) then, inserting these series into both sides of Eqs. (3.38) – (3.39) yields

$$\sum_{m=0}^{\infty} p^m u_m(x, t) = e^x + t - p \left(\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[\begin{array}{l} \sum_{m=0}^{\infty} p^m H_{1m}(u, v) \\ + \sum_{m=0}^{\infty} p^m u_m(x, t) \end{array} \right] \right] \right), \quad (3.42)$$

$$\sum_{m=0}^{\infty} p^m v_m(x, t) = e^{-x} + t + p \left(\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left[\begin{array}{l} \sum_{m=0}^{\infty} p^m H_{2m}(u, v) \\ + \sum_{m=0}^{\infty} p^m v_m(x, t) \end{array} \right] \right] \right). \quad (3.43)$$

Where $H_{1m}(u, v)$ and $H_{2m}(u, v)$ are He's polynomials that represents nonlinear terms vu_x and uv_x respectively. We have a few terms of the He's polynomials for vu_x and uv_x , which are given by

$$\begin{aligned} H_{10}(u, v) &= v_0 u_{0x}, \\ H_{11}(u, v) &= v_1 u_{0x} + v_0 u_{1x}, \\ H_{12}(u, v) &= v_2 u_{0x} + v_1 u_{1x} + v_0 u_{2x}, \\ &\vdots \end{aligned} \quad (3.44)$$

$$\begin{aligned} H_{20}(u, v) &= u_0 v_{0x}, \\ H_{21}(u, v) &= u_1 v_{0x} + u_0 v_{1x}, \\ H_{22}(u, v) &= u_2 v_{0x} + u_1 v_{1x} + u_0 v_{2x}, \\ &\vdots \end{aligned} \quad (3.45)$$

Comparing the coefficients of like powers of p , we have

$$p^0 : \begin{cases} u_0(x, t) = e^x + t, \\ v_0(x, t) = e^{-x} + t, \end{cases} \quad (3.46)$$

$$p^1 : \begin{cases} u_1(x, t) = -\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L}[H_{10}(u, v) + u_0(x, t)] \right] = -t - \frac{t^2}{2!} - te^x - \frac{t^2}{2!}e^x, \\ v_1(x, t) = \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L}[H_{20}(u, v) + v_0(x, t)] \right] = -t - \frac{t^2}{2!} - te^{-x} - \frac{t^2}{2!}e^{-x}, \end{cases} \quad (3.47)$$

Preceeding in a similar manner, we have

$$p^2 : \begin{cases} u_2(x, t) = \frac{t^2}{2!} + t^2e^x, \\ v_2(x, t) = \frac{t^2}{2!} + t^2e^{-x}. \end{cases} \quad (3.48)$$

Similarly, we can find other components. The series solutions are therefore given by

$$\begin{cases} u(x, t) = e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right), \\ v(x, t) = e^{-x} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right). \end{cases} \quad (3.49)$$

By using the Taylor expansion for e^t and e^{-t} we can find the exact solutions of the above system od inhomogeneous nonlinear PDES

$$\begin{cases} u(x, t) = e^{x-t}, \\ v(x, t) = e^{-x+t}. \end{cases} \quad (3.50)$$

4 Concluding Remarks

Homotopy perturbation transform method has been known to be a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations and so on. Here we used this method for solving systems of partial differential equations. It is demonstrated that this method has the ability of solving systems of both linear and nonlinear partial differential equations. In Example 1 and 2, the system was a linear systems and we derived the exact solutions. For inhomogeneous nonlinear systems, we usually control nonlinearity with the help of He's polynomials and derive a exact solutions easily as in Example 3. Extension of the method for solving higher order systems of partial differential equations offers an excellent oppurtunity for future research.

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