Abstract

The problem of a satellite attitude control is considered. Satellite is considered as a solid body in the Newton field of forces. Motion is described by the well-known system of Euler-Poisson equations. By original change of variables this system is reduced to the normal form with explicitly expressed linear part. The system solution is accepted as the non-perturbed motion and it is investigated on stability on first approximation. Control moments are found for obtaining of asymptotically steady motion in general case.

Keywords: Satellite, attitude control, Newton field, Euler-Poisson equations.

1 INTRODUCTION

One of the widespread mathematical models for describing of a satellite motion is a solid body with one fixed point, placed in the field of some forces (gravity, magnetic, Sun and Moon gravity, solar radiation pressure and so on) [1]. The equations of motion don’t have analytic solution and usually they are solved numerically. And any analytic analysis of qualitative features of motion may be useful for obtaining of more precise solution.

2 ANALYSIS OF CONTROL POSSIBILITY

2.1 Problem Definition

The problem of dynamics of solid body with one fixed point in the
Newton field of forces is researched. Let the body’s motionless point $O$ is fixed on the distance $R$ from the centre of gravity $O'$. We combine with the body fixed point the origins of two coordinate systems: the immovable coordinate system $OXYZ$ with the axis $OZ$, directed along the vector $\vec{R} = \overrightarrow{OO'}$, and the mobile coordinate system $Oxyz$ with the axes, directed along the body’s main axes of inertia. Let $\alpha, \beta, \gamma$ are the direct cosines of the mobile axes with axis $OZ$. It is known that the force function $U$ of the Newton field of forces, acting on the solid body, is given by the formula:

$$U = \int \varphi'(r) dm,$$

$$r^2 = R^2 + 2R(x\alpha + y\beta + z\gamma) + \rho^2,$$

$$\rho^2 = x^2 + y^2 + z^2,$$

where $\varphi'(r)$ denotes the force function, acting on the body element $dm$ and depending only on the distance from the body’s point to the centre of gravity, $\rho$ is the distance from the body’s point to the fixed point $O$; the integral is taken on all the body volume.

The solid body motion around the fixed point $O$ is described by the well-known system of the Euler dynamic equations and the Poisson equations [1]:

$$A\ddot{p} + (C - B)qr = \gamma \frac{\partial U}{\partial \beta} - \beta \frac{\partial U}{\partial \gamma},$$

$$B\ddot{q} + (A - C)rp = \alpha \frac{\partial U}{\partial \gamma} - \gamma \frac{\partial U}{\partial \alpha},$$

$$C\ddot{r} + (B - A)pq = \beta \frac{\partial U}{\partial \alpha} - \alpha \frac{\partial U}{\partial \beta},$$

$$\dot{\alpha} = r\beta - q\gamma; \quad \dot{\beta} = p\gamma - r\alpha; \quad \dot{\gamma} = q\alpha - p\beta,$$

where $p, q, r$ are the projections of an angular speed of the body rotation on the mobile axes; $A, B, C$ are the body’s main moments of inertia.

If distance $R$ is great in comparison with the body sizes, the function $U$ may be presented in the form:

$$U = U_1(\alpha, \beta, \gamma) + U_2(\alpha, \beta, \gamma) + \ldots + U_n(\alpha, \beta, \gamma) + \ldots,$$

where $U_n(\alpha, \beta, \gamma)$ denotes the homogeneous function of the $n$-th degree. As is known only first two terms of this series have the significant influence on motion of the satellite, so we take:
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\[ U(\alpha, \beta, \gamma) = \frac{\lambda M}{R^2} (a\alpha + b\beta + c\gamma) - \frac{3}{2} \frac{\lambda}{R^2} (A\alpha^2 + B\beta^2 + C\gamma^2) + ..., \]  

(3)

where \( M \) is the satellite’s mass; \( a, b, c \) are the coordinates of the satellite centre of mass concerning the mobile coordinate system.

Substituting the expression (3) in the equations (1), we obtain the motion equations in the following form:

\[ A\dot{p} + (C - B)qr = Mg(c\beta - b\gamma) + 3 \frac{g}{R} (C - B)\beta\gamma, \]

\[ B\dot{q} + (A - C)rp = Mg(a\gamma - c\alpha) + 3 \frac{g}{R} (A - C)\gamma\alpha, \]  

(4)  

\[ C\dot{r} + (B - A)pq = Mg(b\alpha - a\beta) + 3 \frac{g}{R} (B - A)\alpha\beta, \]

where \( g = \frac{\lambda}{R^2} \) is an acceleration of the gravitation force on distance \( R \) from the centre of gravity. Now the satellite motion is described by the differential equations (2) and (4).

2.2 Research of Stability on the First Approximation

Let’s apply the linear change of variables \( (p, q, r, \alpha, \beta, \gamma) \rightarrow (u_1, u_2, u_3, u_4, u_5, u_6) \) of the next sort [2]:

\[ p = \frac{1}{\sqrt{|A|}} u_1, \quad q = \frac{1}{\sqrt{|B|}} u_2, \quad r = \frac{1}{\sqrt{|C|}} u_3, \]

\[ \alpha = \frac{1}{\sqrt{1 + \frac{3gl}{R} A}} \left( u_4 - \frac{Mgal}{\sqrt{1 + \frac{3gl}{R} A}} \right), \quad \beta = \frac{1}{\sqrt{1 + \frac{3gl}{R} B}} \left( u_5 - \frac{Mgbl}{\sqrt{1 + \frac{3gl}{R} B}} \right), \]

\[ \gamma = \frac{1}{\sqrt{1 + \frac{3gl}{R} C}} \left( u_6 - \frac{Mgcl}{\sqrt{1 + \frac{3gl}{R} C}} \right), \]

where \( I \) is the unit, having the following dimension: \([I] = \frac{s^2}{kg \cdot m^2}\).

Then the equations (4), (2) are reduced to the system of the non-dimensional equations with explicitly expressed linear part:
\[ \ddot{u} = \ddot{A}u + F(u), \]

where \( \ddot{u} = \{u_1, ..., u_6\} \) is the column vector; \( \ddot{A} \) is the constant square matrix of the 6th order; \( F(u) \) is the 6th order vector function, consisting of non-linear members of equations:

\[
F(u) = \begin{bmatrix}
- \frac{C - B}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} u_3 + \frac{l}{A} \sqrt{\lambda_1} \left( \frac{3g(C - B)}{\sqrt{\lambda_1}} \right) u_5 u_6, \\
- \frac{A - C}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} u_4 + \frac{l}{B} \sqrt{\lambda_2} \left( \frac{3g(A - C)}{\sqrt{\lambda_2}} \right) u_6 u_4, \\
- \frac{B - A}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} u_2 + \frac{l}{C} \sqrt{\lambda_3} \left( \frac{3g(B - A)}{\sqrt{\lambda_3}} \right) u_4 u_2, \\
1 \frac{R + 3glA}{\sqrt{\lambda_4}} u_4 - \frac{l}{\sqrt{\lambda_4}} \left( \frac{R + 3glB}{\sqrt{\lambda_4}} \right) u_5 u_3, \\
1 \frac{R + 3glB}{\sqrt{\lambda_5}} u_5 - \frac{l}{\sqrt{\lambda_5}} \left( \frac{R + 3glC}{\sqrt{\lambda_5}} \right) u_4 u_3, \\
1 \frac{R + 3glC}{\sqrt{\lambda_6}} u_6 - \frac{l}{\sqrt{\lambda_6}} \left( \frac{R + 3glA}{\sqrt{\lambda_6}} \right) u_5 u_3
\end{bmatrix}.
\]

The system of the first approximation will be:

\[ u = \ddot{A}u \quad \text{(5)} \]

and it has the next characteristic equation

\[ \lambda^6 + N \lambda^3 + P^2 \lambda^2 = 0. \quad \text{(6)} \]

Here under the constants \( N \) and \( P^2 \) the following expressions were designated:

\[
N = M^2 g^2 l \begin{bmatrix}
a^2 \left[ B \left( 1 + \frac{3gl}{R} \right) + C \left( 1 + \frac{3gl}{R} \right) \right] + b^2 \left[ C \left( 1 + \frac{3gl}{R} \right) + A \left( 1 + \frac{3gl}{R} \right) \right] + \\
b \left[ B \left( 1 + \frac{3gl}{R} \right) \right] + c^2 \left[ A \left( 1 + \frac{3gl}{R} \right) + B \left( 1 + \frac{3gl}{R} \right) \right] + \\
AB \left( 1 + \frac{3gl}{R} \right)
\end{bmatrix}
\]

\[
= M^2 g^2 l \begin{bmatrix}
a^2 \left[ B \left( 1 + \frac{3gl}{R} \right) + C \left( 1 + \frac{3gl}{R} \right) \right] + b^2 \left[ C \left( 1 + \frac{3gl}{R} \right) + A \left( 1 + \frac{3gl}{R} \right) \right] + \\
b \left[ B \left( 1 + \frac{3gl}{R} \right) \right] + c^2 \left[ A \left( 1 + \frac{3gl}{R} \right) + B \left( 1 + \frac{3gl}{R} \right) \right] + \\
AB \left( 1 + \frac{3gl}{R} \right)
\end{bmatrix}
\]

\[
= M^2 g^2 l \begin{bmatrix}
a^2 \left[ B \left( 1 + \frac{3gl}{R} \right) + C \left( 1 + \frac{3gl}{R} \right) \right] + b^2 \left[ C \left( 1 + \frac{3gl}{R} \right) + A \left( 1 + \frac{3gl}{R} \right) \right] + \\
b \left[ B \left( 1 + \frac{3gl}{R} \right) \right] + c^2 \left[ A \left( 1 + \frac{3gl}{R} \right) + B \left( 1 + \frac{3gl}{R} \right) \right] + \\
AB \left( 1 + \frac{3gl}{R} \right)
\end{bmatrix}
\]
The equation (6) gives immediately two zero characteristic roots and the biquadrate equation:

\[ \lambda_1 = \lambda_2 = 0, \quad \lambda^4 + N\lambda^2 + P^2 = 0. \]  

(7)

The roots of biquadrate equation (7) depend on sign of its discriminant. If discriminant \( D \) is positive, all the characteristic roots have zero real parts, and we can say nothing about stability of the unperturbed motion. If \( 0 < D \), it was shown [2] the biquadrate equation has two roots with positive real parts and according to the Lyapunov theorem the unperturbed motion is unstable in this case [3].

2.3 Attitude Control

Now we’ll try to make unstable motion in the last case the stable one with a help of some additive moments of forces. Let’s add to the right part of the equations of the system (5) the terms of the form \( \mu Ku_i \), where \( \mu \), \( i = 1, ..., 6 \), and \( \mu < 1 \), \( K = -\sqrt{2P - N} \).

We will obtain the following system of the differential equations:

\[ \ddot{u} = \bar{B}u + \bar{F}(u), \]  

(8)

where, as before, \( \bar{u} \) is the column vector; \( \bar{F}(u) \) is the 6th order vector function, consisting of non-linear members of equations; \( \bar{B} = \bar{A} + \mu K \bar{E} \) is the constant square matrix and \( \bar{E} \) is the unit matrix.

The system of first approximation has the following characteristic equation:

\[ (\mu K - \lambda)^2 \left[ (\mu K - \lambda)^4 + N(\mu K - \lambda)^2 + P^2 \right] = 0 \]
with the roots $\lambda_i$ of this equation are:

$$\lambda_1 = \lambda_2 = \mu K,$$

$$\lambda_{3,4} = -\sqrt{2P - N (2\mu - 1)} \pm \sqrt{2P + Ni} \over 2,$$

$$\lambda_{5,6} = -\sqrt{2P - N (2\mu + 1)} \pm \sqrt{2P + Ni} \over 2$$

Taking into account that $K < 0$ and constraining $\mu$ within the limits $0.5 < \mu < 1$, we will obtain the negative real parts for all the roots of the characteristic equation. Hence, the Lyapunov's theorem about asymptotical stability of the unperturbed motion of the system (8) is true [3].

It was shown [2] that additive control moments are created by the dissipative forces.

In case when the discriminant $D$ is positive, the proposed method can be applied too. But there we have to constrain $\mu$ within the limits $-1 < \mu < 0$ and we will obtain the negative real parts for all the roots of the characteristic equations. Hence, the Lyapunov's theorem about asymptotical stability of the unperturbed motion of the system (8) by the above-stated restrictions is true.

3 NUMERICAL CALCULATIONS

The results of theory were checked by numerical calculations. At first the geostationary satellite with mass 2.5 ton and moments of inertia $A = B = 1.2 \cdot 10^4, C = 5 \cdot 10^7$ (kg×m$^2$) was considered. For this satellite the transformed system of equations was solved numerically without controlling moments (fig. 1) and with them (fig. 2-4).

![Fig. 1 Changing of variables $u_1,..,u_6$ in time for the first satellite](image-url)
Fig. 2 Changing of variables $u_1, ..., u_6$ in time under controlling moment with $\mu = -1$

Fig. 3 Changing of variables $u_1, ..., u_6$ in time under controlling moment with $\mu = -0,1$

Fig. 4 Changing of variables $u_1, ..., u_6$ in time under controlling moment with $\mu = -0,001$
As the second example the small spacecraft was considered with mass 250 kg, moments of inertia $A = 65, B = 90, C = 75$ (kg×m$^2$) and altitude of 750 km. Figure 5 shows the calculated functions $u_1,\ldots,u_6$ depending on time without controlling moments. Influence of controlling moments is adduced in figure 6.

![Fig. 5 Changing of variables $u_1,\ldots,u_6$ in time for the second satellite](image5)

![Fig. 6 Changing of variables $u_1,\ldots,u_6$ in time under controlling moment with $\mu = -0.01$](image6)

As both satellites belong to the case when the discriminant $D$ is positive, $\mu$ is constrained within the limits $-1 < \mu < 0$. Figures show that all variables reach steady state very quickly depending on the magnitude of parameter $\mu$. 
4 CONCLUSIONS

One way of the satellite attitude control was considered. With a help of linear change of the variables we reduced the initial system to the system of the non-dimensional equations with explicitly expressed linear part. The transformed system of the equations was investigated on stability on the first approximation. It was shown that it is enough to add small dissipative forces, which create moments, stabilizing the body’s attitude. The results were corroborated by numerical calculations.

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References


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