

Geometrical Behavior in Elasticity Problems by Topological Optimization

Coumba DIALLO^{*,+}

^{*}Université Cheikh Anta DIOP de Dakar

⁺Laboratoire de Mathématiques de la Décision et d'Analyse Numérique
Ecole Doctorale de Mathématique et Informatique
coumbdiallo@yahoo.fr

Ibrahima FAYE⁺

Université de Bambey, Bp 30, Bambey
Ecole Doctorale de Mathématique et Informatique
UMMISCO-UMI,209, IRD, France
grandmbodj@hotmail.com

Diaraf SECK^{*,+}

Ecole Doctorale de Mathématique et Informatique
UMMISCO-UMI,209, IRD, France
diaraf.seck@ucad.edu.sn

Abstract

In this paper we study thermoelasticity and Michell trusses problems by using topological optimization tools. After the modelling of the thermoelasticity problem, we give the topological derivative in both Dirichlet and Neumann conditions. We will finish this work by giving a linking between the trusses Michell and topological optimization. And the last part of this work is devoted to numerical simulations.

Mathematics Subject Classification: 74P05, 74P10, 74P15, 65F30, 49, 49Q10

Keywords: Topological optimization, Topological gradient, Thermoelasticity, Elasticity, Michell trusses, Numerical simulations

1 Introduction

In this work we consider the system as the following form:

$$\left\{ \begin{array}{l} -\mu\Delta\vec{u}_\Omega - (\lambda + \mu)\overrightarrow{grad} \operatorname{div} \vec{u}_\Omega - 3k\alpha\overrightarrow{grad} \theta_\Omega = \vec{f} \quad \text{in } \Omega \\ -\Delta\theta_\Omega = g \quad \text{in } \Omega \\ \frac{\partial\theta_\Omega}{\partial n} = h \quad \text{on } \partial\Omega \\ \mathbf{B}\vec{u}_\Omega = v \quad \text{on } \partial\Omega. \end{array} \right. \quad (1)$$

\mathbf{B} is a Dirichlet or Neumann operator defined on $\partial\Omega$, $\vec{f} \in L^2(\Omega, \mathbb{R}^3)$, $g \in L^2(\Omega)$, $v \in L^2(\partial\Omega, \mathbb{R}^3)$ and $h \in L^2(\partial\Omega)$. This system models phenomena of thermoelasticity where the vector $\vec{u}_\Omega = (u_\Omega^1, u_\Omega^2, u_\Omega^3)$ is the deformations vector and θ_Ω is the temperature in the domain Ω . For additional details see for instance [4, 6].

In this model of thermoelasticity problem, we are interested by a problem of identification of the topological and geometrical variations of the materials. These effects can generate deformations and even cracks in a domain Ω under the constraints of the heat and loads.

In this work, we are going to study situations where cracks do not appear. Therefore we suppose that there is a reference (a target $u_0 = (u_0^1, u_0^2, u_0^3)$) for the deformations \vec{u}_Ω which satisfy a system of PDE. And then we are going to use the tools of topological optimization to do this study. Let us consider the functional J from which we evaluate the asymptotic expansion.

The aim of the topological sensitivity analysis is to obtain an asymptotic expansion of shape functional with respect to the creation of a small hole inside the domain. The principle is the following. One considers a cost function $j(\Omega) = J(\Omega, u_\Omega)$ where u_Ω is solution to a partial differential equation defined in the domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, a point $x_0 \in \Omega$ and a fixed domain $B \subset \mathbb{R}^d$, containing the origin. One looks for an asymptotic expansion of $j(\Omega \setminus \overline{(x_0 + \epsilon B)})$ when ϵ tends to zeros. In most cases, it reads in the form

$$j(\Omega \setminus \overline{(x_0 + \epsilon B)}) - j(\Omega) = f(\epsilon)g(x_0) + o(f(\epsilon)). \quad (2)$$

Here $f(\epsilon)$ is an explicit positive function going to zeros with ϵ and the function g , the topological gradient or topological derivative, is in general not easy to compute. Expression (2) is called the asymptotic analysis. Hence to minimize the criterion J , one must to create holes at some points \bar{x} where the topological gradient is lowest. The topological sensitivity analysis was introduced by Schumacher [15], Sokolowski and Zochowski [16] and Masmoudi [7] for the minimization of the compliance in linear elasticity. Actually the topological sensitivity was obtained for various domains. About topological optimization tools we refer the reader to [7, 8, 9, 14].

The paper is organized as follows: in section 2 after presenting the thermoelasticity problem by using topological optimization tools, we give the topological gradient in both Dirichlet and Neumann cases. In Section 3 we are interested by Michell trusses problem by using topological optimization tools. We show that this problem can be reduced to a particular case of topological thermoelasticity problem under some hypotheses. In fact we consider a decomposition of \vec{f} expressed as follows:

$$\vec{f} = \sum_{i,j=1}^n \lambda_{i,j} (\delta_{A_i} - \delta_{A_j}) \frac{A_i - A_j}{|A_i - A_j|} \quad (3)$$

where $\lambda_{i,j}$ are given constants parameters, A_i , $i = 1, \dots, n$ are given points in \mathbb{R}^3 and δ_{A_i} is the Dirac mass concentrated at the point A_i . And we transform the Michell trusses problem in an optimization problem. Then the results obtained show that the compliance $C(\Lambda, \Gamma)$ has to be minimal. For another interesting mathematical approach to study trusses Michell, we invite the reader to see the works due to Bouchitte et al [3]. In their study authors used optimal mass transportation theory and calculus of variations to introduce a new approach to give a meaning of trusses Michell problem. To be completed, one can see also the references in [3]. Section 4 is devoted to numerical simulations of the two problems.

2 Study of a thermoelasticity problem

2.1 Presentation of thermo elasticity problem

Let Ω , ω two regular open and bounded sets of \mathbb{R}^N , $N = 2, 3$ and let $x_0 \in \omega \subset \Omega$. ϵ is a small positive real. Let us defined the hall $\omega_\epsilon = x_0 + \epsilon\omega$ and the perturbed domain $\Omega(\epsilon) = \Omega \setminus \overline{\omega_\epsilon}$. For the theoretical study let us suppose $x_0 = 0$. Consider

$$\left\{ \begin{array}{l} -\mu\Delta\vec{u}_\Omega - 3k\alpha\overrightarrow{\text{grad}}\theta_\Omega = \vec{f} \text{ in } \Omega \\ \text{div}(u_\Omega) = 0 \text{ in } \Omega \\ -\Delta\theta_\Omega = g \text{ in } \Omega \\ \frac{\partial\theta_\Omega}{\partial n} = h \text{ on } \partial\Omega \\ \frac{\partial\vec{u}_\Omega}{\partial n} = v \text{ on } \partial\Omega. \end{array} \right. \quad (4)$$

In $\Omega(\epsilon) = \Omega \setminus \overline{\omega_\epsilon}$ we have

$$\left\{ \begin{array}{l} -\mu\Delta\vec{u}_{\Omega(\epsilon)} - 3k\alpha\overrightarrow{\text{grad}}\theta_{\Omega(\epsilon)} = \vec{f} \text{ in } \Omega(\epsilon) \\ \text{div}(u_{\Omega(\epsilon)}) = 0 \text{ in } \Omega(\epsilon) \\ -\Delta\theta_{\Omega(\epsilon)} = g \text{ in } \Omega(\epsilon) \\ \frac{\partial\theta_{\Omega(\epsilon)}}{\partial n} = h \text{ on } \partial\Omega \\ \frac{\partial\vec{u}_{\Omega(\epsilon)}}{\partial n} = v \text{ on } \partial\Omega \\ \vec{u}_{\Omega(\epsilon)} = 0 \text{ or } \frac{\partial\vec{u}_{\Omega(\epsilon)}}{\partial n} = 0 \text{ on } \partial\omega_\epsilon \\ \theta_{\Omega(\epsilon)} = 0 \text{ or } \frac{\partial\theta_{\Omega(\epsilon)}}{\partial n} = 0 \text{ on } \partial\omega_\epsilon. \end{array} \right. \quad (5)$$

In the following we set $\mu = 1$ without losing generality. In fact we can divide the first vectorial equation of (4) and (5) by μ . Let us consider the functional $J(\Omega, \vec{u}_\Omega, \theta_\Omega)$ defined by

$$\begin{aligned} J(\Omega, \vec{u}_\Omega, \theta_\Omega) = & \sum_{i=1}^3 \alpha_i \int_{\Omega} |u_i - u_0^i|^2 dx + \sum_i^3 \beta_i \int_{\Omega} |\nabla u_i|^2 dx \\ & + r \int_{\Omega} |\theta_\Omega - \theta_0|^2 dx + \delta \int_{\Omega} |\nabla\theta_\Omega|^2 dx \end{aligned} \quad (6)$$

where $\alpha_i, \beta_i, i = 1, 2, 3, r$ and δ are constants.

For any positive real $\epsilon > 0$ we consider $j(\epsilon) = J(\Omega(\epsilon), \vec{u}_{\Omega(\epsilon)}, \theta_{\Omega(\epsilon)})$ where $(\vec{u}_{\Omega(\epsilon)}, \theta_{\Omega(\epsilon)})$ is solution to (5) and $(\vec{u}_\Omega, \theta_\Omega)$ solution to (4).

The aim of this section is to detimine the asymptotic analysis of functional $j(\epsilon)$ as $\epsilon \rightarrow 0$.

Let $v = (\vec{u}_\Omega, \theta_\Omega)$, then problems (4) and (5) become respectively

$$\left\{ \begin{array}{l} -\Delta\vec{v}_\Omega + \mathcal{F}(\vec{v}_\Omega) = \vec{F} \text{ in } \Omega \\ \text{div}(\vec{u}_\Omega) = 0 \text{ in } \Omega \\ \frac{\partial\vec{v}_\Omega}{\partial n} = \tilde{h} \text{ on } \partial\Omega, \end{array} \right. \quad (7)$$

$$\left\{ \begin{array}{l} -\Delta\vec{v}_{\Omega(\epsilon)} + \mathcal{F}(\vec{v}_{\Omega(\epsilon)}) = \vec{F} \text{ in } \Omega(\epsilon) \\ \text{div}(\vec{u}_{\Omega(\epsilon)}) = 0 \text{ in } \Omega \\ \frac{\partial\vec{v}_{\Omega(\epsilon)}}{\partial n} = \tilde{h} \text{ on } \partial\Omega \\ \mathbf{B}\vec{v}_{\Omega(\epsilon)} = 0 \text{ on } \partial\omega_\epsilon, \end{array} \right. \quad (8)$$

where $\vec{F} = (\vec{f}, g)$, $\tilde{h} = (v, h)$, \mathbf{B} is a Dirichlet or Neumann condition on $\partial\omega_\epsilon$ and functional (6) becomes;

$$J(\Omega, u_\Omega, \theta_\Omega) = J(\Omega, \vec{v}_\Omega) = \sum_{i=1}^4 \alpha_i \int_{\Omega} |v_i - v_0^i|^2 dx + \sum_{i=1}^4 \beta_i \int_{\Omega} |\nabla v_i|^2 dx. \tag{9}$$

The case where Ω is a bounded connected of \mathbb{R}^N , $N = 2, 3$ with smooth boundary and v a scalar function was studied in [2] , [7].

The case where the functional J does not depend on the deformation $(u_\Omega^i, i = 1, 2, 3)$ and on the gradient of the deformation ∇u_Ω^i was studied in [6].

In this paper u_Ω is a vector function and functional J depends on the deformation $u_\Omega^i, i = 1, 2, 3$ its gradient ∇u_Ω^i and the temperature $u_\Omega^4 = \theta_\Omega$ and its gradient $\nabla \theta_\Omega$. We use topological optimization tools to study asymptotic analysis of the functional J by using Nazarov and Sokolowski approach [14].

The exterior problem is given by

$$\begin{cases} -\Delta v_\omega + \mathcal{F}(v_\omega) = \vec{F} & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \text{div}(\vec{u}_\omega) = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \frac{\partial v_\omega}{\partial n} = 0 & \text{on } \partial\omega \end{cases} \tag{10}$$

In problems (7),(8) and (10), the function \mathcal{F} is given by

$$\mathcal{F}(v_\Omega) = \begin{pmatrix} -3k\alpha \frac{\partial \theta}{\partial x_1} \\ -3k\alpha \frac{\partial \theta}{\partial x_2} \\ -3k\alpha \frac{\partial \theta}{\partial x_3} \\ 0 \end{pmatrix} = \begin{pmatrix} -3k\alpha \overrightarrow{\text{grad}}\theta \\ 0 \end{pmatrix} \tag{11}$$

where

$$\overrightarrow{\text{grad}}\theta = \begin{pmatrix} \frac{\partial \theta}{\partial x_1} \\ \frac{\partial \theta}{\partial x_2} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix} \tag{12}$$

Existence and uniqueness of solutions to (8) and (10) are studied in [6].

2.2 Case of Dirichlet boundary condition on ω_ϵ

In this case u_Ω depends only on the deformations $u_i, i = 1, 2, 3$. In this case we take $N = 3$.

Let P the space of polynomials in $x = (x_1, \dots, x_n)$. Let

$$P(\Omega) = \{p = (p_1, p_2, p_3) \in P : \frac{\partial p_i}{\partial n} = 0, i = 1, 2, 3 \text{ on } \partial\Omega\} \tag{13}$$

$$P(\omega) = \{p = (p_1, p_2, p_3) \in P : p_i = 0, i = 1, 2, 3 \text{ on } \partial\omega_\epsilon\} \tag{14}$$

Let p be a polynomial in \mathbb{R}^3 with coefficients in \mathbb{R}^3 satisfying $\mathcal{L}p = 0$ where \mathcal{L} designates the operator to (5) with $p = (p^1, p^2, p^3)$ and $p^i \in \mathbb{R}$. Let U^1, \dots, U^N a basis of homogeneous polynomials satisfying

$$U^h(zx) = z^{\tau_h} U^h(x), \quad \text{deg}(U^h) = \tau_h, \quad \tau_h \leq \tau_{h+1}, \quad (15)$$

$$\sum_{j=1}^4 U_j^h(\nabla_x) U_j^k = \delta_{h,k}. \quad (16)$$

In \mathbb{R}^3 the set of homogeneous polynomials of degree 1 is a linear combination of linear functions in x_1, x_2, x_3 . As we have a Dirichlet condition in the boundary of ω_ϵ we can choose U^h , $h = 1, \dots, 3$ in the following way

$$U^h(x) = U^h(x_1, x_2, x_3) = x_h e^h, \quad h = 1, 2, 3, \quad (17)$$

where e^h designates the $h^{\text{ième}}$ vector of the canonical basis of \mathbb{R}^3 . In this case $N = 3$ see [14]. The vectors U^h verify the relations (15) and (16).

Let $\Phi = (\Phi^1, \Phi^2, \Phi^3)$ be the matrix related with the operator \mathcal{L} in \mathbb{R}^3 , satisfying

$$-\Delta \Phi^j = \delta(x), \quad x \in \mathbb{R}^3, \quad j = 1, 2, 3; \quad (18)$$

$\delta(x)$ designates the Dirac mass at the origin. Φ^j is the fundamental solution of the Laplace's operator in \mathbb{R}^3 . The expression of Φ^j is given by

$$\Phi^j(x) = -\frac{1}{K_3|x|}, \quad K_3 = \frac{-1}{\beta_3}, \quad (19)$$

where β_3 the total surface of the unit sphere in \mathbb{R}^3 .

We define the powers solutions U^{-k} , $k = 1, 2, 3$ in the following way

$$U^{-k}(x) = \sum_j^3 U_j^k(\nabla_x) \Phi^j(x). \quad (20)$$

They are defined by

$$U^{-k} = x_k \Phi^k e^k. \quad (21)$$

2.2.1 Topological derivative

In this section we study the asymptotic expansion of functional J .

Theorem 2.1 *Let us suppose that in the functional (6) $\beta_i = 0$, $i = 1, 2, 3$, $\delta = 0$ ie the functional is not depending of ∇u_Ω , $\nabla \theta_\Omega$ and the Dirichlet conditions are prescribed in $\partial\omega_\epsilon$. Then functional $J(\Omega_\epsilon, u_{\Omega_\epsilon}, \theta_{\Omega_\epsilon})$ defined by (6) where $(u_{\Omega_\epsilon}, \theta_{\Omega_\epsilon})$ is solution to (5) admits the following asymptotic expansion*

$$\left| J(\Omega_\epsilon) - J(\Omega) - \epsilon \int_\Omega \mathfrak{F}'_u(x, v(x)) \eta(x) m^\omega v(x) dx \right| \leq o(\epsilon^2) \quad (22)$$

where $\eta(x) = (\eta^1(x), \dots, \eta^4(x))$ is the Green matrix and $v = (u_\Omega, \theta_\Omega)$ is solution to problem (4) and $\mathfrak{F}'_u = 2 \sum_{i=1}^4 (u_\omega^i - u_0^i)$

Proof 1 The first and second expansion of solution u_{Ω_ϵ} of (5) writes

$$v_{\Omega_\epsilon}^i \sim \mathcal{V}(\epsilon, x) = v^i + \sum_{j=1}^N a_j(\epsilon)\eta^j(x) \tag{23}$$

$$u_{\Omega_\epsilon}^i \sim \mathcal{W}(\epsilon, x) = \sum_{j=1}^N b_j(\epsilon)\zeta^j(\epsilon^{-1}x) \tag{24}$$

where $v(x) = (v^1, \dots, v^n)(x)$ is the solution of (7), ζ^j the solution defined in theorem 2.2 and η^j the generalized Green solution.

Injecting (23) in the expression of functional we get

$$J(\Omega_\epsilon) = \sum_{i=1}^4 \int_{\Omega_\epsilon} \left(v^i + \sum_{j=1}^4 a_j(\epsilon)\eta^j(x) - u_0^i \right)^2 dx = \sum_{i=1}^4 \int_{\Omega_\epsilon} \left(v^i - u_0^i + \sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right)^2 \tag{25}$$

Developping we have directly

$$J(\Omega_\epsilon) = \sum_{i=1}^4 \int_{\Omega_\epsilon} (v^i(x) - u_0^i)^2 + \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right)^2 + 2(v^i(x) - u_0^i) \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right) dx \tag{26}$$

$$\begin{aligned} J(\Omega_\epsilon) &= \sum_{i=1}^4 \int_{\Omega} (v^i(x) - u_0^i)^2 + \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right)^2 + 2(v^i(x) - u_0^i) \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right) dx \\ &- \sum_{i=1}^4 \int_{\omega_\epsilon} (v^i(x) - u_0^i)^2 + \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right)^2 + 2(v^i(x) - u_0^i) \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right) dx. \end{aligned} \tag{27}$$

Then

$$\begin{aligned} J(\Omega_\epsilon) - J(\Omega) &= \sum_{i=1}^4 \int_{\Omega} \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right)^2 \\ &+ 2(v^i(x) - u_0^i) \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right) dx \\ &- \sum_{i=1}^4 \int_{\omega_\epsilon} (v^i(x) - u_0^i)^2 + \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right)^2 + 2(v^i(x) - u_0^i) \left(\sum_{j=1}^4 a_j(\epsilon)\eta^j(x) \right) dx. \end{aligned} \tag{28}$$

Following the ideas in [10, 11, 12, 14], the coefficients $a(\epsilon)$ and $b(\epsilon)$ can be written in the form:

$$a(\epsilon) = \{I - \epsilon^{n-2}\epsilon m^\omega \epsilon m^\Omega\}^{-1} \epsilon^{n-2}\epsilon m^\omega \epsilon c \tag{29}$$

$$b(\epsilon) = \{I - \epsilon^{n-2} \epsilon m^\omega \epsilon m^\Omega\}^{-1} \epsilon c \tag{30}$$

where I is the identity matrix and m^ω , and m^Ω are polarization matrices for ω and Ω . As boundary condition on ω_ϵ are different from Neumann condition the matrix ϵ is the unit identity matrix and the coefficient $a(\epsilon)$ can be written as follow

$$a(\epsilon) = \frac{1}{1 - \epsilon m^\omega m^\Omega} \epsilon m^\omega c. \tag{31}$$

Putting the expression (31) in (28) we get

$$\begin{aligned} J(\Omega_\epsilon) - J(\Omega) - \epsilon \sum_{i=1}^4 \int_{\Omega} 2(v^i(x) - u_0^i) \eta(x) m^\omega v(x) dx = \\ \epsilon^2 \sum_{i=1}^4 \int_{\Omega} \left(\sum_{j=1}^4 \frac{1}{1 - \epsilon m^\omega m^\Omega} m^\omega c \eta^j(x) \right)^2 + - \sum_{i=1}^4 \int_{\omega_\epsilon} (v^i(x) - u_0^i)^2 \\ + \left(\sum_{j=1}^4 \frac{1}{1 - \epsilon m^\omega m^\Omega} \epsilon m^\omega c \eta^j(x) \right)^2 + 2(v^i(x) - u_0^i) \left(\sum_{j=1}^4 \frac{1}{1 - \epsilon m^\omega m^\Omega} \epsilon m^\omega c \eta^j(x) \right) dx \end{aligned} \tag{32}$$

Using the fact that functions η^j , v^j are bounded and taking the limit as $\epsilon \rightarrow 0$ we get directly

$$\lim_{\epsilon \rightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon} = \sum_{i=1}^4 \int_{\Omega} 2(v^i(x) - u_0^i) \eta(x) m^\omega v(x) dx. \tag{33}$$

2.3 Case of Neumann condition on ω_ϵ

In this case, $\vec{u}_\Omega = (u_\Omega^1, u_\Omega^2, u_\Omega^3)$ is solution to (4), $J(\Omega)$ defined by (6) or (9) does not depend on θ , ie $r = \delta = 0$ and in (5) we have the boundary condition $\frac{\partial u_{\Omega(\epsilon)}}{\partial n} = 0$ on $\partial \omega_\epsilon$. Let us consider the following spaces

$$P_N(\Omega) = \{p = (p_1, p_2, p_3) \in P, \frac{\partial p_i}{\partial n} = 0 \text{ } \partial \Omega\},$$

$$P_N(\omega) = \{p = (p_1, p_2, p_3) \in P, \frac{\partial p_i}{\partial n} = 0 \text{ } \partial \omega\},$$

where P a linear space of homogeneous polynomials. As we have a Neumann condition on $\partial \omega$ the space $P(\omega)$ contains constants polynomials. Let us suppose that there exists a polynomial p such that $\frac{\partial p_i}{\partial n} \neq 0$. Considering the space of linear polynomials on x in \mathbb{R}^3 and let U^1, \dots, U^N the basis of homogeneous polynomials of degree ≤ 1 and satisfying the following conditions:

$$U^j(\alpha x) = \alpha^{\tau_j} U^j(x), \sum_{i=1}^3 U_h^j U_j^k = \delta_{kh}. \tag{34}$$

Let us define the functions U^{-j} , $j = 1, \dots, N$

$$U^{-j}(x) = \sum_{k=1}^3 U_k^j(-\nabla_x)\Phi^k(x), \tag{35}$$

where Φ^k designates the rows of the fundamental matrix defined by

$$-\Delta\Phi^k = e^k\delta(x), \quad k = 1, 2, 3; \quad x \in \mathbb{R}^3, \tag{36}$$

e^k , $k = 1, 2, 3$ designates the k -th vector of the canonical basis of \mathbb{R}^n . In this case $N = 12$ and we can define the power solutions in the following way

$$U^h(x) = U^h(x_1, x_2, x_3) = x_h e^h, \tag{37}$$

$$U^4(x) = \frac{\sqrt{2}}{2}(x_2 e^3 + x_3 e^2), \quad U^5(x) = \frac{\sqrt{2}}{2}(x_3 e^1 + x_1 e^3), \quad U^6 = \frac{\sqrt{2}}{2}(x_1 e^2 + x^2 e^1), \tag{38}$$

$$U^{6+h}(x) = e^h, \quad U^{9+h}(x) = \frac{\sqrt{2}}{2}x e^h. \tag{39}$$

Consider z^j the solution of the homogeneous exterior problem

$$\begin{cases} -\Delta z^j = 0 \\ \frac{\partial z^j}{\partial n} = -\frac{\partial U^j}{\partial n}. \end{cases} \tag{40}$$

Then $\zeta^j = U^j + z^j$ is also a solution of the homogeneous problem (73). The solution z^j to problem (40) is uniqueness if we add a constant and is given by potentials theory by

$$z^j = \int_{\partial\omega} \mu(y)\partial_{n_y}E(x-y)ds(y) \quad \forall x \in \mathbb{R}^3 \setminus \bar{\omega}, \tag{41}$$

where $\mu = T(-\frac{\partial U^j}{\partial n})$ with T an isomorphism of $H_{00}^{1/2}(\Sigma)' \rightarrow H_{00}^{1/2}(\Sigma)$. For all details see [4]. Using Nazarov's theory [13, 14], we proof that z^j is given by:

$$z^j(\xi) = \sum_{j=1}^{12} m_{ji}^\omega U^j(\xi) = \sum_{j=1}^{12} m_{ji}^\omega \sum_{i=1}^3 U_h^j(-\nabla_h)\Phi^h + \tilde{z}^j(\xi), \tag{42}$$

$\tilde{z}^j(\xi)$ is the remainder of the taylor expansion. Then for function z^j we have

$$z^j(\xi) = \sum_{j=1}^6 m_{ji}^\omega \sum_{i=1}^3 U_h^j(-\nabla_h)\Phi^h + \tilde{z}^j(\xi). \tag{43}$$

The coefficients m_{ij}^ω is the coefficients of polarization matrix. In the following we characterise the coefficients of such matrix.

2.3.1 Matrice de Polarisation

Theorem 2.2 *The space of solutions to homogeneous problem (10) is a linear combination of functions $\zeta^j = U^j + z^j$, $j = 1, \dots, N$ where z^j is solution to problem*

$$\begin{cases} -\Delta z^j = 0 & \mathbb{R}^3 \setminus \bar{\omega} \\ \frac{\partial z^j}{\partial n} = -\frac{\partial U^j}{\partial n} & \partial\omega. \end{cases} \quad (44)$$

Futhermore ζ^j writes

$$\zeta^j = U^j + \sum_{h=1}^N m_{jh}^\omega U^{-h}(\xi) + \tilde{\zeta}^j(\xi)$$

Proof 2 *Let z^j , $j = 1, \dots, N$ the solution to (44). Then we have $\Delta(U^j + z^j) = 0$ and $\frac{\partial(U^j + z^j)}{\partial n} = 0$. Then ζ^j , $j = 1, \dots, N$ is solution to problem*

$$\begin{cases} -\Delta \zeta^j = 0 \\ \frac{\partial \zeta^j}{\partial n} = 0. \end{cases} \quad (45)$$

As Δu is a linear operator every combination of $\zeta^j = U^j + z^j$, $j = 1, \dots, N$ is also a solution of (45). Using (43) z^j , $j = 1, \dots, N$ takes the form

$$z^j = \sum_{h=1}^N m_{jh}^\omega U^{-h}(\xi) + \tilde{\zeta}^j(\xi). \quad (46)$$

Then ζ^j can be writes in the form

$$\zeta^j = U^j + \sum_{h=1}^4 m_{jh}^\omega U^{-h}(\xi) + \tilde{\zeta}^j(\xi). \quad (47)$$

Multiplying (44) by z^k , $k = 1, 2, \dots, N$ and integrating we have

$$\int_{\mathbb{R}^3 \setminus \omega} \nabla \zeta^j \nabla z^k - \underbrace{\int_{\partial\omega} \frac{\partial \zeta^j}{\partial n} z^k}_{=0} = 0.$$

Using the Green formula in the first term of the last equality we obtain:

$$- \int_{\mathbb{R}^3 \setminus \omega} \Delta z^k \zeta^j + \int_{\partial\omega} \frac{\partial z^k}{\partial n} \zeta^j = 0. \quad (48)$$

Using also the relations (46) and (47) yields:

$$- \int_{\mathbb{R}^3 \setminus \omega} \Delta \left(\sum_{h=1}^N m_{hk}^\omega U^{-h} \right) (U^j + \sum_{h=1}^N m_{hj}^\omega U^{-h}) = - \int_{\partial\omega} \frac{\partial z^k}{\partial n} \zeta^j.$$

Randomizing term by term we have

$$\begin{aligned}
 & - \sum_{h=1}^N m_{hk}^\omega \int_{\mathbb{R}^3 \setminus \omega} \Delta U^{-h} \cdot U^j - \underbrace{\sum_{1 \leq h, l \leq N} m_{hk}^\omega m_{lj}^\omega \int_{\mathbb{R}^3 \setminus \omega} \Delta U^{-h} \cdot U^{-l}}_{=0} \\
 & \sum_{h=1}^N m_{hk}^\omega \delta_{hj} = m_{jk}^\omega = - \int_{\partial\omega} \frac{\partial z^k}{\partial n} \cdot \zeta^j
 \end{aligned}$$

Proposition 1 *The polarization matrix coefficients are given by*

$$m_{jk}^\omega = - \int_{\partial\omega} \frac{\partial z^k}{\partial n} \cdot \zeta^j$$

for a Neumann condition in $\partial\omega$.

We are going to built $U(\epsilon, x)$ an approximation of solution $u_\epsilon(x) = (u_\epsilon^1, u_\epsilon^2, u_\epsilon^3)$ of problem (5) in the following way

$$U(\epsilon, x) = u(x) + \epsilon \sum_{j=1}^6 a_j \left\{ \zeta^j \left(\frac{x}{\epsilon} \right) - U^j \left(\frac{x}{\epsilon} \right) + \epsilon^2 \sum_{i=1}^6 m_{ij}^\omega \eta^{j0}(x) \right\} \tag{49}$$

The approximation $U(\epsilon, x)$ defined by $u^\epsilon(x)$ satisfies

$$\|u^\epsilon(\cdot) - U(\epsilon, \cdot)\|_{H^1(\Omega_\epsilon)} \leq C_\delta \epsilon^\alpha, \quad \alpha > 0 \tag{50}$$

2.3.2 Asymptotic expansion of functionals

We evaluate the variation of the functional (6) with $r = \delta = 0$ and $\alpha_i = \beta_i = 1, i = 1, 2, 3$ over the variation of the domain. We use the approximation $U(\epsilon, x)$ of $u(\epsilon, x)$ solution to (40). We have

$$\begin{aligned}
 & J_\epsilon(u_\epsilon) - J(u) = J_\epsilon(u_\epsilon) - J_\epsilon(U^\epsilon) + J_\epsilon(U^\epsilon) - J(u) \\
 & |J_\epsilon(u_\epsilon) - J_\epsilon(U^\epsilon)| = \left| \sum_{i=1}^3 \int_{\Omega_\epsilon} \left[(u_i^\epsilon - u_0^i)^2 + |\nabla u_i^\epsilon|^2 - (U_i^\epsilon - u_0^i)^2 - |\nabla U_i^\epsilon|^2 \right] dx \right| \\
 & = \left| \sum_{i=1}^3 \int_{\Omega_\epsilon} \left[(u_i^\epsilon - 2u_0^i + U_i^\epsilon)(u_i^\epsilon - U_i^\epsilon) + (|\nabla u_i^\epsilon|^2 - |\nabla U_i^\epsilon|^2) \right] dx \right|.
 \end{aligned}$$

As the inequality (50) is valid and for all $\epsilon > 0$,

$$|\nabla u_i^\epsilon| \leq C_1, \quad |\nabla U_i^\epsilon| \leq C_2;$$

we obtain directly

$$|J_\epsilon(u_\epsilon) - J_\epsilon(U^\epsilon)| \leq C \epsilon^\delta, \quad \delta > 0 \tag{51}$$

It remains to evaluate the difference $J_\epsilon(U^\epsilon) - J(u)$.
Evaluating functional J at U^ϵ yields

$$J_\epsilon(U^\epsilon) = \int_{\Omega_\epsilon} \left[(U^\epsilon - u_0)^2 + |\nabla U^\epsilon|^2 \right] dx = \int_{\Omega_\epsilon} \left[(U^\epsilon - u_0)^2 + (\nabla_x u(x) + \pi^+ u \nabla_\xi z(\xi) + \epsilon^2 m^\omega \nabla \eta^0(x))^2 \right] dx. \tag{52}$$

Let

$$\mathcal{F}(x, u(x), \nabla u(x)) = \sum_{i=1}^n (U_i^\epsilon - u_0^i)^2 + |\nabla u^i|^2, \tag{53}$$

and $\eta(\xi) + Z(\xi) = \nabla_\xi z(\xi) + \epsilon^2 m^\omega \nabla \eta^0(x)$, the functional $J_\epsilon(U)$ writes

$$J_\epsilon(U) = \int_{\Omega_\epsilon} \mathcal{F}(x, U(x), \nabla u(x) + \eta(\xi) + Z(\xi)) dx. \tag{54}$$

Applying lemma (voir [14] in appendix) to the functional $J_\epsilon(U)$ defined by (54), we get

$$J_\epsilon(U^\epsilon) - J_0(u) = -\epsilon^3 \mathcal{F}(0, u(0), \nabla u(0)) \text{mes}(\omega) + \epsilon^n \int_{\mathbb{R}^3 \setminus \omega} 2 \nabla u(0) \cdot \nabla z(\xi) d\xi - \epsilon^3 V m^\omega u + o(\epsilon^{3+\delta}). \tag{55}$$

We are now able to give the following theorem which give us the topological derivative in the Neumann case.

Theorem 2.3 *Let u_ϵ be the solution of problem (5) and B^ω is a Neumann condition on $\partial\omega$. Then functional $J_\epsilon(u_\epsilon)$ defined by (6) with $r = \delta = 0$ and $\alpha_i = \beta_i = 1, i = 1, 2, 3$ admits the following asymptotic expansion*

$$J_\epsilon(u_\epsilon) = J_0(u) + \epsilon^3 2\pi \left((u - u_0)^2 + |\nabla u|^2 - \int_{\mathbb{R}^3 \setminus \omega} 2 \nabla u \cdot \nabla z(\xi) d\xi \right) - \epsilon^3 m^\omega V \cdot u + o(\epsilon^{3+\delta}) \tag{56}$$

where u is the solution to problem (4) and V solution to the adjoint problem

$$\begin{cases} -\Delta V = \mathcal{F}'_u(x, u(x), \nabla u(x)) - \nabla_x \mathcal{F}'_{\nabla u}(x, u(x), \nabla u(x)) \text{ in } \Omega \\ \frac{\partial V}{\partial n} = 0 \text{ on } \partial\Omega \end{cases} \tag{57}$$

m^ω is the polarization matrix, $z = (z^1, \dots, z^3)$ solution to problem (44), $\mathcal{F}'_u(x, u(x), \nabla u(x)) = 2(u - u_0)$, and $\mathcal{F}'_{\nabla u}(x, u(x), \nabla u(x)) = -\Delta u$.

Proof 3 *The proof is essentially based on the asymptotic expansion of the functional; then the relation (56) is gotten directly by (51) and (55).*

3 Michell trusses problem

Let us begin this section by a presentation of the problem. We ask the reader to see for example the interesting and meaning full work due to Bouchitte et al [3] and their references. Our aim is to link this problem to shape and topological optimization and we will end our work by numerical simulations.

3.1 Presentation

For the presentation of the Michell trusses problem, we are going to give some elements which can be found in [3] and for more details see this reference and the others therein.

A truss is a finite union of bars (A_i, A_j) , $i \neq j$, $i, j \in \{1, \dots, n\}$ subjected to a force $F = \sum_{i=1}^n F_i \delta_{M_i}$ and result of a tension σ :

$$\sigma = \sum_{i,j=1}^n \lambda_{ij} \sigma^{[A_i, A_j]}, \tag{58}$$

where $A_i \in \mathbb{R}^3$, $i = 1, \dots, n$ and $\sigma^{[A_i, A_j]}$ is given by

$$\sigma^{[A_i, A_j]} = \frac{A_i - A_j}{|A_i - A_j|} \otimes \frac{A_i - A_j}{|A_i - A_j|} \mathcal{H}_{[A_i - A_j]}^1, \tag{59}$$

δ_{M_i} is the Dirac mass at a point $M_i \in \mathbb{R}^3$. The truss is in equilibrium when

$$div \sigma + F = 0. \tag{60}$$

The problem of Michell trusses is to find all points $A = (A_i)_{i=1}^n \subset \mathbb{R}^n$ and all reals $\lambda = (\lambda_{ij})_{i,j=1}^n \subset \mathbb{R}$, which minimize

$$C(A, \Lambda) = \sum_{i,j=1}^n |\lambda_{i,j}| |A_j - A_i| \tag{61}$$

such that

$$\begin{cases} \sigma = \sum_{i,j=1}^n \lambda_{ij} \sigma^{[A_i, A_j]} \\ -div \sigma = F \end{cases} \tag{62}$$

Using the second equation of (61) the problem is equivalent to a decomposition of F as

$$F = \sum_{i,j=1}^n \lambda_{i,j} (\delta_{A_i} - \delta_{A_j}) \frac{A_i - A_j}{|A_i - A_j|} \tag{63}$$

with $C(A, \Lambda)$ minimal.

Let us introduce

$$\sum_F^T(\tilde{s}) = \{ \sigma \in M(\tilde{s}, S^n * S^n) \text{ such that } \sigma = \sum_{i,j=1}^n \lambda_{ij} \sigma^{[A_i, A_j]} \text{ and } -div \sigma = F \}.$$

Our aim is to see the Michell trusses as a topological optimization problem. For a deformation (displacement) u for all pair (A, Γ) solution to the problem of Michell trusses we have

$$\int_{\Omega} \langle F, u \rangle dx = \sum_{i,j}^n \lambda_{i,j} \langle u(A_i) - u(A_j); \frac{A_i - A_j}{|A_i - A_j|} \rangle \leq \sum_{i,j}^n \lambda_{i,j} |A_i - A_j| \leq \sum_{i,j}^n |\lambda_{i,j}| |A_i - A_j| = C(A, \Lambda). \tag{64}$$

Let us mention that in [3], they showed that

$$\min\left\{\int_{\Omega} |\sigma|, \sigma \in \Sigma_F(s)\right\} = \max\left\{\int_{\Omega} \langle F, u \rangle, u \in U_1(s)\right\}, \tag{65}$$

where

$$U_1(\Omega) = \{u : \bar{\Omega} \rightarrow \mathbb{R}^n, u \in \mathcal{C}(\bar{\Omega}) \text{ and } \|u\|_{\Omega} \leq 1\}$$

and

$$\|u\|_{\Omega} = \sup\left\{\frac{|\langle u(x) - u(y); x - y \rangle|}{|x - y|^2}, \forall x \neq y (x, y) \in \Omega^2\right\}.$$

To go back to the objective of this section which is to study a type of Michell trusses problem as a topological optimization problem. Let us introduce the classical model in elasticity in the stationary case: this means that $-div\sigma = F$ where $\sigma(u) = \lambda Tr(\varepsilon(u)) + 2\mu\varepsilon(u)$ with $\sigma_{ij}(u) = \lambda(divu)\delta_{ij} + 2\mu\varepsilon_{ij}(u)$ and $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $div(u) = 0$. Finally we have $-\Delta u_i = F_i$ where $F = (F_1, F_2, F_3)$ and $u = (u_1, u_2, u_3)$. Multiplying by u_i and integrating we get

$$\int_{\Omega} \nabla u_i \nabla u_i dx - \int_{\partial\Omega} \frac{\partial u_i}{\partial n} u_i d\sigma = \int_{\Omega} F_i u_i dx, \quad i = 1, 2, 3.$$

Taking $\frac{\partial u_i}{\partial n} = 0$ on $\partial\Omega$ we have

$$\int_{\Omega} \sum_{i=1}^3 F_i u_i dx = \sum_{i=1}^3 \int_{\Omega} |\nabla u_i|^2 dx = \int_{\Omega} \langle F, u \rangle dx. \tag{66}$$

Let

$$J(u, \Omega) = \sum_{i=1}^3 \int_{\Omega} |\nabla u_i|^2 \tag{67}$$

under the constraints $div\sigma = F$.

Remark 1 *In the compressible case the topological derivative for a point $x \in \Omega$ of the compliance is given by [1, 7, 16, 17]*

$$g(x_0) = \frac{\pi(\lambda + 2\mu)}{2\mu(\lambda + \mu)} \left\{ 4\mu Ae(u) \cdot e(u) + (\lambda - \mu)tr(Ae(u))tr(e(u)) \right\} (x_0) \text{ in } \mathbb{R}^2 \tag{68}$$

and

$$g(x_0) = \frac{\pi(\lambda + 2\mu)}{\mu(9\lambda + 14\mu)} \left\{ 20\mu Ae(u) \cdot e(u) + (3\lambda - 2\mu)tr(Ae(u))tr(e(u)) \right\} (x_0) \text{ in } \mathbb{R}^3. \tag{69}$$

where $\omega = B(0, 1)$.

In the incompressible case ie when $div u = 0$ in Ω , solving the Michell trusses is therefore to maximize the functional (67) along a set of fields, so using the topology optimization. For mathematical convenience we minimize $-J(u, \Omega)$ where J is defined by (67). This reduces to reconsider cases of topological optimization problem related to the thermoelasticity because the functional (67) is a particular case of general functional (6) with $\alpha_i = 0, \beta_i = 1, r = \delta = 0$ and we consider only the elasticity problem with $F = (F_i), i = 1, 2, 3$.

Taking into account that $div u = 0$ in Ω , then $u_i, i = 1, 2, 3$ is solution in Ω of the following equation

$$\begin{cases} -\Delta u_i = F_i & i = 1, 2, 3, \Omega \\ div u = 0 & \text{in } \Omega \\ \frac{\partial u_i}{\partial n} = 0 & \partial\Omega. \end{cases} \tag{70}$$

Since Ω_ϵ is defined by $\Omega_\epsilon = \Omega \setminus \bar{\omega}_\epsilon, u_\epsilon$ is solution to

$$\begin{cases} -\Delta u_i^\epsilon = F_i & \Omega(\epsilon) \\ div u^\epsilon = 0 & \Omega(\epsilon) \\ \frac{\partial u_i^\epsilon}{\partial n} = 0 & \partial\Omega \\ \frac{\partial u_i^\epsilon}{\partial n} = 0 & \partial\omega_\epsilon, \end{cases} \tag{71}$$

and the functional $J(u_\epsilon)$ is defined by

$$J(u_\epsilon) = \sum_{i=1}^n \int_{\Omega_\epsilon} |\nabla u_i^\epsilon|^2 = \int_{\Omega_\epsilon} \langle F, u^\epsilon \rangle. \tag{72}$$

The exterior problem is given by

$$\begin{cases} -\mu \Delta w_i = F_i & \mathbb{R}^3 \setminus \bar{\omega} \\ div w_i = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \frac{\partial w_i}{\partial n} = 0 \text{ ou } w_i = 0 & \partial\omega. \end{cases} \tag{73}$$

We give the following theorem which characterizes the development of the functional. The proof is in the same way than the Theorem 2.3.

Theorem 3.1 *Let $J(u_\epsilon)$ the functional defined by (72) where $u^\epsilon = (u_1^\epsilon, \dots, u_N^\epsilon)$ and u_i^ϵ is solution to (73). Then we have the following asymptotic expansion*

$$J_\epsilon(u^\epsilon) = J(u) + 2\pi\epsilon^3 \left(|\nabla u|^3 - \nabla u \cdot \nabla V - u \cdot V \right) + o(\epsilon^{3+\delta}), \tag{74}$$

where δ is a positive integer and $u = (u_1, \dots, u_N)$; u_i is solution to (70) and V is solution to the adjoint state

$$\begin{cases} -\Delta V = -\nabla_x \mathcal{F}'_{\nabla u}(x, u(x), \nabla u(x)) & \Omega \\ \frac{\partial V}{\partial n} = 0 & \partial\Omega. \end{cases} \tag{75}$$

Proof 4 *The proof of this theorem is similar to the proof of theorem 2.3.*

4 Numerical Simulations

4.1 Case of the thermoelasticity problem

In this section we consider

$$\begin{cases} -\Delta\theta = h \text{ in } \Omega \\ \frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \quad (76)$$

where $h = 10^{-2}$ is a given function and the deformation vector $u = (u_1, u_2, u_3)$ is solution to

$$\begin{cases} -\Delta u_i - 3k\alpha \frac{\partial\theta}{\partial x_i} = f_i \text{ in } \Omega \\ \operatorname{div}(u) = 0 \text{ in } \Omega \\ \frac{\partial u_i}{\partial n} = h_i \text{ on } \partial\Omega \end{cases} \quad (77)$$

and $u_0^i = 2x + y - 2z$ is given.

We consider also the topological derivative of functional (6) defined in theorem 2.3 by

$$-2\pi \left((u - u_0)^2 + |\nabla u|^2 + \int_{\mathbb{R}^3 \setminus \omega} 2\nabla u \cdot \nabla z(\xi) d\xi \right) - 4\pi V \cdot u,$$

where V is the adjoint state associated to functional (6).

Let us take $\Omega = [-1, 1] \times [-1, 1] \times [-1, 1]$, $\alpha = k = 1$, $f_i = 10^{-2}$, $h_i = 0$, $i = 1, 2, 3$. Using finite elements methods and Getfeem++ we obtain the following numerical simulations for the topological derivative and the temperature θ in Ω .

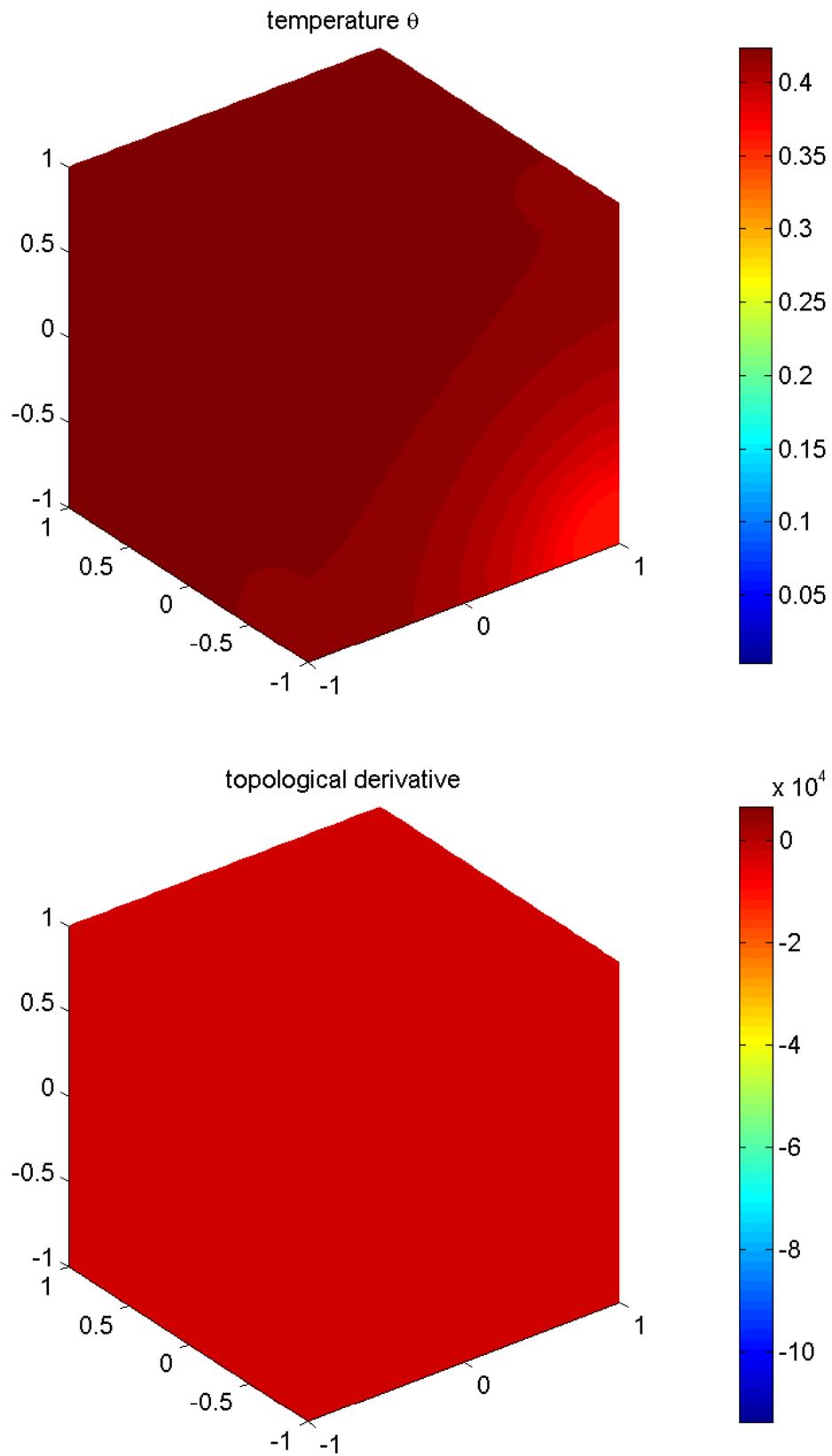


Figure 1: Representation of the temperature θ at the top of this page and representation of the topological derivative at the bottom.

4.2 Case of Michell trusses problem's

Let us consider a square in \mathbb{R}^2 and three points A_1, A_2 and A_3 in the square. Let us consider also a decomposition of F under the form

$$F = \alpha_1\delta_{A_1} + \alpha_2\delta_{A_2} + \alpha_3\delta_{A_3}. \tag{78}$$

We first give the solution $u = (u_1, \dots, u_n)$ where u_i is solution to

$$\begin{cases} -\Delta u_i = f_i \text{ in } \Omega \\ \operatorname{div}(u) = 0 \text{ in } \Omega \\ \frac{\partial u_i}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{79}$$

Given a load F with finite support, we minimize numerically the functional J where u is solution to (79). Let us consider the square $\Omega = [-1, 1] \times [-1, 1]$ and a decomposition of F in the form (78). Let us consider two cases:

$$A_1 = (0, 0), A_2 = (1/2, 1/2), A_3 = (1, 0) \tag{80}$$

and

$$B_1 = (-1, -1), B_2 = (-1/2, 1/2), B_3 = (1/2, -1/2). \tag{81}$$

The results obtained for the topological derivative in the case where points defined by (80) and (81) are given in figure2. The numerical simulations show that $C(\lambda)$ is minimum in the considered points.

These figures show that the topological derivative is smaller at the considered points, ie the points where we have the Dirac distributions.

We consider also a decomposition of $F = \sum_{i=1}^5 \alpha_i A_i$ where the points A_i are given by

$$A_1 = ((-1, -1), A_2 = (0, 1/2), A_3 = (1, 1) A_4 = (0, -1/2) \text{ and } A_5 = (1, -1) \tag{82}$$

and let $\lambda_{ij} = 1, 1 \leq i, j \leq 5$ and $\lambda_{ij} = \frac{i}{j}, i \neq j$. Then we obtain for the topological sensitivity figure 3.

In the case where $\Omega = [-1, 1]^3 \subset \mathbb{R}^3$ and $F = \sum_{i=1}^5 \alpha_i \delta_{A_i}$ where the points $A_i \in \mathbb{R}^3, i = 1, \dots, 5$ are given by

$$A_1 = (0, 0, 0), A_2 = (0, 1, 0), A_3 = (0, 0, 1), A_4 = (0, 0, 1), A_5 = (1, 1, 1) \tag{83}$$

the topological derivative is given in figure 4.

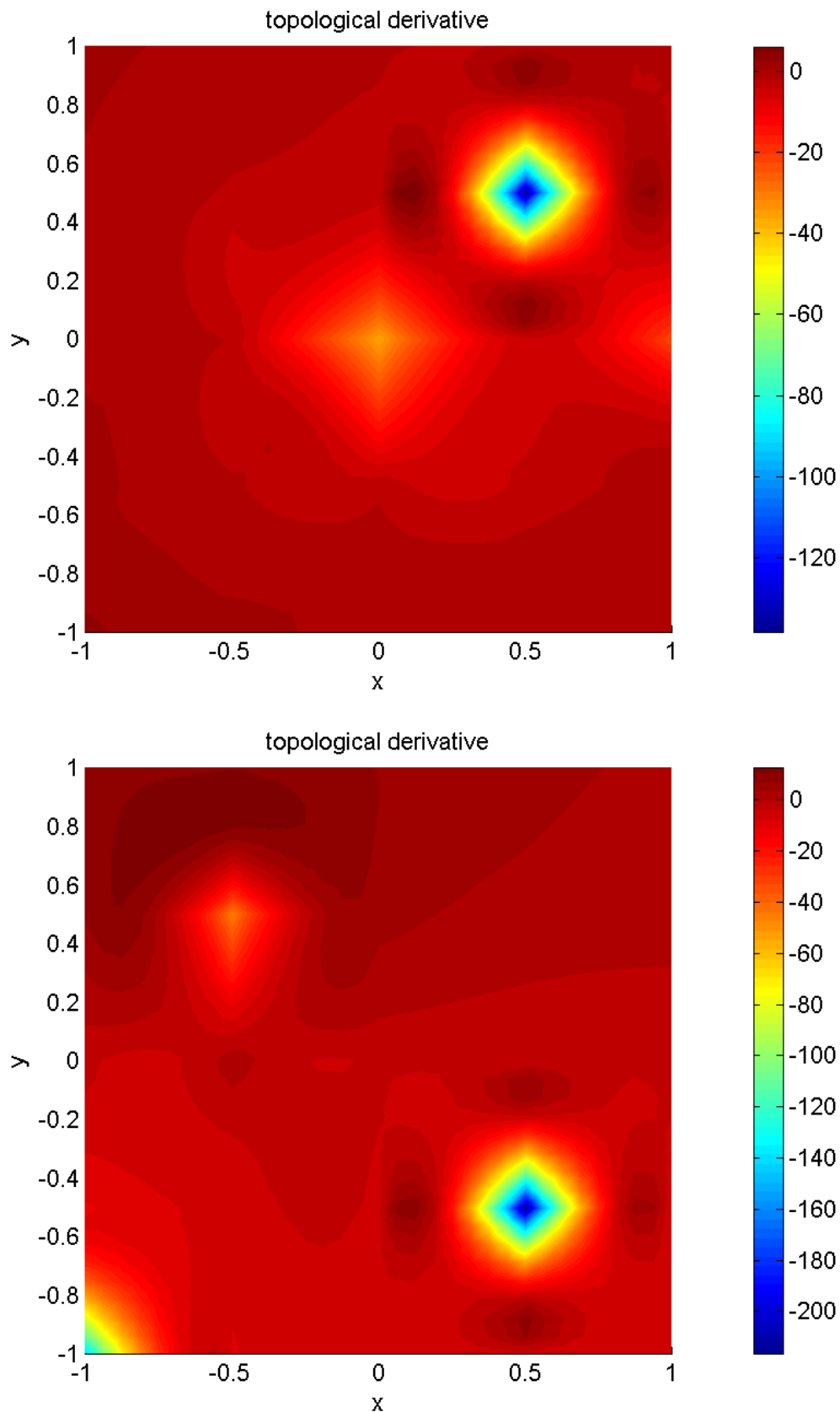


Figure 2: Representation of the topological derivative with in the top $F = \sum_{i=1}^3 \alpha_i \delta_{A_i}$ and $F = \sum_{i=1}^3 \alpha_i \delta_{B_i}$ at the bottom, $\lambda_{ij} = \frac{j}{i}$, A_i given by (80) and b_i given by (81).

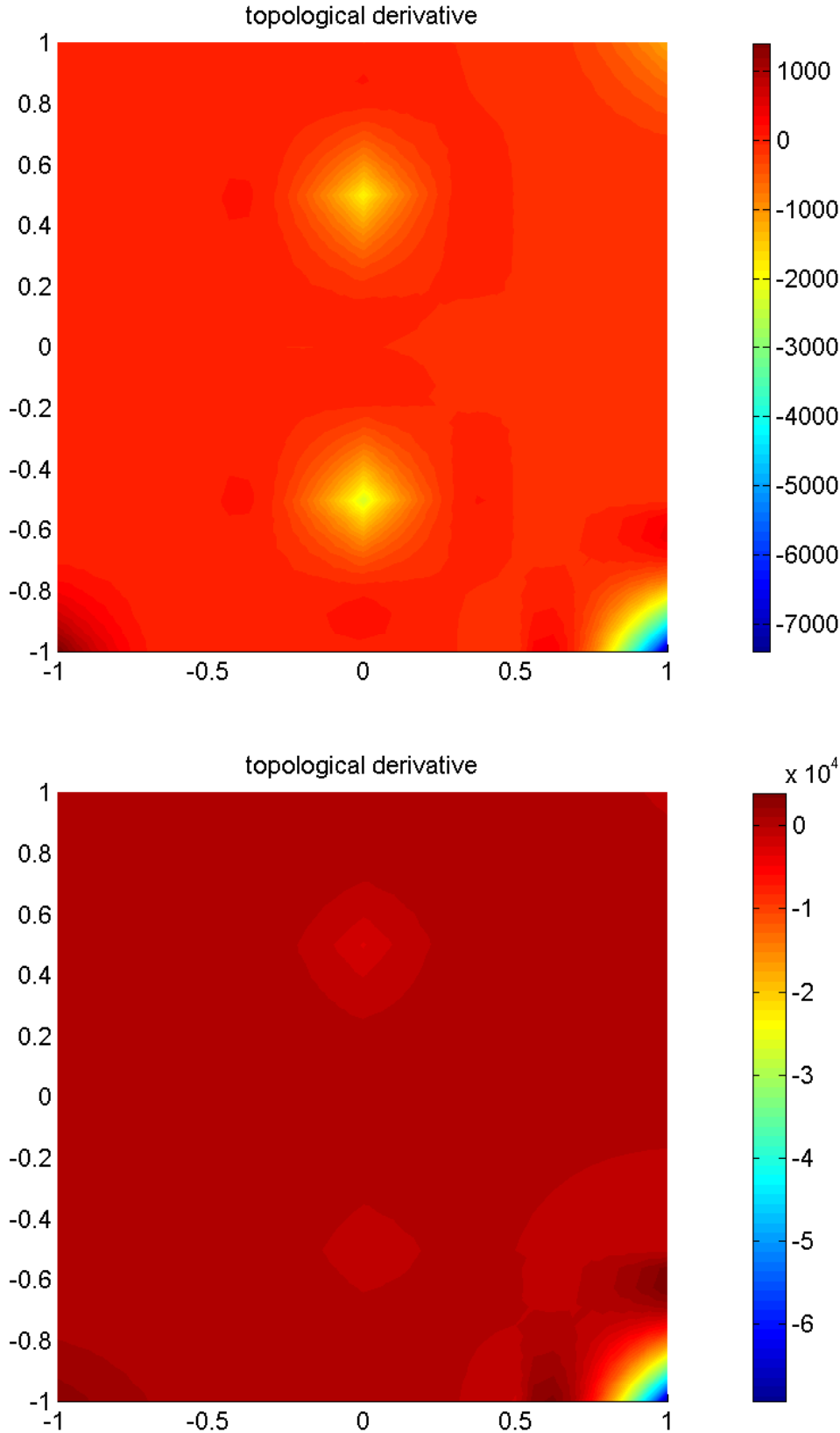


Figure 3: Representation of the topological derivative with in the top $F = \sum_{i=1}^5 \alpha_i \delta_{A_i}$, $\lambda_{ij} = \frac{j}{i}$ and $F = \sum_{i=1}^5 \alpha_i \delta_{A_i}$, $\lambda_{ij} = 1$ at the bottom, A_i given by (82).

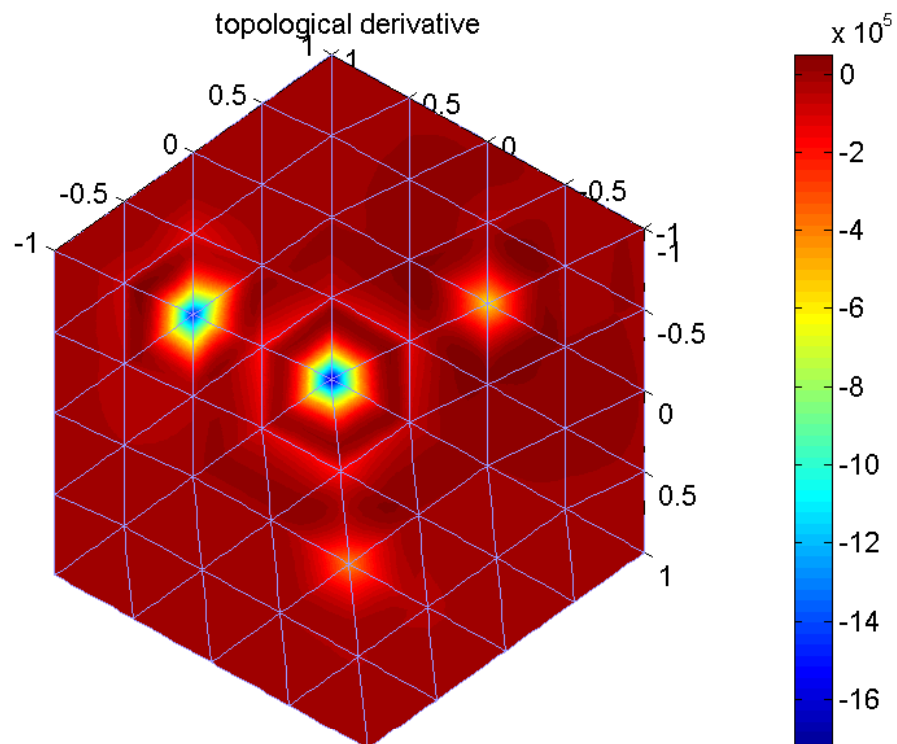


Figure 4: Representation of the topological derivative with $\lambda_{ij} = \frac{j}{i}$ and $F = \sum_{i=1}^5 \alpha_i \delta_{A_i}$, A_i given by (83).

References

- [1] G. Allaire, *Introduction à l'optimisation de formes*, CMAP, Ecole Polytechnique.
- [2] S. Amstutz, *Aspects théoriques et numériques en optimisation de forme topologique*, Thèse de Doctorat, Insa toulouse 2003.
- [3] G. Bouchitte, W. Gangbo, P. Seppecher, *Micell Trusses and Lines of Principal Action Mathematical Models and Methods in Applied Sciences* Vol. 18, No. 9 (2008) 1571-1603.
- [4] R. Dautray, J.L. Lions, *Analyse mathématiques et calcul numérique pour les sciences et les techniques*, Volume 2, Masson 1987.
- [5] R. Dautray, J.L. Lions *Analyse mathématiques et calcul numérique pour les sciences et les techniques*, Volume 1, Masson 1987.
- [6] I. FAYE *Etude d'un problème de pollution et de thermo-élasticité par l'optimisation de forme topologique*, Thèse de doctorat de troisième cycle, UCAD 2008.
- [7] S. Garreau, Ph. Guillaume and M.Masmoudi, *The topological asymptotics for PDE systems: the elastic case* SIAM control and optim. 39(6)pp1756-1778,2001.
- [8] S. Garreau, Ph. Guillaume and M.Masmoudi, *The topological sensitivity for linear isotropic elasticity*, European conference on computational Mechanics(ECCM99), 1999, rapportMIP n°03.45.
- [9] Ph. Guillaume, K.Sid Idris, *The topological expansion for Dirichlet problem*, SIAM J. Control and optim.41(4), pp1042-1072,2002.
- [10] V.G. Maz'ya, S.A. Nazarov,B.A. Plamenevskii *Asymptotics of solutions to elliptic boundary value problems under a singular perturbation of the domain* Tbilisi: Tbilisi Univ.1981. Russian.
- [11] A.M.IL' IN *Mathching of asymptotics expansion of solutions of boundary value problems*, Translations of mathematical monographs Vol 102, AMS 1992.
- [12] D. Leguillon, E.Sanches-Palancia, *Computations of singular solutions in elliptic problems and elasticity*, Paris, Masson.1987.
- [13] S.A.Nazarov, *Asymptotic conditions at points, self adjoint extensions of operators and the method of matched asymptotic expansions* trudy St Peterburg Mat.Obshch.5(1996)112-183,in Russian ;English transl :Trans,Amer.Math.Soc.Ser.2193(1999)77-126.
- [14] S.A. Nazarov, J. Sokolowski *Asymptotic analysis of shape functionals* J Math. Pures Appl. 82 (2003) pp 125-196.
- [15] A. Schumacher, *Topologieoptimierung von banteilstrukturen unter verwendung von lopchpositionierungskriterien*, thesis, universitat-Gesamthochschule-Siegen 1995.
- [16] J. Sokolowski and A. Zochowski, *On topological derivative in shape optimization*, SIAM J.Control Optim. n° 37 (4),1999,1251-1272.
- [17] A. Sy and Y. Renard, *A Fictitious domain approach for structural optimization with a coupling between shape and topological gradient*, Far East Journal of mathematical Sciences, 47, Number 1, 2010, pp 33-50.

Received: October, 2010