The Logarithmic Function as a Limit

Alvaro H. Salas

Department of Mathematics
Universidad de Caldas, Manizales, Colombia
Universidad Nacional de Colombia, Manizales
FIZMAKO Research Group
asalash2002@yahoo.com

Abstract

In this paper we define the logarithmic function of base $e$ and we establish its basic properties. We also define the exponential function of base $e$ and we prove the basic properties of these functions.

1 Introduction

There are several ways to define the logarithmic function of base $e$. Usually, it is defined as the inverse function of the exponential function of base $e$. In this paper we first define the logarithmic function as the limit of a sequence of functions and then we derive its basic properties without using any calculus tools.

2 Auxiliary lemmas

Lemma 1. Given a positive integer $n$ and a positive real number $x$ there exists a positive real number $y$ uniquely defined such that $y^n = x$. This $y$ is called the $n$–th root of $x$ and it is denoted by $\sqrt[n]{x}$ or $x^{1/n}$.

Proof. We proceed by induction on $n$. The assertion is valid for $n = 1$ since $x^1 = x$ and then $y = x$. Suppose that our lemma is valid for some $n$. Let us consider the set

$$E = \{t \geq 0 \mid t^{n+1} < x\}.$$ 

This set is not empty since $0 \in E$. On the other hand, if $t \in E$ then $t^{n+1} < x < (x + 1)^{n+1}$. From this, $t < x + 1$ and $x + 1$ is an upper bound of $E$. Let $ho = \sup E \geq 0$. We claim that $\rho^{n+1} = x$. 


Indeed, suppose that \( \rho^{n+1} < x \). Choose \( c \) with \( \rho^{n+1} < c < x \). Then \( \rho^n < c/\rho \).

By the inductive hypothesis, we may find \( b > 0 \) such that \( b^n = c/\rho \). From \( \rho^n < b^n \) we get \( \rho < b \). Choose \( a \) so that \( \rho < a < \min(b, xp/c) \). Then \( a^{n+1} = a^n a < b^n a = a c/\rho < x \), so that \( a \in E \) and then \( a \leq \rho \). But \( a > \rho \) and we get a contradiction.

Now, suppose that \( \rho^{n+1} > x \). Choose \( c \) with \( \rho^{n+1} > c > x \). Then \( \rho^n > c/\rho \).

By hypothesis, we may find \( b > 0 \) such that \( b^n = c/\rho \) and then \( \rho > b \). Choose \( a \) so that \( \rho > a > \max(b, xp/c) \). We have \( a^{n+1} = a^n a > b^n a = a c/\rho > x \), so that if \( t \in E \) then \( t^{n+1} < x < a^{n+1} \) and this gives \( t < a \) for any \( t \in E \), that is, \( a \) is an upper bound of \( E \) and \( \rho = \sup E \leq a \). But \( a < \rho \). We again get a contradiction.

Thus, \( \rho^{n+1} = x \) and the lemma is also valid for \( n+1 \). Finally, \( \sqrt{x} \) is unique. This follows from the fact that if \( y, z > 0 \) and \( z^n = y^n = x \) then \( z = y \).

**Lemma 2.** Let \( p_{n+1}(z) = n(z^{n+1} - 1) - (n + 1)(z^n - 1) \). Then

\[
p_{n+1}(z) = (z - 1)^2(1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1}) \text{ for } n = 1, 2, 3, \ldots \tag{2.1}
\]

In particular,

\[
n(z^{n+1} - 1) - (n + 1)(z^n - 1) > 0 \text{ for any } z > 0. \tag{2.2}
\]

**Proof.** By induction on \( n \). The identity (2.1) is valid for \( n = 1 \) since \( p_2(z) = z^2 - 1 - 2(z - 1) = z^2 - 2z + 1 = (z - 1)^2 \).

Suppose that it holds for some \( n \). Observe that

\[
p_{n+2}(z) - p_{n+1}(z) = (n + 1)(z - 1)^2 z^n.
\]

Then

\[
p_{n+2}(z) &= p_{n+1}(z) + (n + 1)z^n(z - 1)^2 \\
&= (z - 1)^2(1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1}) + (n + 1)z^n(z - 1)^2 \\
&= (z - 1)^2(1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1} + (n + 1)z^n)
\]

and then equation (2.1) is also valid for \( n + 1 \).

**Lemma 3.** Given any real number \( x > 0 \) the sequence \( f_n(x) \) defined by

\[
f_n(x) = n(\sqrt{x} - 1), n = 1, 2, 3, \ldots \tag{2.3}
\]

has the following properties

1. \( f_n(x) > f_{n+1}(x) \) for \( n = 1, 2, 3, \ldots \), that is, this sequence is strictly decreasing.
2. For any \( n = 1, 2, 3, \ldots \) we have

\[
\frac{x - 1}{x} \leq f_n(x) \leq x - 1 \tag{2.4}
\]
and the equality holds only if \( x = 1 \).

3. If \( n \geq m \) then

\[
f_n(x) \leq m(\sqrt[n]{x} - 1).
\]

(2.5)

**Proof.** 1. Let \( z = x^{\frac{1}{n(n+1)}} \) in (2.2). Then

\[
n(x^{\frac{1}{n}} - 1) - (n + 1)(x^{\frac{1}{n+1}} - 1) > 0,
\]

which is equivalent to \( f_n(x) > f_{n+1}(x) \).

2. Observe that

\[
\sqrt[n]{x} - 1 = \frac{x - 1}{\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2} + \cdots + \sqrt[n]{x} + 1} < x - 1 \text{ for any } x > 0 \text{ and } x \neq 1.
\]

(2.6)

On the other hand,

\[
\sqrt[n]{x} - 1 = \frac{x - 1}{\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2} + \cdots + \sqrt[n]{x} + 1} > \frac{x - 1}{nx} \text{ for any } x > 1.
\]

(2.7)

From (2.6) and (2.7) it is clear that

\[
\frac{x - 1}{x} < n(\sqrt[n]{x} - 1) < x - 1 \text{ for any } x > 1.
\]

(2.8)

Let \( 0 < x < 1 \). Then \( \frac{1}{x} > 1 \) and (2.8) gives

\[
n(\sqrt[n]{\frac{1}{x}} - 1) < \frac{1}{x} - 1
\]

from where

\[
n(1 - \sqrt[n]{x}) < \frac{1-x}{x} \sqrt[n]{x} < \frac{1-x}{x} \text{ and then } \frac{x-1}{x} < n(\sqrt[n]{x} - 1).
\]

(2.9)

Inequalities (2.6) and (2.9) give

\[
\frac{x - 1}{x} < n(\sqrt[n]{x} - 1) < x - 1 \text{ for } 0 < x < 1.
\]

Finally, for \( x = 1 \) we obtain

\[
\frac{x - 1}{x} = n(\sqrt[n]{x} - 1) = x - 1 = 0.
\]

3. This follows from 1 since \( f_n(x) \leq f_m(x) = m(\sqrt[n]{x} - 1) \) for any \( n \geq m \).
3 The logarithmic function.

Since \( n(\sqrt[n]{x} - 1) = f_n(x) > f_{n+1}(x) \geq \frac{x-1}{x} \) for \( n = 1, 2, 3, \ldots \) we see that there exists \( \lim_{n \to \infty} n(\sqrt[n]{x} - 1) \). We will call it the natural logarithm of \( x \) and we will denote it by \( \log x \). Thus,

\[
\log x = \lim_{n \to \infty} n(\sqrt[n]{x} - 1), \quad x > 0. \tag{3.1}
\]

Equation (2.9) defines a function on \((0, \infty)\) as the limit of the sequence of functions \( f_n(x) \) given by (2.3). This function is called logarithmic function. It is also denoted by \( \ln x \).

3.1 Properties of logarithmic function.

Property 1. \( \log 1 = 0 \). This is obvious since \( \log 1 = \lim_{n \to \infty} n(\sqrt[n]{1} - 1) = 0 \).

Property 2. For any \( t > 0 \) and \( x > 0 \),

\[
\log(tx) = \log t + \log x. \tag{3.2}
\]

Indeed, from (2.4) we obtain

\[
|f_n(x)| \leq c_x, \text{ where } c_x = \max\{|x-1|, \frac{|x-1|}{x}\} \text{ for any } x > 0. \tag{3.3}
\]

On the other hand,

\[
|f_n(tx) - f_n(t) - f_n(x)| = n \left| \sqrt[n]{t} - 1 \right| \sqrt[n]{x} - 1 \]
\[
= n \cdot \frac{|f_n(t)|}{n} \cdot \frac{|f_n(x)|}{n} \leq \frac{c_t c_x}{n}.
\]

Thus,

\[
0 \leq |f_n(tx) - f_n(t) - f_n(x)| \leq \frac{c_t c_x}{n} \text{ for any } t, x > 0 \ (n = 1, 2, 3, \ldots) \tag{3.4}
\]

Letting \( n \to \infty \) in (3.4) and taking into account (3.1) we obtain

\[
|\log(tx) - \log t - \log x| = 0,
\]

which proves (3.2).

Property 3. For any positive integer \( n \), \( \log(x^n) = n \log x \). This may be proved by induction by using Property 2.

Property 4. For any \( x > 0 \), \( \log \frac{1}{x} = -\log x \). Indeed, \( 0 = \log 1 = \log(x \cdot \frac{1}{x}) = \log x + \log \frac{1}{x} \).
Property 5. For any \( t > 0 \) and \( x > 0 \),
\[
\log \left( \frac{t}{x} \right) = \log t - \log x.
\] (3.5)
Indeed, \( \log \left( \frac{t}{x} \right) = \log (t \cdot \frac{1}{x}) = \log t + \log \frac{1}{x} = \log t - \log x. \)

Property 5. The logarithmic function is strictly increasing on \((0, \infty)\).
Indeed, letting \( n \to \infty \) in (2.4) gives
\[
\frac{x - 1}{x} \leq \log x \leq x - 1 \quad \text{for all } x > 0.
\] (3.6)

Let \( 0 < x_1 < x_2 \). Then \( x = \frac{x_2}{x_1} > 1 \) and from (3.6) we obtain
\[
0 < \frac{x - 1}{x} < \log \frac{x_2}{x_1} = \log x_2 - \log x_1, \quad \text{from where } \log x_1 < \log x_2.
\]

Property 6. The logarithmic function is continuous on \((0, \infty)\).
Indeed, \( \varepsilon > 0 \) and consider any \( t > 0 \) subject to \( |t - 1| < \frac{\varepsilon}{\varepsilon + 1} \). Then
\[
1 - \varepsilon/(\varepsilon + 1) < t < 1 + \varepsilon/(\varepsilon + 1).
\]
From (3.6) we see that
\[
\log t \leq t - 1 < \frac{\varepsilon}{\varepsilon + 1} < \varepsilon. \quad \text{(3.7)}
\]
On the other hand, \( t > 1 - \varepsilon/(\varepsilon + 1) = 1 - \frac{\varepsilon}{\varepsilon + 1} = \frac{1}{\varepsilon + 1} \) and then by (3.6) we have
\[
\log t \geq \frac{t - 1}{t} = 1 - \frac{1}{t} > 1 - (\varepsilon + 1) = -\varepsilon. \quad \text{(3.8)}
\]
Conditions (3.7) and (3.8) give
\[
|\log t| < \varepsilon \quad \text{for any } t > 0 \text{ satisfying } |t - 1| < \frac{\varepsilon}{\varepsilon + 1}. \quad \text{(3.9)}
\]
Now, let \( a > 0 \) be a fixed number and \( \varepsilon > 0 \). Define \( \delta = \frac{a\varepsilon}{\varepsilon + 1} \). Consider any \( x \) such that \( |x - a| < \delta \). Let \( t = \frac{x}{a} \). Then
\[
\left| \frac{x}{a} - 1 \right| = |t - 1| = \frac{|x - a|}{a} < \frac{\delta}{a} = \frac{\varepsilon}{\varepsilon + 1}. \quad \text{(3.10)}
\]
and then from (3.9) and (3.10) it follows that \( |\log x - \log a| = \left| \log \frac{x}{a} \right| = |\log t| < \varepsilon \) if \( |x - a| < \frac{\varepsilon}{\varepsilon + 1}. \) This proves that \( \lim_{x \to a} \log x = \log a. \)

Property 7. Given any real number \( \xi \) there exists a positive real number \( x \) such that \( \log x < \xi \).
Indeed, choose a positive integer \( m \) for which \( \xi > -m \). Then \( 1 + \xi/m > 0 \). We define \( x = \frac{1}{2}(1 + \xi/m)^m \). Taking into account (2.5) we obtain \( f_n(x) \leq m(\sqrt[n]{x} - 1) \) for any \( n \geq m \). Letting \( n \to \infty \) in this inequality gives \( \log x \leq m(\sqrt[m]{1 + \xi/m} - 1) \).

Then
\[
\log x \leq m(\sqrt[m]{x} - 1) < m(\sqrt[m]{1 + \xi/m} - 1) = \xi.
\] (3.11)

**Property 8.** The range of logarithmic function is the set of real numbers. Indeed, let \( y \) be any real number. By Property 7, we may find a number \( a > 0 \) and a number \( b' \) such that \( \log a < y \) and \( \log b' < -y \). Then
\[
\log a < y < -\log b' = \log b, \text{ where } b = \frac{1}{b'}.
\]

It is clear that \( a < b \). Let \( H = \{ t > 0 \text{ such that } \log t < y \} \). This set is not empty since \( a \in H \). If \( t \in H \) then \( \log t < y < \log b \) and this gives \( t < b \). Thus, \( H \) is bounded from above by \( b \). Let \( x = \sup H \). We claim that \( \log x = y \).

Indeed, suppose that \( \log x < y \). Let \( \varepsilon = y - \log x \). There exists \( \delta > 0 \) such that \( |\log x - \log t| < \varepsilon \) if \( |x - t| < \delta \). Let \( t = x + \delta/2 > x \). Then \( \log t < \log x + \varepsilon = y \) and \( t \in H \) so that \( t \leq x \). But \( t > x \). Contradiction.

On the other hand, suppose that \( \log x > y \). Let \( \varepsilon = \log x - y \). There exists \( \delta > 0 \) such that \( |\log x - \log s| < \varepsilon \) if \( |x - s| < \delta \). Choose some \( s \) such that \( 0 < s < x \) and \( |x - s| < \delta \). Then \( \log s > \log x - \varepsilon = y > \log t \) so that \( t < s \) for any \( t \in H \). We conclude that \( s \) is an upper bound of \( H \) and then \( x \leq s \). But \( x > s \). Contradiction.

We have proved that \( \log x = y \). Since \( y \) was arbitrarily chosen, we conclude that the range of \( \log \) is the set of real numbers.

### 4 The exponential function.

By Properties 5 and 6 of previous section, the logarithmic function \( \log \) defined by (3.8) is strictly increasing and continuous. On the other hand, by virtue of Property 8, its range is the set of real numbers. Thus, \( \log : (0, \infty) \to (-\infty, \infty) \) is one to one and onto. The inverse function of \( \log \) is denoted by \( \exp \) so that \( \exp : (-\infty, \infty) \to (0, \infty) \) is strictly increasing, one to one and onto. Moreover,
\[
\exp(\log(x)) = x \text{ for any } x > 0 \text{ and } \log(\exp(x)) = x \text{ for any real } x. \quad (4.1)
\]

Observe that
\[
\exp(x_1 + x_2) = \exp(x_1) \exp(x_2) \text{ for any real numbers } x_1 \text{ and } x_2. \quad (4.2)
\]

Indeed, let \( y_1 = \exp(x_1) \) and \( y_2 = \exp(x_2) \) so that \( x_1 = \log y_1 \) and \( x_2 = \log y_2 \). We have
\[
\exp(x_1 + x_2) = \exp(\log y_1 + \log y_2) = \exp(\log(y_1 y_2)) = y_1 y_2 = \exp(x_1) \exp(x_2).
\]
In particular, \( \exp(x) \exp(-x) = \exp 0 = 1 \). This gives \( \exp(-x) = 1/\exp(x) \).

Then

\[
\exp(x_1 - x_2) = \exp(x_1) \exp(-x_2) = \frac{\exp(x_1)}{\exp(x_2)}. \tag{4.3}
\]

The function \( \exp \) is continuous. To prove this, let \( \varepsilon > 0 \). Let us fix a real number \( a \) and define \( \delta = \log(1 + \varepsilon \exp(-a)) \). Consider any real \( x \) for which \( |x - a| < \delta \). Then \(-\delta < x - a < \delta \) and \( \exp(-\delta) < \exp(x - a) < \exp(\delta) \). This inequality gives

\[
\frac{1}{1 + \varepsilon/\exp(a)} < \exp(x - a) = \frac{\exp(x)}{\exp(a)} < 1 + \varepsilon \exp(-a) \tag{4.4}
\]

From this inequality it is easy to see that

\[
-\varepsilon < -\frac{\varepsilon}{1 + \varepsilon \exp(-a)} < \exp(x) - \exp(a) < \varepsilon.
\]

We have proved that for any \( \varepsilon > 0 \) and any real number \( a \) there exists \( \delta = \log(1 + \varepsilon \exp(-a)) \) such that if \( |x - a| < \delta \) then \( |\exp(x) - \exp(a)| < \varepsilon \), that is, \( \lim_{x \to a} \exp(x) = \exp(a) \).

In view of Property 8, the equation \( \log x = y \) has unique solution on \((0, \infty)\) for any real number \( y \). The solution of the equation \( \log x = 1 \) is denoted by \( e \). This is the base of natural logarithms.

## 5 Conclusions.

We have defined the logarithmic function as a limit of sequence of functions and we established its basic properties. We also defined the exponential function as the inverse of the logarithmic function. These facts allow to establish the law of exponents starting from the equation \( a^x = \exp(x \log a) \) for any real number \( x \) and \( a > 0 \). In particular, \( e^x = \exp(x \log e) = \exp x \). Finally, inequality (3.6) allows us to prove following important limits:

\[
\lim_{t \to 0} \frac{\log(1 + t)}{t} = 1, \quad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e
\]

and

\[
e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \text{ for any real number } x.
\]

## References


Received: April, 2012