A Common Fixed Point Theorem in Dislocated Metric Space

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Abstract

In this paper we establish a common fixed point theorem for two pairs of weakly compatible maps in dislocated metric space which generalizes and improves similar fixed point results.

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1 Introduction

In 1922, S. Banach proved a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by different authors and many generalizations of this theorem have been established. In 2000, P. Hitzler and A. K. Seda [8] introduced the notion of dislocated metric space in which self distance of a point need not be equal to zero. They also generalized the famous Banach contraction principle in this space. The study of common fixed points of mappings in dislocated metric space
satisfying certain contractive conditions has been at the center of vigorous re-
search activity. Dislocated metric space plays very important role in topology,
logical programming and in electronics engineering. C. T. Aage and J. N. Salunke [2], A. Isufati [1] established some important fixed point theorems in
single and pair of mappings in dislocated metric space. The purpose of this
paper is to establish a common fixed point theorem for two pairs of weakly
compatible mappings in dislocated metric space. Our result generalizes and
improves the similar results of fixed points.

2 Preliminaries

Now we start with the following definitions, lemmas and theorems.

Definition 2.1 [4] Let $X$ be a non empty set and let $d : X \times X \to [0, \infty)$
be a function satisfying the following conditions:
(i) $d(x, y) = d(y, x)$
(ii) $d(x, y) = d(y, x) = 0$ implies $x = y$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then $d$ is called dislocated metric(or simply d-metric) on $X$.

Definition 2.2 [8] A sequence $\{x_n\}$ in a d-metric space $(X, d)$ is called a
Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in N$ such that for all
$m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.

Definition 2.3 [8] A sequence in d-metric space converges with respect to
$d$ (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
In this case, $x$ is called limit of $\{x_n\}$ (in d) and we write $x_n \to x$.

Definition 2.4 [8] A d-metric space $(X, d)$ is called complete if every Cauchy
sequence in it is convergent with respect to $d$.

Definition 2.5 [8] Let $(X, d)$ be a d-metric space. A map $T : X \to X$ is
called contraction if there exists a number $\lambda$ with $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$.

We state the following lemmas without proof.

Lemma 2.6 Let $(X, d)$ be a d-metric space. If $T : X \to X$ is a contraction
function, then $\{T^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.

Lemma 2.7 [8] Limits in a d-metric space are unique.

Definition 2.8 [5] Let $A$ and $S$ be mappings from a metric space $(X, d)$
into itself. Then, $A$ and $S$ are said to be weakly compatible if they commute at
their coincident point; that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

Theorem 2.9 [8] Let $(X, d)$ be a complete d-metric space and let $T : X \to X$ be a contraction mapping, then $T$ has a unique fixed point.
3 Main Results:

**Theorem 3.1** Let \((X, d)\) be a complete \(d\)-metric space. Let \(A, B, S, T : X \to X\) be continuous mappings satisfying,

(i) \(T(X) \subset A(X), S(X) \subset B(X)\)

(ii) The pairs \((S, A)\) and \((T, B)\) are weakly compatible and

(iii) \(d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By)\)

for all \(x, y \in X\) where \(\alpha, \beta, \gamma \geq 0\), \(0 \leq \alpha + \beta + \gamma < \frac{1}{2}\).

Then \(A, B, S,\) and \(T\) have a unique common fixed point.

**Proof:**
Using condition(i), we define sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) by the rule,

\[
y_{2n} = Bx_{2n+1} = Sx_{2n}, \quad \text{and} \quad y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, ...
\]

If \(y_{2n} = y_{2n+1}\) for some \(n\), then \(Bx_{2n+1} = Tx_{2n+1}\). Therefore \(x_{2n+1}\) is a coincidence point of \(B\) and \(T\). Also, if \(y_{2n+1} = y_{2n+2}\) for some \(n\), then \(Ax_{2n+2} = Sx_{2n+2}\). Hence \(x_{2n+2}\) is a coincidence point of \(S\) and \(A\).

Assume that \(y_{2n} \neq y_{2n+1}\) for all \(n\). Then, we have

\[
d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})
\leq \alpha d(Ax_{2n}, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Sx_{2n}) + \gamma d(Ax_{2n}, Bx_{2n+1})
\leq \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n}, y_{2n}) + \gamma d(y_{2n-1}, y_{2n})
\leq \alpha [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \beta [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \gamma d(y_{2n-1}, y_{2n})
\]

\[
= (\alpha + \beta + \gamma)d(y_{2n-1}, y_{2n}) + (\alpha + \beta)d(y_{2n}, y_{2n+1})
\]

So,

\[
d(y_{2n}, y_{2n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta}d(y_{2n-1}, y_{2n})
= h d(y_{2n-1}, y_{2n})
\]

where, \(h = \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta} < 1\)

This shows that,

\[
d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n) \leq \ldots \leq h^n d(y_0, y_1)
\]
For every integer \( q > 0 \), we have

\[
d(y_n, y_{n+q}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{n+q-1}, y_{n+q})
\]

\[
\leq (1 + h + h^2 + \ldots + h^{q-1})d(y_n, y_{n+1})
\]

\[
\leq \frac{h^n}{1-h}d(y_0, y_1)
\]

Since, \( 0 < h < 1 \), \( h^n \to 0 \) as \( n \to \infty \).

So, we get \( d(y_n, y_{n+q}) \to 0 \). This implies \( \{y_n\} \) is a Cauchy sequence in a complete dislocated metric space. So, there exists a point \( z \in X \) such that \( \{y_n\} \to z \).

Therefore, the subsequences,

\[
\{Sx_{2n}\} \to z, \{Bx_{2n+1}\} \to z, \{Tx_{2n+1}\} \to z \quad \text{and} \quad \{Ax_{2n+2}\} \to z.
\]

Since \( T(X) \subset A(X) \), there exists a point \( u \in X \) such that \( z = Au \). So,

\[
d(Su, z) = d(Su, Tx_{2n+1})
\]

\[
\leq \alpha d(Au, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Su) + \gamma d(Au, Bx_{2n+1})
\]

\[
= \alpha d(z, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Su) + \gamma d(z, Bx_{2n+1})
\]

Now, taking limit as \( n \to \infty \), we get,

\[
d(Su, z) \leq \alpha d(z, z) + \beta d(z, Su) + \gamma d(z, z)
\]

\[
= (\alpha + \gamma)d(z, z) + \beta d(z, Su)
\]

\[
\leq 2(\alpha + \gamma)d(z, Su) + \beta d(z, Su)
\]

\[
= (2\alpha + \beta + 2\gamma)d(z, Su)
\]

which is a contradiction, since \( 2\alpha + \beta + 2\gamma < 1 \).

So, we have \( Su = Au = z \).

Again, since \( S(X) \subset B(X) \), there exists a point \( v \in X \) such that \( z = Bv \).

We claim that \( z = Tv \). If \( z \neq Tv \), then

\[
d(z, Tv) = d(Su, Tv)
\]

\[
\leq \alpha d(Au, Tv) + \beta d(Bv, Su) + \gamma d(Au, Bv)
\]

\[
= \alpha d(z, Tv) + \beta d(z, z) + \gamma d(z, z)
\]

\[
\leq \alpha d(z, Tv) + 2(\beta + \gamma)d(z, Tv)
\]

\[
= (\alpha + 2\beta + 2\gamma)d(z, Tv)
\]

a contradiction, since \( \alpha + 2\beta + 2\gamma < 1 \). So, we get \( z = Tv \).

Hence, we have \( Su = Au = Tv = Bv = z \).
Since the pair \((S, A)\) are weakly compatible so by definition \(SAu = ASu\) implies \(Sz = Az\).

Now, we show that \(z\) is the fixed point of \(S\). If \(Sz \neq z\), then
\[
d(Sz, z) = d(Sz, Tv) \\
\leq \alpha d(Az, Tv) + \beta d(Bv, Sz) + \gamma d(Az, Bv) \\
= (\alpha + \beta + \gamma)d(Sz, z)
\]
which is a contradiction. So, we have \(Sz = z\).

This implies that \(Az = Sz = z\).

Again, the pair \((T, B)\) are weakly compatible, so by definition \(TBv = BTv\) implies \(Tz = Bz\).

Now, we show that \(z\) is the fixed point of \(T\). If \(Tz \neq z\), then
\[
d(z, Tz) = d(Sz, Tz) \\
\leq \alpha d(Az, Tz) + \beta d(Bz, Sz) + \gamma d(Az, Sz) \\
\leq (\alpha + \beta + 2\gamma)d(z, Tz)
\]
which is a contradiction. This implies that \(z = Tz\).

Hence, we have \(Az = Bz = Sz = Tz = z\).

This shows that \(z\) is the common fixed point of the self mappings \(A, B, S\) and \(T\).

**Uniqueness:**

Let \(u \neq v\) be two common fixed points of the mappings \(A, B, S\) and \(T\). Then we have,
\[
d(u, v) = d(Su, Tv) \\
\leq \alpha d(Au, Tv) + \beta d(Bv, Su) + \gamma d(Au, Bv) \\
= \alpha d(u, v) + \beta d(v, u) + \gamma d(u, v) \\
= (\alpha + \beta + \gamma)d(u, v).
\]
a contradiction. This shows that \(d(u, v) = 0\)

Since \((X, d)\) is a dislocated metric space, so we have \(u = v\). This establishes the theorem.

**Example:**

Let \(X = [0, 1]\) and \(d\) be a usual metric. Let the mappings \(A, B, S\) and \(T\) be defined by
\[
Sx = 0, \quad Ax = x, \quad Tx = \frac{x}{5} \quad \text{and} \quad Bx = x.
\]

Then, for \(\alpha = \frac{1}{5}, \beta = \frac{1}{6}, \gamma = \frac{1}{8}\),

The mappings \(A, B, S,\) and \(T\) satisfy all conditions of above Theorem 3.1 and so, \(x = 0\) is the unique common fixed point of the four mappings \(A, B, S\) and \(T\).
Corollary 3.2 Let \((X, d)\) be a complete d-metric space. Let \(S, T : X \to X\) be continuous mappings satisfying, \(d(Sx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y)\) for all \(x, y \in X\), where \(\alpha, \beta, \gamma \geq 0\), \(0 \leq \alpha + \beta + \gamma < \frac{1}{2}\). Then \(S\) and \(T\) have a unique common fixed point.

**Proof:** If we take \(A = B = I\) an identity mapping in above theorem 3.1, and follow the similar proof as that in theorem, we can establish this corollary 3.2.

If we take \(S = T\) then the above corollary is reduced to,

Corollary 3.3 Let \((X, d)\) be a complete d-metric space. Let \(T : X \to X\) be a continuous mapping satisfying, \(d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Tx) + \gamma d(x, y)\) for all \(x, y \in X\), where \(\alpha, \beta, \gamma \geq 0\), \(0 \leq \alpha + \beta + \gamma < \frac{1}{2}\). Then \(T\) has a unique fixed point.

This is the Theorem 3.2 established by A. Isufati [1].

Corollary 3.4 Let \((X, d)\) be a complete d-metric space. Let \(S, T : X \to X\) be continuous mappings satisfying, \(d(Sx, Ty) \leq \alpha d(Tx, Ty) + \beta d(Sy, Sx) + \gamma d(Tx, Sy)\) for all \(x, y \in X\), where \(\alpha, \beta, \gamma \geq 0\), \(0 \leq \alpha + \beta + \gamma < \frac{1}{2}\). Then \(S\) and \(T\) have a unique common fixed point.

**Proof:** If we take \(A = T\) and \(B = S\) in Theorem 3.1 and apply the similar proof, we can establish this corollary 3.4.


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References


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