Applied Mathematical Sciences, Vol. 6, 2012, no. 92, 4589 - 4600

Common Fixed Points of Weakly Compatible Mappings in G-Metric Spaces

Zead Mustafa

Department of Mathematics
The Hashemite University
P.O. 330127, Zarqa 13115, Jordan
zmagablh@hu.edu.jo

Abstract. In this paper, we prove several common fixed points results for pair of weakly compatible mapping satisfying certain contractive conditions on G-metric space. Also we present some examples to support our results.

Mathematics Subject Classification: Primary 47H10, Secondary 46B20

Keywords: Metric space, G-metric space, Fixed points (common), Commuting Mappings

1. Introduction and preliminaries

Recently, Several authors have proved a good number of common fixed points theorem for two three or four maps satisfying certain contractive condition.

The commutativity of pairs of maps is vital for proving the common foxed point theorems and jungck ([2]) in 1976 first used it in ordinary metric space to prove common fixed point theorem for commuting maps, but his results required the continuity of one of the maps.

In ordinary metric space, Sessa ([9]) first introduced a weaker version of the commutativity for a pair of selfmaps and it is shown in Sessa ([9]) that weakly commuting pair of maps in metric pace is commuting, but the converse may not be true.

Jungck ([3]) introduce the notion of compatible mappings in order to generalized the concepts of weak commutativity and showed that weak commuting map are compatible, but the reverse implication may not hold.

Jungck ([5]) defined a pair of self mappings to be weakly compatible if they commute at their coincidence points.

Thus, we have one way implication namely, Commuting maps $\Rightarrow$ Weakly Commuting maps $\Rightarrow$ compatible maps $\Rightarrow$ Weakly Compatible maps. Recently

Various Authors have introduced a coincidence points results for a various classes of mappings on metric spaces, for more detail of coincidence point theory and related results see ([4, 6, 8]).

In 2005 Zead Mustafa and Brailey Sims introduce the notion of G-metric spaces as generalization of the concept of ordinary metric spaces and they have obtained some fixed point results for mappings satisfying different contractive conditions.

In this paper we will obtain several common fixed point results for two weakly compatible mapping satisfying certain contractive condition on G-metric space. Also we present some examples to support our results.

The following definitions and results will be needed in the sequel.

Definition 1. *G-metric space is a pair (X, G), where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$ we have:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$;

(G2) $0 < G(x, x, y)$; *for all* $x, y \in X$, *with* $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, *(symmetry in all three variables); and*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$, *(rectangle inequality).*

*The function G is called a G-*metric *on X.*

Definition 2. ([11]) *A sequence (x_n) in a G-metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is G-*convergent *to x. (through this paper we mean by $\mathbf{N}$ the set of all natural numbers).*

Proposition 1. ([11]) *Let X be G-metric space. Then the following statements are equivalent.*

(1) (x_n) *is G-convergent to x.*

(3) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$.

(4) $G(x_n, x, x) \to 0$, *as* $n \to \infty$.

(5) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 3. ([11]) *In a G-metric space X, a sequence (x_n) is said to be G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$. That is $G(x_n, x_m, x_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.*

Proposition 2. ([11]) *In a G-metric space X, the following statements are equivalent.*

1. *The sequence (x_n) is G-Cauchy.*
2. *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 4. ([11]) *Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G*-continuous at a point *$a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous on X if it is G-continuous at all $a \in X$.*

Proposition 3. ([11]) *Let (X, G), and $(X^{'}, G^{'})$ be two G-metric spaces. Then a function $f : X \longrightarrow X^{'}$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x we have $(f(x_n))$ is $G^{'}$-convergent to $f(x)$.*

Definition 5. ([11]) *a G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$, and called nonsymmetric if its not symmetric.*

Proposition 4. ([11]) *Let X be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 6. ([11]) *A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.*

Definition 7. ([19]) *Let f and g be self maps of a set X. If $w = fx = gx$ for some $x \in X$. Then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*

Recall that, a pair of self mappings called weakly compatible if they commute at their coincidence points.

Proposition 5. ([19]) *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Proof. Since $w = fx = gx$ and f and g are weakly compatible, we have $fw = fgx = gfx = gw$ is a point of coincidence of f and g. But w is the only point of coincidence of f and g, so $w = fw = gw$. More over if $z = fz = gz$, then z is a point of coincidence of f and g, and therefor $z = w$ by uniqueness. Thus w is the unique common fixed point of f and g. □

Example 1. *Let $X = \mathbf{R}$ and define $f, g : \mathbf{R} \to \mathbf{R}$, by $f(x) = x^3$ and $g(x) = \frac{x^2}{4}$ for $x \in \mathbf{R}$. Clearly there are two coincidence points for the maps f and g in $\mathbf{R}$, namely 0 and $\frac{1}{4}$. We see that f and g are commute at 0; i.e. $f(g(0)) = g(f(0))$, but $f(g(\frac{1}{4})) = \frac{1}{(64)^3} \neq g(f(\frac{1}{4})) = \frac{1}{4}(\frac{1}{64})^2$, and so f and g are not weakly compatible on $\mathbf{R}$.*

In [7], the authors proved the following theorem.

Theorem 1.1. *Let (X, d) be a metric space, g be a continuous self mapping of X and $f : X \to X$ satisfying the following conditions:*

1. *$f(g(x)) = g(f(x))$ for every $x \in X$,*

2. $f(X) \subset g(X)$.

If there exists a constant $0 \leq \alpha < 1$ *such that for every* $x, y \in X$

$$d(fx, fy) \leq \alpha \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\} \tag{1.1}$$

then f *and* g *have a unique common fixed point.*

In [14], Mustafa Proved the following theorem.

Theorem 1.2. *Let* (X, G) *be a complete* G*-metric space, and suppose that* $T : X \longrightarrow X$ *satisfying the following condition*

$$G(T(x), T(y), T(z)) \leq k\, G(x, y, z), \text{ for all } x, y, z \in X \tag{1.2}$$

where $k \in [0, 1)$, *then.*

1. T *has a unique fixed point, say* u.
2. *For each* $x_0 \in X$, *the sequence* $(T^n x_0)$ *is* G*-convergent to* u.
3. $G(T^n x_0, u, u) \leq \frac{k^n}{1-k} G(x_0, T(x_0), u)$.
4. $G(T^n x_0, T^{n+p} x_0, u) \leq \frac{k^n}{1-k}\, G(x_0, T(x_0), u)$, *for all* $p \in \mathbf{N}$.
5. $G(T^n x_0, u, u) \leq \frac{k^n}{1-k} G(x_0, T(x_0), T(x_0))$.

2. Main Results

Theorem 2.1. *Let* (X, G) *be a* G*-metric space, suppose mappings* $f, g : X \longrightarrow X$ *satisfy the following condition*

$$G(f(x), f(y), f(z)) \leq kM(x, y, z);$$

$$M(x, y, z) = \max \left\{ \begin{array}{l} G(g(x), g(y), g(z)), G(g(x), f(y), g(z)), \\ G(g(y), f(x), g(z)), G(g(x), f(x), g(z))), \\ G(g(y), f(y), g(z)) \end{array} \right\} \tag{2.1}$$

for all $x, y, z \in X$, *where* $k \in [0, 1/2)$. *If the range of* g *contains the range of* f *and* $g(X)$ *is a* G*-complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$, this can be done since the range of g contains the range of f. Continue in this process having chosen $(x_n) \in X$ such that $f(x_n) = g(x_{n+1})$. We may assume that $g(x_n) \neq g(x_{n+1})$ for each n. Since if there exists an n such that $g(x_n) = g(x_{n+1})$, then $g(x_n) = g(x_{n+1}) = f(x_n)$, yields f and g have a coincidence point. Then from (2.1) we have

$G(g(x_n), g(x_n), g(x_{n+1})) = G(f(x_{n-1}), f(x_{n-1}), f(x_n)) \leq$
$k \max\{G(g(x_{n-1}), g(x_{n-1}), g(x_n)), G(g(x_{n-1}), f(x_{n-1}), g(x_n))\}$,

Thus,

$$G(g(x_n), g(x_n), g(x_{n+1})) \leq k \max \left\{ \begin{array}{l} G(g(x_{n-1}), g(x_{n-1}), g(x_n)), \\ G(g(x_{n-1}), g(x_n), g(x_n)) \end{array} \right\} \tag{2.2}$$

but $G(g(x_{n-1}), g(x_n), g(x_n)) \leq 2G(g(x_{n-1}), g(x_{n-1}), g(x_n))$,
therefore (2.2) implies that

$$G(g(x_n), g(x_n), g(x_{n+1})) \leq 2kG(g(x_{n-1}), g(x_{n-1}), g(x_n)).$$

Let $q = 2k$, then $0 \leq q < 1$, continue the above process we obtain

$$G(g(x_n), g(x_n), g(x_{n+1})) \leq q^n G(g(x_0), g(x_0), g(x_1)).$$

For every $m, n \in \mathbf{N}; m > n$, we have by repeated use the rectangle inequality that

$$\begin{aligned} G(g(x_n), g(x_n), g(x_m)) &\leq \sum_{j=n}^{m-1} G(g(x_j), g(x_j), g(x_{j+1})) \\ &\leq \sum_{j=n}^{m-1} q^j G(g(x_0), g(x_0), g(x_1)) \\ &\leq \frac{q^n}{1-q} G(g(x_0), g(x_0), g(x_1)). \end{aligned}$$

Therefore, limit $G(g(x_n), g(x_n), g(x_m)) = 0$ as $m, n \to \infty$, hence $(g(x_n))$ is G-cauchy sequence.

Since $g(X)$ is G-complete there exists a point u in $g(X)$ such that $(g(x_n)) \to u$, as $n \to \infty$, consequently we can find $p \in X$ such that $g(p) = u$, we claim that $f(p) = g(p)$. If not then (2.1) implies that

$G(f(p), f(p), g(x_n)) = G(f(p), f(p), f(x_{n-1})) \leq$
$k \max\{G(g(p), g(p), g(x_{n-1})), G(g(p), f(p), g(x_{n-1}))\}$,

that is,

$$G(f(p), f(p), g(x_n)) \leq k \max\{G(g(p), g(p), g(x_{n-1})), G(g(p), f(p), g(x_{n-1}))\}. \tag{2.3}$$

Taking the limit of both sides of (2.3) as $n \to \infty$ we get,

$$G(f(p), f(p), g(p)) \leq k \max\{0, G(g(p), f(p), g(p))\} = kG(g(p), f(p), g(p)). \tag{2.4}$$

similarly we can get,

$$G(g(p), g(p), f(p)) \leq kG(f(p), f(p), g(p)). \tag{2.5}$$

So, (2.4) and (2.5) implies that $G(f(p), f(p), g(p)) \leq k^2 G(f(p), f(p), g(p))$, a contradiction, hence $f(p) = g(p)$.

Now we will show that p is unique, assume that there exists another point $v \in X$; $f(v) = g(v)$ then if $g(v) \neq g(p)$ we deduce from (2.1) that

$$G(g(v), g(p), g(p)) = G(f(v), f(p), f(p)) \leq k \max \left\{ \begin{array}{l} G(g(v), g(p), g(p)), G(g(v), f(p), g(p)), \\ G(g(p), f(v), g(p)), G(g(v), f(v), g(p)), \\ G(g(p), f(p), g(p)) \end{array} \right\} \tag{2.6}$$

yields,

$G(g(v), g(p), g(p)) \leq k \max\{G(g(v), g(p), g(p)), G(g(v), g(v), g(p))\} = kG(g(v), g(v), g(p))$, similarly we get, $G(g(v), g(v), g(p)) \leq kG(g(v), g(p), g(p))$ yields, $G(g(v), g(p), g(p)) \leq k^2 G(g(v), g(p), g(p))$, which is a contradiction. Hence $g(v) = g(p)$.

So, from Proposition (5), f and g have unique common fixed point. □

Now we give some examples to support Theorem 2.1.

Example 2. *Let $X = [0, 1]$, $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$, $f(x) = \frac{1}{8}x^2$, $g(x) = \frac{1}{2}x^2$ and $k = \frac{1}{3}$ for all $x, y, z \in X$.*

We have $f(X) = [0, \frac{1}{8}]$, $g(X) = [0, \frac{1}{2}]$ and $g(X)$ is G-complete subspace of X.

Also we have,

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{8} \max\{|x^2 - y^2|, |y^2 - z^2|, |x^2 - z^2|\} \\ &\leq \frac{1}{6} \max\{|x^2 - y^2|, |y^2 - z^2|, |x^2 - z^2|\} \\ &= \frac{1}{3}(\frac{1}{2} G(gx, gy, gz)) \\ &\leq \frac{1}{3}(M(x, y, z)). \end{aligned}$$

Note that $w = 0 = f(0) = g(0)$, is the unique point of coincidence in X. Moreover f and g are weakly compatible. Hence, all conditions of Theorem 2.1 are satisfied and $u = 0$ is the unique common fixed point of f and g.

Note that Theorem 1.1 is not applicable because f doesn't commute with g. Indeed, $f(g(x)) = \frac{1}{32}x^4 \neq g(f(x)) = \frac{1}{128}x^4$ for any $x \neq 0$ in X.

Theorem 2.2. *Let (X, G) be G-metric space, suppose mappings $f, g : X \longrightarrow X$ satisfy the following condition*

$$G(f(x), f(y), f(z)) \leq M(x, y, z);$$

where,

$$M(x, y, z) = \left\{ \begin{array}{l} a_1 G(g(x), g(y), g(z)) + a_2 G(g(x), g(x), f(x)) + \\ a_3 G(g(y), g(y), f(y)) + a_4 G(g(z), g(z), f(z)) + \\ a_5\, G(g(x), g(x), f(y)) + a_6\, G(g(y), g(y), f(z)) + \\ a_7 G(g(z), g(z), f(x)) \end{array} \right\} \tag{2.7}$$

for all $x, y, z \in X$, where $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 + a_7 < 1$. If the range of g contains the range of f and $g(X)$ is a G-complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$, this can be done since the range of g contains the range of f, continue this process having chosen (x_n) we obtain $x_{n+1} \in X$ such that

$f(x_n) = g(x_{n+1})$. We may assume that $g(x_n) \neq g(x_{n+1})$ for each n, since otherwise f and g have a coincidence point. Then (2.7) implies that

$G(g(x_n), g(x_n), g(x_{n+1})) = G(f(x_{n-1}), f(x_{n-1}), f(x_n)) \leq$
$\{a_1 G(g(x_{n-1}), g(x_{n-1}), g(x_n)) + (a_2 + a_3 + a_5)G(g(x_{n-1}), g(x_{n-1}), f(x_{n-1}) +$
$a_4 G(g(x_n), g(x_n), g(x_{n+1})) + a_6 G(g(x_{n-1}), g(x_{n-1}), g(x_{n+1})) + a_7 G(g(x_n), g(x_n), f(x_{n-1}))\}$,

Thus,

$$G(g(x_n), g(x_n), g(x_{n+1})) \leq \left\{ \begin{array}{l} kG(g(x_{n-1}), g(x_{n-1}), g(x_n)) + \\ a_4 G(g(x_n), g(x_n), g(x_{n+1})) + \\ a_6 G(g(x_{n-1}), g(x_{n-1}), g(x_{n+1})) \end{array} \right\} \tag{2.8}$$

where $k = (a_1 + a_2 + a_3 + a_5)$, but from $G(5)$ we have

$G(g(x_{n-1}), g(x_{n-1}), g(x_{n+1})) \leq G(g(x_{n-1}), g(x_{n-1}), g(x_n)) + G(g(x_n), g(x_n), g(x_{n+1}))$,

therefore, (2.8) implies that

$G(g(x_n), g(x_n), g(x_{n+1})) \leq qG(g(x_{n-1}), g(x_{n-1}), g(x_n))$

where $q = \frac{a_1+a_2+a_3+a_5+a_6}{1-(a_4+a_6)} < 1$. Continuing the above process, we obtain

$G(g(x_n), g(x_n), g(x_{n+1})) \leq q^n G(g(x_0), g(x_0), g(x_1))$

For every $m, n \in \mathbf{N}; m > n$ we have by repeated use the rectangle inequality that

$$\begin{aligned} G(g(x_n), g(x_n), g(x_m)) &\leq \sum_{j=n}^{m-1} G(g(x_j), g(x_j), g(x_{j+1})) \\ &\leq \sum_{j=n}^{m-1} q^j G(g(x_0), g(x_0), g(x_1)) \\ &\leq \frac{q^n}{1-q} G(g(x_0), g(x_0), g(x_1)). \end{aligned}$$

Therefor, limit $G(g(x_n), g(x_n), g(x_m)) \to 0$ as $m, n \to \infty$, hence $(g(x_n))$ is G-cauchy sequence.

Since $g(X)$ is G-complete there exists a point u in $g(X)$ such that $(g(x_n)) \to u$, as $n \to \infty$, consequently we can find $p \in X$ such that $g(p) = u$, we claim that $f(p) = g(p)$, if not then (2.7) implies that

$G(g(x_n), g(x_n), f(p)) = G(f(x_{n-1}), f(x_{n-1}), f(p)) \leq$
$\{a_1 G(g(x_{n-1}), g(x_{n-1}), g(p)) + (a_2 + a_3 + a_5)G(g(x_{n-1}), g(x_{n-1}), f(x_{n-1})) +$
$a_4 G(g(p), g(p), f(p)) + a_6 G(g(x_{n-1}), g(x_{n-1}), f(p)) + a_7 G(g(p), g(p), f(x_{n-1}))\}$

Taking the limit of both sides of the above equation as $n \to \infty$ we get, $G(g(p), g(p), f(p)) \leq (a_6 + a_4)G(g(p), g(p), g(p))$, which is a contradiction. Therefore $f(p) = g(p)$.

Now we will show that p is unique, assume that there exists another point $v \in X$; $f(v) = g(v)$, then if $g(v) \neq g(p)$ we deduce from (2.7) that

$G(g(v), g(v), g(p)) = G(f(v), f(v), f(p)) \leq \frac{a_7}{1-(a_1+a_6)} G(g(p), g(p), g(v))$,

similarly we get

$G(g(p), g(p), g(v)) \leq \frac{a_7}{1-(a_1+a_6)} G(g(v), g(v), g(p))$, therefore

$G(g(v), g(v), g(p)) \leq (\frac{a_7}{1-(a_1+a_6)})^2 G(g(v), g(v), g(p))$, which is a contradiction. Hence $g(v) = g(p)$.

So, from Proposition (5), f and g have unique common fixed point. □

Now we give some examples to support Theorem 2.2.

Example 3. *Let $X = [1, \infty)$ be endowed with the G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$ for all $x, y, z \in X$. Define $f, g : X \to X$ by $f(x) = 3x - 2$ and $g(x) = 4x - 3$ for each $x \in X$. Set $a_1 = \frac{4}{5}$ and $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = \frac{25}{1000}$.*

It is clear that the range of g contains the range of f and $g(X)$ is a G-complete subspace of X, also f and g have a unique point of coincidence $w = 1$ in X. Moreover f and g are weakly compatible.

Note that for all $x, y, z \in X$ we have

$$\begin{aligned} G(f(x), f(y), f(z)) =& 3[|x-y| + |x-z| + |y-z|] \\ \leq& \frac{16}{5}[|x-y| + |x-z| + |y-z|] \\ =& \frac{4}{5}(4[|x-y| + |x-z| + |y-z|]) \\ =& \frac{4}{5} G(g(x), g(y), g(z)) \leq M(x, y, z) \end{aligned}$$

Thus, all conditions of Theorem 2.2 are satisfied and $u = 1$ is the unique common fixed point of f and g.

Note that Theorem (1.2) is not applicable in this case. Indeed, for $y = z = 1$ and $x = 2$

$$G(f(2), f(1), f(1)) = 6 > 2k = kG(2, 1, 1) \quad \textit{for all } k \in [0, 1).$$

Also, the Banach principle [1] is not applicable. Indeed, for $d(x, y) = |x - y|$ for all $x, y \in X$ we have for $x \neq y$

$$d(f(x), f(y) = 3|x - y| > k|x - y| \quad \textit{for all } k \in [0, 1).$$

Theorem 2.3. *Let (X, G) be a G-metric space, suppose mappings $f, g : X \longrightarrow X$ satisfy the following condition*

$$G(f(x), f(y), f(z)) \leq kM(x, y, z);$$

$$(2.9) \quad M(x, y, z) = \max \left\{ \begin{array}{l} [G(gx, gx, fy) + G(gy, gy, fz) + G(gz, gz, fx)], \\ [G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz)], \\ [G(gy, fx, fx) + G(gz, fy, fy) + G(gx, fz, fz)] \end{array} \right\}$$

for all $x, y \in X$, where $k \in [0, \frac{1}{6})$. If the range of g contains the range of f and $g(X)$ is a G-complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Argument similar in previous theorem we can choose a sequence $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. We may assume that $g(x_n) \neq g(x_{n+1})$ for each n, since otherwise f and g have a coincidence point. Then from (2.9) we will have

$G(g(x_n), g(x_{n+1}), g(x_{n+1})) = G(f(x_{n-1}), f(x_n), f(x_n)) \leq$

$$k \max \left\{ \begin{array}{l} [G(g(x_{n-1}), f(x_{n-1}), f(x_{n-1})) + 2G(g(x_n), f(x_n), f(x_n))], \\ [G(g(x_{n-1}), g(x_{n-1}), f(x_n)) + G(g(x_n), g(x_n), f(x_n)) + \\ G(g(x_n), g(x_n), f(x_{n-1}))], \ [G(g(x_n), f(x_{n-1}), f(x_{n-1})) + \\ G(g(x_n), f(x_n), f(x_n)) + G(g(x_{n-1}), f(x_n), f(x_n))] \end{array} \right\} \tag{2.10}$$

Therefore,

$G(g(x_n), g(x_{n+1}), g(x_{n+1})) \leq$

$$k \max \left\{ \begin{array}{l} [G(g(x_{n-1}), g(x_n), g(x_n)) + 2G(g(x_n), g(x_{n+1}), g(x_{n+1}))], \\ [G(g(x_{n-1}), g(x_{n-1}), g(x_{n+1})) + G(g(x_n), g(x_n), g(x_{n+1}))], \\ [G(g(x_n), g(x_{n+1}), g(x_{n+1})) + G(g(x_{n-1}), g(x_{n+1}), g(x_{n+1}))] \end{array} \right\} \tag{2.11}$$

using the fact that, from $G(5)$ we have,

$G(g(x_{n-1}), g(x_{n+1}), g(x_{n+1})) \leq G(g(x_{n-1}), g(x_n), g(x_n)) + G(g(x_n), g(x_{n+1}), g(x_{n+1}))$,

also,

$G(g(x_{n-1}), g(x_{n-1}), g(x_{n+1})) \leq 2G(g(x_{n-1}), g(x_n), g(x_n)) + 2G(g(x_n), g(x_{n+1}), g(x_{n+1}))$,

and

$G(g(x_n), g(x_n), g(x_{n+1})) \leq 2G(g(x_n), g(x_{n+1}), g(x_{n+1}))$

so, equation (2.11) reduces to,

$G(g(x_n), g(x_{n+1}), g(x_{n+1})) \leq \max\{\frac{k}{1-2k}, \frac{2k}{1-4k}\}\, G(g(x_{n-1}), g(x_n), g(x_n))$.

Continuing the above process we obtain,

$G(g(x_n), g(x_{n+1}), g(x_{n+1})) \leq q^n G(g(x_0), g(x_1), g(x_1))$, where $q = \frac{2k}{1-4k} < 1$.

For every $m, n \in \mathbf{N}$; $m > n$ we have by repeated use the rectangle inequality that

$$\begin{aligned} G(g(x_n), g(x_m), g(x_m)) &\leq \sum_{j=n}^{m-1} G(g(x_j), g(x_{j+1}), g(x_{j+1})) \\ &\leq \sum_{j=n}^{m-1} q^j G(g(x_0), g(x_1), g(x_1)) \\ &\leq \frac{q^n}{1-q} G(g(x_0), g(x_1), g(x_1)). \end{aligned}$$

Therefore, limit $G(g(x_n), g(x_m), g(x_m)) = 0$ as $m, n \to \infty$, hence $(g(x_n))$ is G-cauchy sequence.

Since $g(X)$ is G-complete there exists a point u in $g(X)$ such that $(g(x_n)) \to u$, as $n \to \infty$, consequently we can find $p \in X$ such that $g(p) = u$, we claim that $f(p) = g(p)$, if not then (2.9) implies that

$G(g(x_n), f(p), f(p)) = G(f(x_{n-1}), f(p), f(p)) \leq$

(2.12)

$$k \max \left\{ \begin{array}{l} [G(g(x_{n-1}), g(x_n), g(x_n)) + 2G(g(p), f(p), f(p))], \\ [G(g(x_{n-1}), g(x_{n-1}), f(p)) + G(g(p), g(p), f(p)) + G(g(p), g(p), f(x_{n-1}))], \\ [G(g(p), f(x_{n-1}), f(x_{n-1})) + G(g(p), f(p), f(p)) + G(g(x_{n-1}), f(p), f(p))] \end{array} \right\},$$

taking the limit of both sides of the above equation as $n \to \infty$ we get,

$$G(g(p), f(p), f(p)) \leq k \max \left\{ \begin{array}{l} [2G(g(p), f(p), f(p))], \\ [2G(g(p), g(p), f(p))] \end{array} \right\}, \tag{2.13}$$

but from $G(5)$ we have $G(g(p), g(p), f(p)) \leq 2G(g(p), f(p), f(p))$, then (2.13) implies that,

$$G(g(p), f(p), f(p)) \leq 4kG(g(p), f(p), f(p)),$$

which is a contradiction. Hence $f(p) = g(p)$.

Now we will show that p is unique, assume that there exists another point $v \in X$; $f(v) = g(v)$ then if $g(v) \neq g(p)$ we deduce from (2.9) that

$G(g(v), g(p), g(p)) = G(f(v), f(p), f(p)) \leq$

(2.14)

$$k \max \left\{ \begin{array}{l} [G(g(v), g(v), f(p)) + G(g(p), g(p), f(p)) + G(g(p), g(p), f(v))], \\ [G(g(p), f(v), f(v)) + G(g(p), f(p), f(p)) + G(g(v), f(p), f(p))] \end{array} \right\}$$

then,

$$G(g(v), g(p), g(p)) \leq \frac{k}{1-k} G(g(v), g(v), g(p)) \tag{2.15}$$

again by the same argument we deduce that,

$$G(g(v), g(v), g(p)) \leq \frac{k}{1-k} G(g(v), g(p), g(p)) \tag{2.16}$$

combining (2.15) and (2.16) gives,

$G(g(v), g(p), g(p)) \leq (\frac{k}{1-k})^2 G(g(v), g(p), g(p))$,

which is a contradiction since $\frac{k}{1-k} < 1$. Hence $g(v) = g(p)$.

So, from Proposition (5), f and g have unique common fixed point. □

Now we give some examples to support Theorem 2.3.

Example 4. *Let* $X = [0, 1]$, *define* $G : X \times X \times X \longrightarrow [0, \infty)$ *by* $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$ *and* $f, g : X \longrightarrow X$ *by* $f(x) = \frac{x^3}{16}$ *and* $g(x) = \frac{x^3}{2}$.

It is clear that the range of g *contains the range of* f *and* $g(X)$ *is a* G-*complete subspace of* X, *also* f *and* g *have a unique point of coincidence* $w = 0$ *in* X. *Moreover* f *and* g *are weakly compatible.*

Note that for $x, y, z \in X$ we have
$|\frac{x^3}{16} - \frac{y^3}{16}| \leq \frac{x^3}{16} + \frac{y^3}{16}$,
$|\frac{y^3}{16} - \frac{z^3}{16}| \leq \frac{y^3}{16} + \frac{z^3}{16}$, *and*
$|\frac{x^3}{16} - \frac{z^3}{16}| \leq \frac{x^3}{16} + \frac{z^3}{16}$. *Then*
$G(f(x), f(y), f(z)) = \max\{|\frac{x^3}{16} - \frac{y^3}{16}|, |\frac{y^3}{16} - \frac{z^3}{16}|, |\frac{x^3}{16} - \frac{z^3}{16}|\} \leq$
$\frac{x^3}{16} + \frac{y^3}{16} + \frac{z^3}{16} = \frac{1}{7}(\frac{7x^3}{16} + \frac{7y^3}{16} + \frac{7z^3}{16}) =$
$\frac{1}{7}(G(g(x), f(x), f(x)) + G(g(y), f(y), f(y)) + G(g(z), f(z), f(z))) \leq kM(x, y, z)$.

Therefore, all hypothesis of Theorem 2.3 are satisfied for $k = \frac{1}{7}$ and $x = 0$ is the unique common fixed point of f and g.

Note that Theorem 1.1 is not applicable because f doesn't commute with g.

Corollary 1. *Some corollaries could be derived from Theorems 2.1, 2.2 and 2.3 by taking $z = y$ or $g = Id_X$.*

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133-181.
[2] G.Jungk, Commuting maps and fixed points, *Am. Math. Monthly,* **83**(1976)261-263.
[3] G.Jungk, Compatible mappings and common fixed points, *Int.J. Math. Sci.,* **9(4)**(1986)771-779.
[4] G.Jungk, Common Fixed points for commuting and Compatible maps on compacta, *Pro. Am. Math. Soc.,* **103**(1988)977-983.
[5] G.Jungk, Common fixed points for noncontinuous nonself maps on nonmetric spaces, *Far East J. Math. Sci.,* **4**(1996)199-215.
[6] G.Jungk, N. Hussain, Compatible maps and invariant approximation, *J.M. M.A,* **325(2)**(2007)1003-1012.
[7] K.M. Das and K.V. Naik, Common fixed point theorems for commuting maps on a metric space, *Proc. Amer. Math. Soc.* **77** (1979) 369-373.
[8] R.P.Pant, Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.* **188**(1994) 436-440.
[9] S. Sessa, On a weak commutativity condition of mappings in dixed point consideration, *Publ. Inst. Math. Soc.,* **32**(1982)149-153.
[10] Zead Mustafa and Brailey Sims, Some Remarks Concerninig D–Metric Spaces, *Proceedings of the Internatinal Conferences on Fixed Point Theorey and Applications, Valencia (Spain),* July (2003). 189–198.
[11] Zead Mustafa and Brailey Sims, A New Approach to Generalized Metric Spaces, *Journal of Nonlinear and Convex Analysis,* **7**, No. 2 (2006). 289–297.
[12] Z.Mustafa, H.Obiedat, F. Awawdeh, Some fixed point theorem for mappings on complete G-metric spaces, *Fixed Point Theory and Applications* (2008)(article ID 189870,12 pages).
[13] Zead Mustafa, Hamed Obiedat, A Fixed Point Theorem of Reich in G- metric Spaces, *CUBO A Mathematical Journal*, vol12,No 01,(83-93). March 2010. 2.
[14] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, (2005).
[15] Zead Mustafa,Brailey Sims, Fixed Point Theorems for Contractive Mappings in Complete G-Metric Spaces, *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009. Doi:10.1155/2009/917175.

[16] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces, Inter. J. Math. Math. Sci. Volume 2009, Article ID 283028, 10 pages, (2009).

[17] Z. Mustafa, M. Kandagjy and W.Shatanawi, Fixed Point Results on Complete G-metric Spaces,*Studia Scientiarum Mathematicarum Hungarica*, vol 48, 2011, no 3, 304-319, DOI: 10.1556/SScMath.2011.1170.

[18] Z. Mustafa, F. Awawdeh, W. Shatanawi, Fixed point Theorem for expansive mappings in G-metric Spaces, *Int. Journal of Contemp. Math. Sciences*, Vol. 5, 2010, no.49-52.

[19] M. Abbas, B.E. Rhoades, Common fixed point results for noncomuting mappings without continuity in generalized metric spaces, *Applied Mathematics and Computation,* **215**(2009)262-269.

Received: April, 2012