Sensitivity Analysis of Costs
in Interval Transportation Problems

K. Kavitha and P. Pandian

Department of Mathematics, School of Advanced Sciences
VIT University, Vellore-632 014, India
k.kavitha@vit.ac.in, pandian61@rediffmail.com

Abstract

Sensitivity analysis of cost coefficients of an interval transportation problem is discussed. A new solution method namely, upper-lower method is proposed to determine the ranges of costs in the interval transportation problem such that its optimal basis is invariant. The proposed method is illustrated with the help of a numerical example. Further, the upper-lower method is extended to fuzzy transportation problems.

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Keywords: Sensitivity analysis, Transportation problem, Optimal solution, Intervals, Upper-lower method, Fuzzy numbers

1. Introduction

The transportation problem is one of the fundamental problems of network flow optimization which has a large number of real-life applications. Sensitivity analysis (SA) is one of the most interesting and preoccupying areas in optimization. SA is to analyze the effect of the changes of the parameters in the optimization problems on the optimal value of the objective function as well as the validity ranges of these effects. SA is a useful tool in model building as well as in model evaluation and it can also indicate which parameter values are reasonable to use in the model. Information of SA, in a TP, is usually more important than the optimal solution itself. Gal [5], Srinivasan and Thompson [11], Intrator and Paroush [6] and Arsham [2] studied the conventional SA of a TP and derived some interesting

In this paper, we propose a new method namely, upper-lower method to develop the SA of costs in an interval TP. Since we focus on interval cost parameter sensitivity, we will show that the optimal basis of the interval TP is invariant when the interval coefficients vary between the interval limits. An illustrative example is presented to clarify the idea of the upper-lower method. Then, we extend the proposed method to fully fuzzy TP. The SA of costs in a TP by the upper-lower method can help the decision makers to determine what level of accuracy is necessary for a parameter to make the model sufficiently useful and valid when they are handling distribution problem having imprecise parameters.

2. Preliminaries

We need the following definitions of the basic arithmetic operators and partial ordering on closed bounded intervals which can be found in [10].

Let $D = \{[a, b], a \leq b \text{ and } a \text{ and } b \text{ are in } R \}$ denote the set of all closed bounded intervals on the real line $R$.

**Definition 2.1:** Let $A = [a, b]$ and $B = [c, d]$ be in $D$. Then,
(i) $A \oplus B = [a + c, b + d]$ and
(ii) $A \otimes B = [p, q]$ where $p = \min\{ac, ad, bc, bd\}$ and $q = \max\{ac, ad, bc, bd\}$.

**Definition 2.2:** Let $A = [a, b]$ and $B = [c, d]$ be in $D$. Then,
(i) $A \preceq B$ if $a \leq c$ and $b \leq d$;
(ii) $A \succeq B$ if $a \geq c$ and $b \geq d$ and
(iii) $A = B$ if $a = c$ and $b = d$.

3. Fully Interval Integer Transportation Problem

Consider the following fully interval integer transportation problem (IP):

(IP) Minimize $[z_1, z_2] = \sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij}, d_{ij}] \otimes [x_{ij}, y_{ij}]$

subject to
Sensitivity analysis

\[
\sum_{j=1}^{n} [x_{ij}, y_{ij}] = [a_i, p_i], i = 1,2,\ldots, m
\]  \hspace{1cm} (1)

\[
\sum_{i=1}^{m} [x_{ij}, y_{ij}] = [b_j, q_j], j = 1,2,\ldots, n
\]  \hspace{1cm} (2)

\[x_{ij} \geq 0, \ y_{ij} \geq 0, \ i = 1,2,\ldots, m \text{ and } j = 1,2,\ldots, n \text{ and are integers} \] \hspace{1cm} (3)

where \(c_{ij}\) and \(d_{ij}\) are positive real numbers for all \(i\) and \(j\), \(a_i\) and \(p_i\) are positive real numbers for all \(i\) and \(b_j\) and \(q_j\) are positive real numbers for all \(j\).

Now, we need the following theorem which finds a relation between optimal solutions of a fully interval integer TP and a pair of induced TPs and also, is used in the proposed method which can be found in Pandian and Natarajan [10].

**Theorem 3.1:** If the set \([y_{ij}^\text{up}, y_{ij}^\text{lo}, \text{for all } i \text{ and } j]\) is an optimal solution of the upper bound TP (UP) of the problem (IP) where

\[(\text{UP}) \quad \text{Minimize } z_2 = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} y_{ij}\]

subject to

\[\sum_{j=1}^{n} y_{ij} = p_i, \ i = 1,2,\ldots, m ; \quad \sum_{i=1}^{m} y_{ij} = q_j, \ j = 1,2,\ldots, n ;\]

\[y_{ij} \geq 0, \ i = 1,2,\ldots, m \text{ and } j = 1,2,\ldots, n \text{ and are integers} \]

and the set \([x_{ij}^\text{lo}, \text{for all } i \text{ and } j]\) is an optimal solution of the lower bound TP (LP) of the problem (IP) where

\[(\text{LP}) \quad \text{Minimize } z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}\]

subject to

\[\sum_{j=1}^{n} x_{ij} = a_i, \ i = 1,2,\ldots, m ; \quad \sum_{i=1}^{m} x_{ij} = b_j, \ j = 1,2,\ldots, n ;\]

\[x_{ij} \geq 0, \ i = 1,2,\ldots, m \text{ and } j = 1,2,\ldots, n \text{ and are integers}, \]

then the set of intervals \([x_{ij}^\text{lo}, y_{ij}^\text{lo}, \text{for all } i \text{ and } j]\) is an optimal solution of the problem (IP) provided \(x_{ij}^\text{lo} \leq y_{ij}^\text{lo}, \text{for all } i \text{ and } j\).

**4. Sensitivity Analysis for Interval Cost Coefficients**

Consider the SA of interval cost transportation problem having interval cost matrix \([c_{ij}, d_{ij}]\). As per the Theorem 3.1., for analyzing the sensitivity of \((i,j)\)th cost...
of the interval TP, we enough to do the SA of (i,j)th cost of upper bound TP (UP) and the lower bound TP (LP) of the interval TP(IP). So, we separate the given interval TP into two classical TPs namely, upper TP having cost matrix \([d_{ij}]\) and lower TP having cost matrix\([c_{ij}]\).

Now, to analyze the sensitivity range of (i,j)th cost in the problem (UP), we proceed to a re-optimization of the problem, replacing \(d_{ij}\) by \(d_{ij} + t_2\). We compute the minimum and maximum values \(t_2\), \(t_2^0\) and \(t_2^1\) respectively so that the current solution remains optimal provided \((d_{ij} + t_2) - (u_i + v_j) \geq 0\) for all \(i\) and \(j\). Therefore, the sensitivity range of \(d_{ij}\) is \([d_{ij} + t_2^0, d_{ij} + t_2^1]\).

Now, to analyze the sensitivity range of (i,j)th cost in the problem (LP), we proceed to a re-optimization of the problem, replacing \(c_{ij}\) by \(c_{ij} + t_1\). We compute the minimum and maximum values \(t_1\), \(t_1^0\) and \(t_1^1\) respectively so that the current solution remains optimal provided that \((c_{ij} + t_1) - (u_i + v_j) \geq 0\) where \(u_i\) and \(v_j\) are Modi indices and \(c_{ij} + t_1^0 \leq d_{ij} + t_2^0\) and \(c_{ij} + t_1^1 \leq d_{ij} + t_2^1\), for all \(i\) and \(j\). Therefore, the sensitivity range of \(c_{ij}\) is \([c_{ij} + t_1^0, c_{ij} + t_1^1]\).

Thus, the sensitivity range of the interval cost \([c_{ij}, d_{ij}]\) is from the interval \([c_{ij} + t_1^0, c_{ij} + t_1^1]\) to the interval \([c_{ij} + t_1^0, c_{ij} + t_1^1]\).

### 4.1. Upper-Lower Method

We, now introduce a new method namely, upper-lower method to study the SA of interval cost TPs.

The upper-lower method proceeds as follows.

**Step 1.** Construct the upper bound and lower bound TPs for the given interval cost TP.

**Step 2.** Determine the optimal solution to the given interval cost TP using the separation method [10].

**Step 3.** (i) For cost SA, replace the (i,j)th cost value \(d_{ij}\) by \(d_{ij} + t_2\) in which the parameter \(t_2\) may vary.

(ii) Compute the minimum and maximum values of \(t_2\), say \(t_2^0\) and \(t_2^1\) respectively so that the optimal basis to the problem (UP) is not changed.
Step 4. (i) For cost SA, replace the (i,j)th cost value $c_{ij}$ by $c_{ij} + t_1$ in which the parameter $t_1$ may vary.

(ii) Compute the minimum and maximum values of $t_1$, say $t_1^0$ and $t_1^1$ respectively so that the optimal basis to the problem (LP) is not changed and $c_{ij} + t_1^0 \leq d_{ij} + t_2^0$ and $c_{ij} + t_1^1 \leq d_{ij} + t_2^1$.

Step 5. The minimum and maximum interval values of the (i,j)th cost $[c_{ij}, d_{ij}]$ is $[c_{ij} + t_1^0, d_{ij} + t_2^0]$ and $[c_{ij} + t_1^1, d_{ij} + t_2^1]$ respectively.

Now, the upper-lower method is illustrated by the following example.

**Example 4.1:** Consider the following fully interval transportation problem.

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1</td>
<td>[2,6]</td>
<td>[3,5]</td>
<td>[2,4]</td>
<td>[40,50]</td>
</tr>
<tr>
<td>O2</td>
<td>[1,3]</td>
<td>[3,6]</td>
<td>[2,4]</td>
<td>[25,30]</td>
</tr>
<tr>
<td>Demand</td>
<td>[20,25]</td>
<td>[15,20]</td>
<td>[30,35]</td>
<td>[65,80]</td>
</tr>
</tbody>
</table>

Now, using the separation method [12], the basic cells of the given fully interval TP are (1,2), (1,3), (2,1) and (2,3).

Now, we consider the SA of the given problem in the cell (1, 1) which is a non-basic cell.

**Case (i):** We consider the SA of the upper bound TP in the cell (1, 1).

<table>
<thead>
<tr>
<th></th>
<th>$v_1$ = 3</th>
<th>$v_2$ = 5</th>
<th>$v_3$ = 4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_1$ = 0</td>
<td>6 + $t_2$</td>
<td>5</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>$\nu_2$ = 0</td>
<td>3</td>
<td>25</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>20</td>
<td>35</td>
<td></td>
</tr>
</tbody>
</table>

Then, $t_2$ varies from -3 to $\infty$. Therefore, $d_{11}$ varies from 3 to $\infty$.

**Case (ii):** We consider the SA of the lower bound TP in the cell (1, 1).

<table>
<thead>
<tr>
<th></th>
<th>$v_1$ = 1</th>
<th>$v_2$ = 3</th>
<th>$v_3$ = 2</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_1$ = 0</td>
<td>2 + $t_1$</td>
<td>3</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>$\nu_2$ = 0</td>
<td>1</td>
<td>25</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>20</td>
<td>35</td>
<td></td>
</tr>
</tbody>
</table>
Then, \( t_1 \) varies from -1 to \( \infty \). Therefore, \( c_{11} \) varies from 1 to \( \infty \).
Thus, the cell (1,1) interval cost, \([c_{11}, d_{11}]\) varies from [1, 3] to \( (\infty, \infty) \).

Now, we consider the SA of the given problem in the cell (1, 2) which is a basic cell.

**Case (i):** We consider the SA of the upper bound TP in the cell (1, 2).

<table>
<thead>
<tr>
<th></th>
<th>( v_1 = 3 )</th>
<th>( v_2 = 5 + t_2 )</th>
<th>( v_3 = 4 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1 = 0 )</td>
<td>6</td>
<td>5 + ( t_2 )</td>
<td>4</td>
<td>30</td>
</tr>
<tr>
<td>( u_2 = 0 )</td>
<td>3</td>
<td>25</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>20</td>
<td>35</td>
<td>80</td>
</tr>
</tbody>
</table>

Then, \( t_2 \) varies from \(-\infty\) to 1. Now, since \( d_{12} \geq 0 \), \( d_{12} \) varies from 0 to 6.

**Case (ii):** We consider the SA of the lower bound TP in the cell (1, 2).

<table>
<thead>
<tr>
<th></th>
<th>( v_1 = 1 )</th>
<th>( v_2 = 3 + t_1 )</th>
<th>( v_3 = 2 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1 = 0 )</td>
<td>2</td>
<td>3 + ( t_1 )</td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>( u_2 = 0 )</td>
<td>1</td>
<td>25</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>20</td>
<td>35</td>
<td>80</td>
</tr>
</tbody>
</table>

Then, \( t_1 \) varies from \(-\infty\) to 0. Now, since \( c_{12} \geq 0 \), \( c_{12} \) varies from 0 to 3.
Thus, the cell (1,2,) interval cost, \([c_{12}, d_{12}]\) varies from [0,0] to [3, 6].

Similarly, we can find the sensitivity ranges of other costs in the interval transportation problems.

Now, the sensitivity ranges of all costs in the given interval TP such that its optimal solution is invariant, are given in the following table:

<table>
<thead>
<tr>
<th>Cell</th>
<th>Minimum limit</th>
<th>Original value</th>
<th>Maximum limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>[1, 3]</td>
<td>[2, 6]</td>
<td>(( \infty, \infty ))</td>
</tr>
<tr>
<td>(1,2)</td>
<td>[0,0]</td>
<td>[3, 5]</td>
<td>[3, 6]</td>
</tr>
<tr>
<td>(1,3)</td>
<td>[2, 3]</td>
<td>[2, 4]</td>
<td>[3, 7]</td>
</tr>
<tr>
<td>(2,1)</td>
<td>[0,0]</td>
<td>[1, 3]</td>
<td>[2, 6]</td>
</tr>
<tr>
<td>(2,2)</td>
<td>[3, 5]</td>
<td>[3, 6]</td>
<td>(( \infty, \infty ))</td>
</tr>
<tr>
<td>(2,3)</td>
<td>[1, 1]</td>
<td>[2, 4]</td>
<td>[2, 5]</td>
</tr>
</tbody>
</table>
5. Fully Fuzzy Transportation Problem

Consider the following fuzzy integer transportation problem (FFITP) where

\[(\text{FFITP}) \quad \text{Minimize} \quad \tilde{z} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \tilde{c}_{ij} \otimes \tilde{x}_{ij} \right) \]

subject to

\[\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_{i}, \ i=1,2,\ldots,m; \quad \sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_{j}, \ j=1,2,\ldots,n; \]

where \( m \) = the number of supply points; \( n \) = the number of demand points; \( \tilde{x}_{ij} \) is the uncertain number of units shipped from supply point \( i \) to demand point \( j \); \( \tilde{c}_{ij} \) is the uncertain cost of shipping one unit from supply point \( i \) to the demand point \( j \) which is non-negative; \( \tilde{a}_{i} \) is the uncertain supply at supply point \( i \) and \( \tilde{b}_{j} \) is the uncertain demand at demand point \( j \).

A trapezoidal fuzzy number \((a,b,c,d)\) can be represented as an interval number form as follows.

\[(a,b,c,d) = [a + (b - a)\alpha, d - (d - c)\alpha] ; \ 0 \leq \alpha \leq 1. \quad (4)\]

Using the relation (4), we can convert the given fuzzy TP into an interval TP. Using the upper-lower method, we obtain the ranges of costs in the interval TP such that its optimal is invariant. Then, again using the relation (4), we can find the ranges of costs in the given fuzzy TP.

Example 5.1: Consider the following fuzzy transportation problem

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1</td>
<td>(0,1,3,4)</td>
<td>(2,3,5,6)</td>
<td>(10,20,30,40)</td>
</tr>
<tr>
<td>O2</td>
<td>(1,3,5,7)</td>
<td>(2,6,7,9)</td>
<td>(2,4,8,12)</td>
</tr>
<tr>
<td>Demand</td>
<td>(2,8,10,20)</td>
<td>(10,16,28,32)</td>
<td></td>
</tr>
</tbody>
</table>

Now, using the relation (4) and upper-lower method, we obtain the sensitivity ranges of fuzzy costs of the given fuzzy TP as follows:

<table>
<thead>
<tr>
<th>Cell</th>
<th>Minimum limit</th>
<th>Original value</th>
<th>Maximum limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>(0,0,0,0)</td>
<td>(0,1,3,4)</td>
<td>(1,2,3,4)</td>
</tr>
<tr>
<td>(1,2)</td>
<td>(1,2,5,6)</td>
<td>(2,3,5,6)</td>
<td>(\infty, \infty, \infty)</td>
</tr>
<tr>
<td>(2,1)</td>
<td>(0,4,5,7)</td>
<td>(1,3,5,7)</td>
<td>(\infty, \infty, \infty)</td>
</tr>
<tr>
<td>(2,2)</td>
<td>(0,0,0,0)</td>
<td>(2,6,7,9)</td>
<td>(3,7,7,9)</td>
</tr>
</tbody>
</table>
REFERENCES


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