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Equilibrium Analysis for Power Based Flow Control Algorithm

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Abstract

We study a power-based flow control algorithm for virtual circuit flow control in a decentralized network. This algorithm is based on a greedy heuristic. Each virtual circuit, (or player) iteratively optimizes its individual performance measure, *power*, by adjusting its message rate to achieve an ideal tradeoff point between high throughput and low delay. For our equilibrium analysis, we formulate this flow control algorithm into *n*-person noncooperative game.

In our equilibrium analysis, we first provide a new proof for the existence of a Nash equilibrium point for every strictly concave game. A constraninted *n*-person game is considered in which the constraints for each player, as well as his payoff function, may depend on the strategy of every player. It is proved that the existence of a Nash equilibrium point of the power based flow control for an arbitrary network configuration. We then discuss the existence of Nash equilibrium points for generalized flow control for virtual circuit networks.

Keywords: Power-based flow control algorithm, user optimization, noncooperative concave *n*-person game

1. Introduction

In this paper, we consider the problem of decentralized flow control in a network with fixed virtual circuits. Users (who are in one-to-one correspondence with virtual circuits) are modeled to be *greedy*. We assume that user *i*'s utility for a given data rate γ_i is "*power*," which the user wishes to maximize. Power [1] is simply the ratio of throughput to delay, which offers a straightforward way to trade off delay and throughput.

More specifically, we consider a network with users $i = 1, \cdots, I$. The network is modeled as a collection $l = 1, \cdots, L$ of resources, commonly links, channels, or switches. Each user *i* has (is identified with) a path or virtual circuit, $L(i)$. The resources are shared. We denote by $I(l)$, the users sharing resource *l*. Each resource *l* has a capacity C_l, which is shared amongst the users of *l*. We model the behavior of the users as an *I* person noncooperative game. Each user individually tries to maximize its power by choosing its flow value γ_i. We assume that delay is given by a M/M/1 queueing model. Additionally, we make Kleinrock's independence assumption [2]. Thus, the power for user *i* is given by

$$P_i(\gamma) = \frac{\gamma_i^\beta}{D_i(\gamma)} = \frac{\gamma_i^\beta}{\sum_{l \in L(i)} D_{il}(\gamma)} = \frac{\gamma_i^\beta}{\sum_{l \in L(i)} \frac{1}{C_l - \sum_{j \in I(l)} \gamma_j}} , \qquad (1)$$

where β is nonnegative, $D_i(\gamma)$ is the average delay of virtual circuit i, $D_{il}(\gamma)$ is the average delay of link l and $D_i(\gamma) = \sum_{l \in L(i)} D_{il}(\gamma)$.

Without loss generality, we let $\beta = 1$ and then the power for user i is $P_i(\gamma) = \frac{\gamma_i}{D_i(\gamma)}$.

This makes sense for γ in the interior of *K*, where

$$\{\gamma \mid C_l - \sum_{j \in I(l)} \gamma_j \geq 0, l \in L(i); \ \gamma_i \geq 0, i \in I(l)\}. \qquad (2)$$

We assume all the C_l are positive so that *K* has a non-empty interior. We wish to extend the definition of power function, $P(\gamma)$, to all of $E_\oplus^n$. We simply replace

$(C_l - \sum_{j \in I(l)} \gamma_j)$ by $(C_l - \sum_{j \in I(l)} \gamma_j)_{\oplus}$, where $x_{\oplus} = x$ if $x > 0$, and $x_{\oplus} = 0$ if $x \le 0$. When the flow on a link equals its capacity, the delay is infinite, and the power is zero.

Bharath-Kumar and Jaffe [3] proposed a distributed flow control algorithm, where each *user* (i.e., *virtual circuit*) is presented by the network with the total flow on each link that it uses. Based on this information, on occasion, the user greedily updates its flow value, independent of and without the "*cooperation*" of other users. That is, the flow control problem is cast as an *n-person noncooperative game*. As is usual, equilibrium for this process means arriving at a Nash equilibrium point. That is, a set of flows, for which no user can individually improve its lot by changing only its flow.

There are three significant issues. First, does a Nash equilibrium point exist? Second, is this Nash equilibrium point unique? Third, suppose that the answer to the first two questions is yes, do the updating procedures converge to the equilibrium point? Bharath-Kumar and Jaffe [3] couldn't resolve these three issues. This paper focuses on the proof of existence of Nash equilibria in power based flow control for virtual circuits in networks. Chung and Van Slyke showed convergence and uniqueness for two user networks in [7], and for free steering schemes in [8].

In order to have a continuous discussion of the existence of Nash equilibria for the original power-based flow control algorithm proposed by Bharath-Kumar and Jaffe in [3]. We put the related theorems and lemmas of the generalized power function in the Appendix A, where we introduce a definition of a generalized power function $P^G(\gamma)$ and discuss Nash equilibria for the flow control algorithm based on the generalized power function. In particular, we provide a new proof for the existence of a Nash equilibrium point for every strictly concave game based on the generalized power function. Next section, we show that the power function $P(\gamma)$, and constraint set K is a special case of a generalized power function. Thus, the power function defined for flow control on networks has a Nash equilibrium point.

2. Nash Equilibria for the Power Based Flow Control in General Networks

In this section, we show that the specific power function, *P*, and constraint set *K* is an example of a generalized power, P^G, and hence, there is an equilibrium set of flows (i.e., a Nash equilibrium point) in general networks. With the modifications *P* is non-negative on *K*, and 0 outside of *K*. Moreover, *P* is continuous on $E_{\oplus}^{n}$. Now we look at user *i*'s optimal response to the other users. That is, when user *i* maximizes its power, given the flows of the other users.

User i's optimal response:

Suppose $\gamma^0 = (\gamma_i^0, \gamma_{-i}^0) \in K$, then γ_i^* is user *i*'s optimal response to γ_{-i}^0 if:

$$\gamma_i^* \in \arg\max_{\gamma_i \geq 0} P_i(\gamma_i, \gamma_{-i}^0) \tag{3}$$

where $P_i(\gamma_i, \gamma_{-i}^0) = \arg\max_{\gamma_i \geq 0} \dfrac{\gamma_i}{\sum_{l \in L(i)} \dfrac{1}{(C_l - \gamma_i - \sum_{j \in I(l), j \neq i} \gamma_j^0)}}$, for $(\gamma_i, \gamma_{-i}^0) \in K$;

it equals 0 otherwise.

The optimization is over all non-negative γ_i; however, for $(\gamma_i, \gamma_{-i}^0) \notin K$, $P_i(\gamma) = 0$. In the interior of K, $P_i(\gamma) > 0$. Thus we can make the convention that $(\gamma_i^*, \gamma_{-i}^0)$ is always in *K*. Finally, we need to show that, with this convention, γ_i^* is unique. By the convexity of *K*, the intersection of $\{(\gamma_i, \gamma_{-i}) \mid \gamma_i \geq 0, \gamma_{-i} = \gamma_{-i}^0\}$ with *K* is an interval for any γ^0 in *K*. The intersection is clearly non-empty; however, the interval can degenerate to a point, $(0, \gamma_{-i}^0)$. If the interval is non-degenerate, on the interior. In the first case the maximum is clearly unique. In the second case, we need to make an argument that uses the following properties.

Properties of Power and Inverse Power

Let $\gamma = (\gamma_1, \cdots, \gamma_n) \in E_{\oplus}^n$ and $\gamma_{-i} = (\gamma_1, \cdots, \gamma_{i-1}, \gamma_{i+1}, \cdots, \gamma_n) \in E_{\oplus}^{n-1}$, where $E_{\oplus}^n = \{\gamma \mid \gamma \geq 0, \gamma \in E^m, m$ is positive integer$\}$. The inverse power of VC_i is

$$J_i(\gamma) = J_i(\gamma_i, \gamma_{-i}) = P_i^{-1}(\gamma_i, \gamma_{-i})$$
$$= \frac{D_i(\gamma_i, \gamma_{-i})}{\gamma_i} = \sum_{l \in L(i)} P_{il}^{-1}(\gamma_i, \gamma_{-i}) = \sum_{l \in L(i)} J_{il}(\gamma_i, \gamma_{-i})$$
$$= \sum_{l \in L(i)} \frac{D_{il}(\gamma_i, \gamma_{-i})}{\gamma_i}, \text{ where } D_i(\gamma_i, \gamma_{-i}) = \sum_{l \in L(i)} D_{il}(\gamma_i, \gamma_{-i}) \tag{4}$$

$$\text{Where } D_{il}(\gamma_i,\gamma_{-i}) = \frac{1}{(C_l - \sum_{j\in I(l), j\neq i} \gamma_j) - \gamma_i}$$

$$= \frac{1}{C_{Rl}(i) - \sum_{j\in I(l), j\neq i} \gamma_j} = \frac{1}{C_l - \sum_{j\in I(l)} \gamma_j}$$

and $C_{Rl}(i) = C_l - \sum_{j\in I(l), j\neq i} \gamma_j$ is the current residual capacity of link l from user i's point of view. Note that $0 < \gamma_i < C_{R\min}(i) = \min C_{Rl}(i), \text{for } l \in L(i)$.

Clearly,

Lemma 1: Power is maximized if and only if inverse power is minimized.

Lemma 2: Let $P_{il}(\gamma_i,\gamma_{-i})$ be the power of a link l of VC_i, γ_i be the throughput of user i over that link l, and $C_{Rl}(i)$ be the current residual capacity of link l from user *i's* point of view. Then $P_{il}(\gamma_i,\gamma_{-i})$ is uniformly concave in γ_i for fixed γ_{-i}, such that $(\gamma_i,\gamma_{-i}) \in K$, where $\gamma_i \in (0, C_{R\min}(i))$ and $C_{R\min}(i) > 0$ is the least residual link capacity from VC_i's point of view on the virtual circuit.

Proof: Let $D_{il}(\gamma_i,\gamma_{-i}) = \dfrac{1}{C_{Rl}(i) - \gamma_i}$ be the average delay experienced at γ_i for messages that go through link l. The power for that link is:

$$P_{il}(\gamma_i,\gamma_{-i}) = \frac{\gamma_i}{D_{il}(\gamma_i,\gamma_{-i})} = \gamma_i (C_{Rl}(i) - \gamma_i)\,. \tag{5}$$

Then

$$\frac{\partial P_{il}(\gamma_i,\gamma_{-i})}{\partial \gamma_i} = C_{Rl}(i) - 2\gamma_i \tag{6}$$

$$\frac{\partial^2 P_{il}(\gamma_i,\gamma_{-i})}{\partial \gamma^2{}_i} = -2 \tag{7}$$

Thus, the power of a single link is uniformly concave in γ_i for fixed γ_{-i}, where $\gamma_i \in (0, C_{R\min}(i))$. □

Lemma 3: Let $J_{il}(\gamma_i, \gamma_{-i})$ be the inverse power of a link l of VC_i, γ_i be the throughput of user i over that link l, and $C_{Rl}(i)$ be the current residual capacity of link l from user i's point of view. Then $J_{il}(\gamma_i, \gamma_{-i})$ is uniformly convex in γ_i for fixed γ_{-i}, such that $(\gamma_i, \gamma_{-i}) \in K$, where $\gamma_i \in (0, C_{R\min}(i))$ and $C_{R\min}(i) > 0$ is the least residual link capacity from VC_i's point of view on the virtual circuit.

Proof: The inverse power for link l of VC_i is:

$$J_{il}(\gamma_i, \gamma_{-i}) = P_{il}^{-1}(\gamma_i, \gamma_{-i}) = \frac{D_{il}(\gamma_i, \gamma_{-i})}{\gamma_i} = \frac{\frac{1}{C_{Rl}(i) - \gamma_i}}{\gamma_i} = \frac{1}{\gamma_i (C_{Rl}(i) - \gamma_i)} \tag{8}$$

Then (for $0 < \gamma_i < C_{R\min}(i)$):

$$\frac{\partial^2 J_{il}(\gamma_i, \gamma_{-i})}{\partial \gamma_i^2} = \frac{\partial^2}{\partial \gamma_i^2}(P_{il}^{-1}(\gamma_i, \gamma_{-i})) = \frac{\partial^2}{\partial \gamma_i^2}(P_{il}^{-1}) = \frac{\partial}{\partial \gamma_i}(-P_{il}^{-2} \times \frac{\partial P_{il}}{\partial \gamma_i})$$

$$= 2P_{il}^{-3}(\frac{\partial P_{il}}{\partial \gamma_i})^2 - P_{il}^{-2}\frac{\partial^2 P_{il}}{\partial \gamma_i^2} = P_{il}^{-2}[2P_{il}^{-1} \times (\frac{\partial P_{il}}{\partial \gamma_i})^2 - \frac{\partial^2 P_{il}}{\partial \gamma_i^2}] \tag{9}$$

By (9) and by (7), we know that $\frac{\partial^2 P_{il}}{\partial \gamma_i^2} = -2 < 0$ and $P_{il} > 0$, for $\gamma_i \in (0, C_{R\min}(i))$, and fixed γ_{-i}.

Therefore,

$$\frac{\partial^2 J_{il}(\gamma_i, \gamma_{-i})}{\partial \gamma_i^2} \geq 2P_{il}^{-2} \geq 2 \times \frac{16}{C_{Rl}^4(i)} > 0, \tag{10}$$

for a fixed γ_{-i}, where $\gamma_i \in (0, C_{R\min}(i))$, $C_{R\min}(i) > 0$, because

$$J_{il} = J_{il}(\gamma_i, \gamma_{-i}) = \frac{1}{\gamma_i (C_{Rl}(i) - \gamma_i)} \geq \frac{1}{(\frac{C_{Rl}(i)}{2})^2} = \frac{4}{C_{Rl}^2(i)} \tag{11}$$

(i.e., For a fixed γ_{-i}, J_{il} has minimum at $\gamma_i = \frac{C_{Rl}(i)}{2}$). Note that (11) is true for any $\gamma_i \in (0, C_{R\min}(i)) \subseteq (0, C_{Rl}(i))$. Thus, the inverse power of a single link is uniformly convex in γ_i for fixed γ_{-i}, where $\gamma_i \in (0, C_{R\min}(i))$. □

Lemma 4: Let $J_i(\gamma_i, \gamma_{-i})$ be the inverse power of a VC_i and γ_i be the throughput of VC_i. Then $J_i(\gamma_i, \gamma_{-i})$ is uniformly convex in γ_i for fixed γ_{-i} such that $(\gamma_i, \gamma_{-i}) \in K$, where $\gamma_i \in (0, C_{R\min}(i))$ and $C_{R\min}(i) > 0$ is the least residual link capacity from VC_i's point of view on the virtual circuit.

Proof: The inverse power of a virtual circuit is equal to the sum over the links in the circuit of the inverse power of each link. By Lemma 3 and fact that the sum of (uniformly) convex functions is a (uniformly) convex function, we can conclude that $J_i(\gamma_i, \gamma_{-i})$ is uniformly convex in γ_i for fixed γ_{-i}, where $\gamma_i \in (0, C_{R\min}(i))$ and $C_{R\min}(i) > 0$ is the minimum link capacity from VC_i's point of view on the virtual circuit. □

Remark - Analytic Difficulties - Coupled Strategy Spaces:

In [5], Rosen provided an existence theorem for concave *n*-person games. Often, this Rosen's existence theorem has been used for the equilibrium analysis for the flow control problems for the telecommunication networks [6]. However, Rosen's existence theorem requires that the constraint set (i.e., the allowed strategies space) for the game is convex and compact (closed and bounded). Also, it requires the cost function that each user tries to minimize is convex in its own strategy and continuous at every point in the product strategy space.

By Lemma 3, we know that the inverse power of a link relative to a VC_i, $J_{il}(\gamma_i, \gamma_{-i})$, is uniformly convex in γ_i for fixed γ_{-i}, such that $(\gamma_i, \gamma_{-i}) \in K$, where $\gamma_i \in (0, C_{R\min}(i))$ Moreover, by Lemma 4, we know that the inverse power of the entire virtual circuit, $J_i(\gamma_i, \gamma_{-i})$, is uniformly convex in γ_i for fixed γ_{-i} such that $(\gamma_i, \gamma_{-i}) \in K$, where $\gamma_i \in (0, C_{R\min}(i))$. Therefore, the inverse power $J_i(\gamma_i, \gamma_{-i})$ goes to plus infinity as γ_i approaches 0 or $C_{R\min}(i)$.

Note that the user's payoff function for the power-based flow control algorithm is not defined for all points (i.e., plus infinity at zero flow and at link capacity). That is, the inverse power function for a M/M/1 network is not defined at all points of the product strategy space, therefore one cannot apply the Rosen's existence theorem directly to show the existence of Nash equilibria in power based flow control for virtual circuits in networks.

We now show that the existence of a Nash equilibrium point of the power based flow control for an arbitrary network configuration.

Nash Equilibria for Flow Control Problem on Networks

Theorem 1: For the power function defined for flow control on networks, we have, as a function of s_i on $[0,\infty)$, $P_i(s_i,\gamma_{-i})$ has a unique maximum, (s_i^*,γ_{-i}), for all $\gamma \in K$. Moreover, $(s_i^*,\gamma_{-i}) \in K$.

Corollary: The power function defined for flow control on networks has a Nash equilibrium point.

Proof: Theorem 1 shows that the power function defined for flow control on networks is a generalized power (i.e., see Definition A1). Theorem 1 then applies, proving the corollary. □ □

Note that in Appendix B, we further discuss the equilibrium analysis for power based flow control algorithm based on the generalized inverse power function.

3. Some Unsolved Problems

In this section, we propose some unsolved problems for future research.

Problem 1: Does the Nash equilibrium point unique for the power-based flow control for an arbitrary network topology?

Problem 2: Suppose that there is a unique Nash equilibrium point for the power based flow control algorithm, whether the free steering update schemes defined in [8] converge to the equilibrium point in an arbitrary network topology?

Appendix A. Equilibrium Analysis for Power Based Flow Control Algorithm - Using Generalized Power Function

In this Appendix, we introduce a definition of a generalized power function $P^G(\gamma)$ and discuss Nash equilibria for the flow control algorithm based on the generalized power function. Specifically, we provide a new proof for the existence of a Nash equilibrium point for every strictly concave game based on the generalized power function. A constrainted *n*-person game is considered in which the constraints for each player, as well as his payoff function, may depend on the strategy of every player. Without loss generality, we consider a generalized power function defined as follows.

We consider functions of the form $P^G(\gamma)$ where P^G is an *n*-vector valued function of γ which is itself an *n*-vector $(\gamma_1,\cdots,\gamma_n)$. Often we wish to consider P^G as a function of one component, γ_i, of the vector γ. With a slight abuse of notation we then write $P^G(\gamma_i,\gamma_{-i})$. We represent the *j*th component of P^G by $P_j^G(\gamma)$. We assume that $P^G(\gamma)$ is defined over all $E_\oplus^n$, but we will be particularly interested in its behavior over a compact convex set, K in $E_\oplus^n$.

Definition A1: (Generalized Power Function)

An *n* valued vector function $P^G(\gamma)$ of an *n*-vector γ with respect to a compact, convex set K is said to be a generalized power function if :

1. $P^G(\gamma)$ is continuous on $E_\oplus^n$, where $E_\oplus^n = \{\gamma \mid \gamma \ge 0, \gamma \in E^n\}$.
2. As a function of s_i on $[0,\infty)$, $P_i^G(s_i,\gamma_{-i})$ has a unique maximum, (s_i^*,γ_{-i}), for all $\gamma \in K$. Moreover, $(s_i^*,\gamma_{-i}) \in K$.

Definition A2: (Nash Equilibrium Point)

γ^* is a Nash equilibrium point for a generalized power function P^G, if $\gamma^* \in K$ and for each $i = 1,\cdots,n$. We have:

$$P_i^G(\gamma_i^*,\gamma_{-i}^*) \ge P_i^G(s_i,\gamma_{-i}^*) \text{ for all } i \text{ and for all } (s_i,\gamma_{-i}^*) \in K. \quad \text{(A1)}$$

We define the auxiliary function

$$G(s,\gamma) = \sum P_i^G(s_i,\gamma_{-i}) \text{ where } s = (s_1,\cdots,s_n). \quad \text{(A2)}$$

Lemma A1: $G(s,\gamma)$ is continuous on $E_\oplus^{2n}$.
Proof: Obvious. □

Lemma A2: $G(s,\gamma)$ has a unique maximum in s for any γ in K.
Proof: Follows immediately from Property 2 of Definition A1, since $G(s,\gamma)$ can be maximized term by term. □

Lemma A3: γ^* is a Nash equilibrium point if and only if
$\gamma^* \in \arg\max_{s \in K} G(s,\gamma^*)$.

Proof: Suppose $\gamma^* \notin \arg\max_{s\in K} G(s,\gamma^*)$ and $\gamma^0 \in \arg\max_{s\in K} G(s,\gamma^*)$; (i.e., $G(\gamma^0,\gamma^*) >$ $G(\gamma^*,\gamma^*)$). By Lemma A2, we know that γ^0 is unique. For some *i*, then, $P_i^G(\gamma_i^0,\gamma_{-i}^*) > P_i^G(\gamma_i^*,\gamma_{-i}^*)$. By Property 2 of Definition A1, $(\gamma_i^0,\gamma_{-i}^*) \in K$. Therefore we know that $P_i^G(\gamma_i^0,\gamma_{-i}^*)$ provides greater power than $P_i^G(\gamma_i^*,\gamma_{-i}^*)$ when both $(\gamma_i^0,\gamma_{-i}^*)$ and $(\gamma_i^*,\gamma_{-i}^*)$ are in *K*. Thus, γ^* can't be a Nash equilibrium point. Conversely, suppose, for some *i*, $\gamma_i{}^0 \in \arg\max_{s_i} P_i^G(s_i,\gamma^*)$ and $\gamma_i^0 \neq \gamma_i^*$; again we have $(\gamma_i^0,\gamma_{-i}^*) \in K$ (i.e., γ^* is not a Nash equilibrium point). Define s *by* $s_i = \gamma_i^0$ and $s_{-i}^* = \gamma_{-i}^*$, then $G(s,\gamma^*) > G(\gamma^*,\gamma^*)$, so that $\gamma^* \notin \arg\max_{s\in K} G(s,\gamma^*)$. By Lemma A2, we can define a single valued function, $f: K \to K$ by $f(\gamma) = \arg\max_{s\in K} G(s,\gamma^*)$. □

Lemma A4: $f(\gamma)$ is continuous on K.

Proof: Suppose f is not continuous at some $\gamma^0 \in K$. Then there exists an $\varepsilon > 0$, and a sequence $\{\gamma^k\}$ in *K* converging to γ^0 with $\left\| f(\gamma^k) - f(\gamma^0) \right\| > \varepsilon$ for all k. Let $s^k = f(\gamma^k)$ for all k. Since all the $s^k \in K$, and *K* is compact there is a convergent subsequence of the $\{s^k\}$, which we also denote $\{s^k\}$ which converges to, say, s^0. Then on this subsequence $(\gamma^k, s^k) \to (\gamma^0, s^0)$. By definition, for any $s \in K$, $G(s^k,\gamma^k) \geq G(s,\gamma^k)$ on the convergent subsequence. Since *G* is continuous we have: $G(s^0,\gamma^0) = \lim_k G(s^k,\gamma^k) \geq \lim_k G(s,\gamma^k) = G(s,\gamma^0)$. Therefore $s^0 \in \arg\min_s G(s,\gamma^0) = f(\gamma^0)$, which establishes a contradiction. □

Theorem A1: There always exists a Nash equilibrium point for a generalized power function.

Proof: f is a continuous function from a compact, convex set into itself. By Brouwer's fixed point theorem [4], f has a fixed point on *K*. By Lemma A3, the fixed point is a Nash equilibrium point. This approach is a modification of Rosen's [5]. □

For the following, we define a generalized inverse power function and consider a generalized flow control problem based on generalized inverse powers.

Appendix B. Equilibrium Analysis for Power Based Flow Control Algorithm - Using Generalized Inverse Power Function

Definition B1: (Generalized inverse power)
We say that $J_i^G(\gamma_i, \gamma_{-i})$ is a generalized inverse power if for every value of γ_{-1} there is an interval $(a(\gamma_{-1}), b(\gamma_{-1}))$, where the function is finite, non-negative, uniformly convex, and approaches infinity as γ_{-1} as approaches $a(\gamma_{-1})$ or $b(\gamma_{-1})$, and is plus infinity outside $(a(\gamma_{-1}), b(\gamma_{-1}))$.

Lemma B1: The finite sum of generalized inverse powers is a generalized inverse power or is everywhere plus infinity.

We can then define a generalized flow control problem based on generalized inverse powers.

Definition B2: (Generalized Flow Control Problem)
The generalized flow control problem is a noncooperative game where VC_i (player i) wishes to minimize $J_i^G(\gamma_i, \gamma_{-i})$ which is a generalized inverse power function for each i.

Theorem B1: For the generalized inverse power function defined for flow control on networks, we have, as a function of s_i on $[0, \infty)$, $J_i^G(s_i, \gamma_{-i})$ has a unique minimum, (s_i^*, γ_{-i}), for all $\gamma \in K$. Moreover, $(s_i^*, \gamma_{-i}) \in K$.

By the similar arguments to show Theorem A1 and Definition B2, we have:

Theorem B2: There always exists a Nash equilibrium point for a generalized inverse power function.

Definition B3: (Nash Equilibrium Point in Generalized Flow Control Problem)
γ^* is a Nash equilibrium point for a generalized inverse power function J^G, if and only is $\gamma^* \in K$ and for each $i = 1, \cdots, n$ we have:

$$J_i^G(\gamma_i^*, \gamma_{-i}^*) \le J_i^G(s_i, \gamma_{-i}^*) \text{ for all } i \text{ and for all } (s_i, \gamma_{-i}^*) \in K. \qquad \text{(B1)}$$

References

[1] A. Giessler, J. Hanle, and A. Koning, and E. Pade, Free buffer allocation - An investigation by simulation, Computer Networks, Vol. 2(1978), 191-208.

[2] L. Kleinrock, Queueing Systems, Vol. 2, Wiley, 1976.

[3] K. Bharath – Kumar, and J. M. Jaffe, A new approach to performance oriented flow control, IEEE Trans. on Communications, Vol. Com-29, No.4 (1981), 427-435.

[4] J. M. Ortega, and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, 1970, (reprinted by SIAM in 2000).

[5] J. B. Rosen, Existence and uniquness of equilibrium points for concave N person games, Econometrica, Vol. 33, No. 3 (1965), 520-534.

[6]. Y. A. Korilis and A. A. Lazar, Why is flow control hard: optimality, fairness, partial and delayed information, Second ORSA Telecommunication Conference, 1992.

[7] P. T. Chung and R. Van Slyke, Noncooperative Bottleneck flow Control in Two User Networks", Vol 175, Congressus Numerantium, Vol. 175, (2005), 189-202.

[8]. P. T. Chung and R. Van Slyke, Free Steering Update Schemes for Distributed Flow Control, Congressus Numerantium, Vol. 181, (2006), 111 – 128.

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