A Note on a Model of Merton Type for Valuing Default Risk

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Abstract

The aim of this note is to investigate the problem of valuing default risk for a firm when its assets value is a jump process. The default probability is established for the case of one liability and is estimated for the case of various liabilities.

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1 Introduction

In 1974 Merton make uses of the Black-Sholes option pricing model to value corporate liabilities. In this model, the firms asset value is assumed to follow a diffusion process given by

\[ dV_t = rV_t dt + \sigma_V V_t dW_t, \]  

where \( \sigma_V \) is the asset volatility, \( r \) is the risk-free interest rate and \( W_t \) is a Brownian motion. It is known that for this model, all probabilities and expectations are taken under the risk neutral measure. In this note we consider
the problem of valuing default risk when the firm’s asset value $V_t$ is a jump process, i.e. this asset is driven by a Poisson process $N_t$ of intensity $\lambda > 0$

$$dV_t = rV_t dt + \sigma V_t dM_t, \quad (1.1.2)$$

where $M_t = N_t - \lambda t$ is the compensated Poisson process under the risk neutral probability.

In fact $V_t$ is a geometric Poisson process which can be written in an explicit form as

$$V_t = V_0 e^{rt - \lambda \sigma t} (\sigma + 1)^{N_t} \quad (1.1.3)$$

refer to [5], chapter 11.

If at some time the asset’s value of a company is less than its total debt that should be paid exactly at that time and the company has not ability to pay for this, it will jump into default state.

2 Default at time $t$ when $V_t$ is less than a liability $L$.

Suppose that the firm’s value $V_t$ is given in (1.2) and that $V_t < L$ at the time $t$. Then it follows from (1.3) that

$$\ln V_t = \ln V_0 + (rt - \lambda \sigma t) + N_t \ln(\sigma + 1) < \ln L$$

or

$$N_t < \frac{1}{\ln(\sigma + 1)} \left[ \ln \frac{L}{V_0} - (r - \lambda \sigma) t \right] : = x \quad (2.2.1)$$

We assume that $\ln(L/V_0) - rt + \lambda \sigma t > 0$ then $x > 0$. In other word the parameter $\lambda$ should satisfy

$$\lambda > \frac{rt - \ln(L/V_0)}{\sigma t}. \quad$$

We can see that the default probability is

$$P(\text{default}) = P(N_t < x) = \sum_{k=0}^{[x]} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad (2.2.2)$$

where $[x]$ is the integer part of $x$.

Example 1. For the initial value $V_0 = 41.3$, a liability $L = 7.2$, the risk-free interest rate $r = 0.07$, a volatility $\sigma = 1$, the Poisson process $N_t$ with intensity $\lambda = 2.5$, $t = 1$. Then the default probability of the company after $t = 1$ is 8.2085%.
3 The case when the total debt consists of liabilities $L_1, L_2, ..., L_n$

Assume that liabilities $L_1, L_2, ..., L_n$ should be paid at times $t_1, t_2, ..., t_n$ respectively, $t_1 < t_2 < ... < t_n$.

Let $T$ be a time such that $T \geq \max(t_1, ..., t_n)$.

The company will be in default state before the time $T$ if and only if at one of time $t_i$ ($i = 1, 2, ..., n$),

$$V_{t_i} < L_i.$$ 

And the probability of default before $T$ is

$$P_{\text{default}}(0, T) = 1 - P(V_{t_i} > L_i, \forall t_i)$$  \hspace{1cm} (3.3.1)

We easily see that

$$(V_{t_i} > L_i) \supset (V_{t_i} > L), \ \forall t_i$$

where $L = \max\{L_1, ..., L_n\}$. Then

$$P_{\text{default}}(0, T) = 1 - P(V_{t_i} > L_i, \forall t_i) \leq 1 - P(V_{t_i} > L, \forall t_i).$$  \hspace{1cm} (3.3.2)

The inequality $V_{t_i} > L$ means that

$$N_{t_i} > \frac{1}{\ln(\sigma + 1)} \left[ \ln \frac{L}{V_0} + (\lambda \sigma - r) t_i \right] := x_i$$  \hspace{1cm} (3.3.3)

As before we assume also that

$$\lambda > \frac{r t_i - \ln(L/V_0)}{\sigma t_i},$$  \hspace{1cm} (3.3.4)

then \{x_1, ..., x_n\} is an increasing sequence by definition because $t_1 < t_2 < ... < t_n$.

Consider the event

$$A = \{V_{t_i} > L, \forall t_i\} = \bigcap_{i=1}^{n} \{N_{t_i} > x_i\}.$$ 

Denoting by $A_i$ the event \{N_{t_i} > x_i\}, $i = 1, 2, ..., n$ we can see that

$$A_1 = \{N_{t_1} > x_1\} = \{N_{t_1} - N_0 > x_1\}$$

$$A_2 = \{N_{t_2} > x_2\} = \{N_{t_2} - N_{t_1} > x_2 - N_{t_1}\} \supset \{N_{t_2} - N_{t_1} > x_2 - x_1\}$$

...
\( A_n = \{ N_{t_n} > x_n \} = \{ N_{t_n} - N_{t_{n-1}} > x_n - N_{t_{n-1}} \} \supset \{ N_{t_n} - N_{t_{n-1}} > x_n - x_{n-1} \}. \)

Put \( B_i = \{ N_{t_i} - N_{t_{i-1}} > x_i - x_{i-1} \} \) for \( i = 1, 2, \ldots, n \) and \( x_0 = 0 \) by convention. It follows that

\[
B_i \subset A_i \ \forall i = 1, \ldots, n
\]

so

\[
\bigcap_{i=1}^n B_i \subset \bigcap_{i=1}^n A_i = A.
\]

Because of the independence of increments of the Poisson process \( N_t \) we have

\[
P(A) \geq P \left( \bigcap_{i=1}^n B_i \right) = \prod_{i=1}^n P(B_i),
\]

where

\[
P(B_i) = P( N_{t_i} - N_{t_{i-1}} > x_i - x_{i-1} ) = 1 - \sum_{k=0}^{\lfloor x_i - x_{i-1} \rfloor} \frac{\lambda(t_i - t_{i-1})^k}{k!} e^{-\lambda(t_i - t_{i-1})}.
\]

Therefore

\[
1 - P(A) \leq 1 - \prod_{i=1}^n P(B_i).
\]

Finally we have

**Theorem 3.1.** The probability of default before \( T \) is estimated by

\[
P_{\text{default}}(0, T) \leq 1 - P(A)
\]

\[
\leq 1 - \prod_{i=1}^n \left( 1 - \sum_{k=0}^{\lfloor x_i - x_{i-1} \rfloor} \frac{\lambda(t_i - t_{i-1})^k}{k!} e^{-\lambda(t_i - t_{i-1})} \right).
\]

**References**


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