The Exponential Function as a Limit

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Abstract

In this paper we define the exponential function of base $e$ and we establish its basic properties. We also define the logarithmic function of base $e$ and we prove its continuity.

Keywords: number $e$, limit of sequence of functions, exponential function, logarithmic function

1 Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers and let $\mathbb{R}$ be the set of real numbers. Suppose that $\{ f_n(x) \}_{n=1}^{\infty}$ is a sequence of functions defined on $E \subseteq \mathbb{R}$. We say that this sequence converges to the function $f(x)$ on $E$ if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for any } x \in E.$$ (1.1)

This means that

$$\forall \varepsilon > 0 \forall x \in E \exists N = N(\varepsilon, x) \in \mathbb{N} \forall n \in \mathbb{N} : n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

In this case we write $f_n(x) \xrightarrow{E} f(x) \ (n \to \infty)$.

Suppose that $x \in \mathbb{R}$. Let us consider the numbers $m_0 = m_0(x)$ and $n_0 = n_0(x)$ defined as follows:

$$m_0 = m_0(x) = \{ k \in \mathbb{N} \mid k > x \} \quad \text{and} \quad n_0 = n_0(x) = \{ k \in \mathbb{N} \mid k > -x \}$$ (1.2)

Thus, $m_0 = 1$ and $n_0 = [-x] + 1$ if $x \leq 0$ and $m_0 = \lfloor x \rfloor + 1$ and $n_0 = 1$ if $x \geq 0$ ( $\lfloor x \rfloor$ is the integer part of $x$). It is clear that $1 + \frac{x}{n} > 0$ for any $n \geq n_0$.
and \(1 - \frac{x}{n} > 0\) for any \(n \geq m_0\). We define the two sequences \(\{f_n(x)\}_{n=1}^{\infty}\) and \(\{g_n(x)\}_{n=1}^{\infty}\) as follows:

\[
f_n(x) = 0 \text{ if } n < n_0 \quad \text{and} \quad f_n(x) = \left(1 + \frac{x}{n}\right)^n \text{ if } n \geq n_0.
\]

Moreover,

\[
g_n(x) = 0 \text{ if } n < m_0 \quad \text{and} \quad g_n(x) = \left(1 - \frac{x}{n}\right)^{-n} \text{ if } n \geq m_0.
\]

**Lemma 1.** Let \(x \in \mathbb{R}\) and consider the sequences defined by (1.3) and (1.4).

- **a.** The sequence \(\{f_n(x)\}_{n=1}^{\infty}\) is increasing for \(n \geq n_0\), that is, \(f_n(x) \leq f_{n+1}(x)\) for any \(n \geq n_0\). In particular it is increasing for \(x \geq 0\) since \(n_0 = 1\).
- **b.** The sequence \(\{g_n(x)\}_{n=1}^{\infty}\) is decreasing for \(n \geq m_0\), that is, \(g_n(x) \geq g_{n+1}(x)\) for any \(n \geq m_0\). In particular it is increasing for \(x \leq 0\) since \(m_0 = 1\).
- **c.** \(0 \leq g_n(x) - f_n(x) \leq \frac{x^2}{n} g_{k_0}(x)\) for any \(n \geq k_0 = \max(m_0, n_0)\).
- **d.** There exist the limits \(\lim_{n \to \infty} f_n(x) = \sup \{f_n(x) \mid n \in \mathbb{N}\}\) and \(\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} f_n(x) = L\). Moreover, \(f_{n_0}(x) \leq L \leq g_{m_0}(x)\)
- **e.** If \(|h| < 1\) then

\[
1 + h \leq \left(1 + \frac{h}{n}\right)^n \leq \left(1 - \frac{h}{n}\right)^{-n} \leq \frac{1}{(1-h)^{-1}} \text{ for all } n \geq 1
\]

**Proof.**

- **a.** Let \(n \geq n_0\). From the AGM inequality

\[
a_1 + a_2 + \cdots + a_{n+1} \geq \frac{n+1}{n} \sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} \quad (a_i > 0, \ i = 1, 2, \ldots, n+1)
\]

with

\[
a_1 = 1, \quad a_2 = a_3 = \cdots = a_{n+1} = 1 + \frac{x}{n} > 0
\]

we obtain

\[
1 + \frac{x}{n+1} = \frac{1 + n \left(1 + \frac{x}{n}\right)}{n+1} \geq \frac{n+1}{n} \sqrt[n+1]{\left(1 + \frac{x}{n}\right)^n}
\]

and then

\[
f_{n+1}(x) = \left(1 + \frac{x}{n+1}\right)^{n+1} \geq \left(1 + \frac{x}{n}\right)^n = f_n(x).
\]

This inequality is strict unless \(x = 0\).
b. Let \( n \geq m_0 \). From the AGM inequality (1.6) with
\[
a_1 = 1, \quad a_2 = a_3 = \cdots = a_{n+1} = 1 - \frac{x}{n} > 0.
\]
it follows that
\[
1 - \frac{x}{n+1} = \frac{1 + n(1 - \frac{x}{n})}{n+1} \geq \sqrt[n+1]{\left(1 - \frac{x}{n}\right)^n},
\]
and then
\[
\left(1 - \frac{x}{n+1}\right)^{n+1} \geq \left(1 - \frac{x}{n}\right)^n > 0,
\]
which implies
\[
g_{n+1}(x) = \left(1 - \frac{x}{n+1}\right)^{-(n+1)} \leq \left(1 - \frac{x}{n}\right)^{-n} = g_n(x).
\]
This inequality is strict unless \( x = 0 \).

c. Let \( n \geq k_0 = \max(m_0, n_0) \). We have
\[
g_n(x) - f_n(x) = g_n(x) \left(1 - \frac{f_n(x)}{g_n(x)}\right) = g_n(x) \left(1 - q^n\right), \quad (1.7)
\]
where \( q = 1 - \frac{x^2}{n^2} \). Observe that \( n \geq k_0 > |x| \) from where \( 0 < q \leq 1 \) and then \( q^n \leq 1 \) and \( 1 - q^n \geq 0 \). It is clear from (1.7) that \( g_n(x) - f_n(x) \geq 0 \) for \( n \geq k_0 \).

On the other hand, by virtue of (1.7),
\[
0 \leq g_n(x) - f_n(x) = g_n(x)(1 - q)(1 + q + \cdots + q^{n-1})
\leq g_{k_0}(x) \cdot \frac{x^2}{n^2} (1 + 1 + \cdots + 1)
= g_{k_0}(x) \cdot \frac{x^2}{n^2} \cdot n = \frac{x^2}{n} g_{k_0}(x).
\]
Thus,
\[
0 \leq g_n(x) - f_n(x) \leq \frac{x^2}{n} g_{k_0}(x) \text{ for } n \geq k_0. \quad (1.8)
\]
From the last inequality we see that given \( \varepsilon > 0 \) if we choose a natural number \( N \) subject to \( N \geq k_0 \) and \( N > x^2 g_{k_0}(x)/\varepsilon \) then
\[
|g_n(x) - f_n(x)| = g_n(x) - f_n(x) < \varepsilon \quad \text{for all } \quad n > N.
\]
We have proved that
\[
\lim_{n \to \infty} (g_n(x) - f_n(x)) = 0. \quad (1.9)
\]
d. Let $k_0 = \max(m_0, n_0) \geq m_0$. By virtue of $b$ and $c$,

$$g_n(x) \leq g_{k_0}(x) \quad \text{and} \quad f_n(x) \leq g_n(x) \leq g_{k_0}(x) \quad \text{for all} \quad n \geq k_0,$$

which proves that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded from above for each $x \in \mathbb{R}$ and then $\lim_{n \to \infty} f_n(x) = L$, where

$$L = \sup \{ f_n(x) \mid n \in \mathbb{N} \} = \sup \{ f_n(x) \mid n \geq n_0 \}.$$

On the other hand, from (1.9),

$$\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} ((g_n(x) - f_n(x)) + f_n(x)) = L.$$

e. Observe that $m_0 = n_0 = 1$ since $|h| < 1$. From $a$, $b$ and $c$ we obtain

$$1 + h = f_1(h) \leq f_n(h) \leq g_n(h) \leq g_1(h) = (1 - h)^{-1} \quad \text{for any} \quad n \geq k_0 = 1.$$

## 2 The exponential function and its properties

In previous section we established the existence of the limits

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^{-n}$$

for each $x \in \mathbb{R}$. This allows us to define a function $\exp : \mathbb{R} \to (0, \infty)$ as follows :

$$\exp(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^{-n}, \quad x \in \mathbb{R}.$$ (2.1)

Is obvious that $\exp(0) = 1$. The value $\exp(1)$ is special and it is denoted by $e$ :

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828182846$$

We will call function defined by (2.1) the exponential function of base $e$. This function is also denoted by $e^x$.

### 2.1 Properties of the exponential function

In this section we establish the main properties of the exponential function starting from (2.1).

**Property 1.** Let $x \in \mathbb{R}$.

i. If $x > -1$ then $\exp(x) > 1 + x$. In particular, $\exp(x) > 1$ for $x > 0$. 
ii. If \( x < 1 \) then \( \exp(x) \leq \frac{1}{1-x} \). In particular, \( \exp(x) < 1 \) if \( x < 0 \).

Proof

i. Since \( x > -1 \) we have \( n_0 = [-x] + 1 = 1 \). By virtue of lemma 1, parts a and d,

\[
\exp(x) \geq \left( 1 + \frac{x}{2} \right)^2 > \left( 1 + \frac{x}{1} \right)^1 = 1 + x.
\]

ii. If \( x < 1 \) then \( m_0 = [x] + 1 = 1 \). In view of lemma 1 parts b, c and d, \( f_n(x) \leq g_n(x) \leq g_1(x) \) for all \( n \geq k_0 = \max(m_0, n_0) \) and letting \( n \to \infty \) we obtain

\[
\exp(x) = \lim_{n \to \infty} f_n(x) \leq g_1(x) = \left( 1 - \frac{x}{1} \right)^{-1} = \frac{1}{1-x}.
\]

**Property 2.** (Multiplicative property)

\[
\exp(x + y) = \exp(x) \exp(y) = \exp(y) \exp(x) \quad \text{for any} \quad x, y \in \mathbb{R}.
\]  \hspace{1cm} (2.2)

In particular,

\[
\exp(-x) = (\exp(x))^{-1} = \frac{1}{\exp(x)} \quad \text{for all} \quad x \in \mathbb{R}.
\]  \hspace{1cm} (2.3)

Proof. Let us consider the sequences

\[
f_n(x) = \left( 1 + \frac{x}{n} \right)^n, \quad f_n(y) = \left( 1 + \frac{y}{n} \right)^n \quad \text{and} \quad f_n(x + y) = \left( 1 + \frac{x + y}{n} \right)^n,
\]

where \( n \geq k_0 > |x| + |y| \). By lemma 1, part d,

\[
\lim_{n \to \infty} f_n(x) = \exp(x), \quad \lim_{n \to \infty} f_n(y) = \exp(y) \quad \text{and} \quad \lim_{n \to \infty} f_n(x + y) = \exp(x + y).
\]

Since \( h(n) \overset{\text{def}}{=} \frac{xy}{n + x + y} \to 0 \ (n \to \infty) \), we may choose \( N \) large enough so that \( |h(n)| < 1 \) for \( n \geq N \). We obtain

\[
\frac{f_n(x)f_n(y)}{f_n(x + y)} = \left( 1 + \frac{xy}{n(n + x + y)} \right)^n = \left( 1 + \frac{h(n)}{n} \right)^n \quad \text{for} \ n \geq N.
\]  \hspace{1cm} (2.4)

In view of lemma 1, part e, from (2.4) it is clear that

\[
1 + h(n) \leq \frac{f_n(x)f_n(y)}{f_n(x + y)} \leq (1 - h(n))^{-1}
\]  \hspace{1cm} (2.5)
Taking into account that \( \lim_{n \to \infty} (1 + h(n)) = \lim_{n \to \infty} (1 - h(n))^{-1} = 1 \) from (2.5) we obtain \( \lim_{n \to \infty} \frac{f_n(x)f_n(y)}{f_n(x + y)} = 1 \), from where

\[
\frac{\exp(x) \exp(y)}{\exp(x + y)} = \lim_{n \to \infty} \frac{f_n(x)}{f_n(x + y)} \cdot \lim_{n \to \infty} \frac{f_n(y)}{f_n(x + y)} = \lim_{n \to \infty} \frac{f_n(x)f_n(y)}{f_n(x + y)} = 1.
\]

We have proved that \( \exp(x) \exp(y) = \exp(x + y) \).

**Property 3.** Given \( t, x \in \mathbb{R} \), if \( t < x \), then \( \exp(t) < \exp(x) \), that is, the exponential function is strictly increasing on \( \mathbb{R} \).

**Proof.** If \( x > t \) then \( x - t > 0 \) and making use of Property 1, \( \exp(x - t) > 1 \). We have

\[
\exp(x) = \exp((x - t) + t) = \exp(x - t) \exp(t) > 1 \cdot \exp(t) = \exp(t).
\]

**Property 4.** If \( x > 0 \) then \( 0 < \exp(x) - 1 \leq x \exp(x) \).

**Proof.** Let \( n \in \mathbb{N} \). We have

\[
0 < \left(1 + \frac{x}{n}\right)^n - 1 = \left(1 + \frac{x}{n} - 1\right) \left(1 + \frac{x}{n}\right)^{n-1} + \left(1 + \frac{x}{n}\right)^{n-2} + \cdots + 1
\]

\[
< \frac{x}{n} \left(\left(1 + \frac{x}{n}\right)^n + \left(1 + \frac{x}{n}\right)^n + \cdots + \left(1 + \frac{x}{n}\right)^n\right)
\]

\[
= \frac{x}{n} \cdot n \left(1 + \frac{x}{n}\right)^n = x \left(1 + \frac{x}{n}\right)^n < x \exp(x).
\]

Thus,

\[
0 < \left(1 + \frac{x}{n}\right)^n - 1 < x \exp(x) \quad \text{for any} \quad n \in \mathbb{N}.
\]

Letting \( n \to \infty \) in the last inequality gives

\[
0 < \exp(x) - 1 \leq x \exp(x). \tag{2.6}
\]

**Property 5.** The exponential function is continuous on \( \mathbb{R} \), i.e, for a given real number \( a \) and any \( \varepsilon > 0 \) we may find \( \delta = \delta(\varepsilon, a) > 0 \) such that if \( |x - a| < \delta \) then \( |\exp(x) - \exp(a)| < \varepsilon \).

**Proof.** Let us first show that

\[
|\exp(t) - 1| \leq 3|t| \quad \text{for} \quad |t| < 1. \tag{2.7}
\]

Indeed, this inequality is obvious if \( t = 0 \). Let \( t \neq 0 \). If \( 0 < t < 1 \) then \( \exp(t) < \exp(1) = e < 3 \). Consequently, in view of Property 4, \( 0 < \exp(t) - 1 < 3t \).
Now, let $-1 < t < 0$. From one hand, by Property 1, $\exp(t) < 1$. On the other hand, $0 < -t < 1$ and then $0 < \exp(-t) - 1 < 3(-t) = 3|t|$, from where

$$|\exp(t) - 1| = |\exp(t)(1 - \exp(-t))| = \exp(t)(\exp(-t) - 1) < 3\exp(t)|t| < 3|t|.$$ 

We have established (2.7). Let $a \in \mathbb{R}$ and consider values of $x$ subject to $|x - a| < 1$. Setting $t = x - a$ in (2.7) we obtain $|\exp(x - a) - 1| < 3|x - a|$. Multiplying this inequality by $\exp(a)$ and making use of Property 2 we obtain

$$|\exp(x) - \exp(a)| < 3\exp(a)|x - a|$$

for any $x \in \mathbb{R}$ such that $|x - a| < 1$.

(2.8)

From condition (2.8) it is clear that choosing $\delta$ such that

$$0 < \delta < \min \left( \frac{1}{2}, \frac{\varepsilon}{3\exp(a)} \right),$$

then $|\exp(x) - \exp(a)|$ for all $x \in \mathbb{R}$ such that $|x - a| < \delta$. This means that

$$\lim_{x \to a} \exp(x) = \exp(a).$$

3 The logarithmic function

In view of Property 5, the exponential function is strictly increasing on $\mathbb{R}$. In view of Property 1, part i, $\exp(x) > 1 + x$ for $x \geq 0$. On the other hand, if $x < 0$ then (2.2) gives $\exp(x)\exp(-x) = \exp(x - x) = \exp(0) = 1$ and then $\exp(x) > 0$. This says that the exponential function $\exp : \mathbb{R} \to (0, \infty)$ is one to one and it admits inverse. We will denote it by $\log$ and we will call it logarithmic function of base $e : \log : (0, \infty) \to \mathbb{R}$.

Let $y \in (0, \infty)$. There exists $x \in \mathbb{R}$, uniquely defined, such that $\exp(x) = y$. Indeed, choose $b > 0$ subject to $b > y - 1$. By Property 1, part i, $\exp(b) > 1 + b > y$.

On the other hand, let $a$ be any negative number such that $a < 1 - 1/y$. By Property 1, part ii,

$$\exp(a) \leq \frac{1}{1 - a} < y.$$ 

We have proved that for any $y \in (0, \infty)$ we may find two real numbers $a$ and $b$ such that $a < b$ and $\exp(a) < y < \exp(b)$. Let us consider the function $\exp(x)$ on the interval $[a, b]$. Since this function is continuous on $[a, b]$ (Property 1), it takes all values between $\exp(a)$ and $\exp(b)$. This allows us to choose $x$ on $[a, b]$ for which $y = \exp(x)$. This number $x$ is unique since the exponential
function is one to one. We have proved that function \( \exp : \mathbb{R} \to (0, \infty) \) is one to one and onto. Thus, \( \log(y) = x \iff y = \exp(x) \). It is clear that
\[
\exp(\log(y)) = y \quad \forall \, y > 0 \quad \text{and} \quad \log(\exp(x)) = x \quad \forall \, x \in \mathbb{R}.
\]

**Theorem.** The logarithmic function \( \log : (0, \infty) \to \mathbb{R} \) is continuous on \((0, \infty)\).

**Proof:** It is easy to see that given \( b > 0 \) and \( \varepsilon > 0 \), if \( |y - b| < \delta = \min \{ b(1 - \exp(-\varepsilon)), b(\exp(\varepsilon) - 1) \} \) then \( |\log(y) - \log(b)| < \varepsilon \). This means that \( \lim_{y \to b} \log(y) = \log(b) \) for any \( b > 0 \).

## 4 Conclusions

We defined two of the most important functions in mathematics: the exponential and logarithmic functions. This allows to define the exponential function of base \( a \) as \( a^x = \exp(x \log(a)), x \in \mathbb{R}, a > 0 \) and \( a \neq 1 \). From this we may establish the laws of exponents:

\[
a. \, a^x a^y = a^{x+y}; \quad b. \, \frac{a^x}{a^y} = a^{x-y}; \quad c. \, a^x b^x = (ab)^x; \quad d. \, \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x; \quad e. \, (a^x)^y = a^{xy}.
\]

Finally, we may define the logarithmic function as a limit as follows
\[
\log(x) = \lim_{n \to \infty} n(\sqrt[n]{x} - 1) = \lim_{n \to \infty} n(1 - x^{-1/n}) \quad \text{for} \quad x > 0.
\]

Starting from this definition we may define the exponential function as the inverse of logarithmic function.

## References


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