Quadratic and Cubic Equations
for the Four-Point Diamond Array

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Abstract
The interpretation of four numbers in a diamond array is seldom encountered in textbooks. A quadratic equation for interpolating the array was introduced several years ago. This paper illustrates new methods for the analysis of four numerical data in a diamond array. They apply quadratic, cubic, and trigonometric equations.

Mathematics Subject Classification: 65D05, 65D07, 65D17

Keywords: interpolation, diamond data array, design of experiment

1. Introduction
The bilinear equation is used to interpolate four numbers in a rectangular array. This array is represented by the vertices A,C,I,G in Fig. 1. The diamond array is represented by vertices B,F,H,D. A quadratic equation for this array was intro-
duced many years ago [1]. It is exact on bilinear numbers and their squares. This paper introduces alternative polynomial and trigonometric equations for the four-point diamond array. The present method uses Fig. 1 below and Fig. 1 in Ref. [2].

\[
\begin{array}{ccc}
G & H & I \\
D & E & F \\
A & B & C
\end{array}
\]

Fig. 1. The nine-point figure contains the four-point rectangle ACIG and the four-point diamond BFHD.

2. Polynomial equations for four data in a diamond array

Quadratic and cubic polynomial equations for the four-point rectangular data array have been illustrated [2]. They derive from quadratic and cubic equations for the eight-point rectangular prism, henceforth described as a cube [3,4]. Four numerical data that are to be represented by a quadratic or a cubic equation are positioned at the vertices on the bottom plane of the cube. The same four data, in the same order, are positioned on the top plane of the cube. The numbers at the top are augmented by an incrementing parameter denoted by the letter “t”. The present approach uses the method in [2]. It is based on Fig. 1 in [2,3].

If the number at vertex A of the cube is 2, the number at the corresponding upper vertex F is 2+t. See Fig. 1 in [2,3]. If \( u(P) \) represents the squaring operation, then the number at vertex A of the cube is \( u(2) \) or 4. The number at the upper vertex F is then \( u(2+t) \). This process is continued for all eight vertices of the cube.

The precision of the calculations is set to a high number such as 30 digits. The incrementing parameter “t” is assigned a value smaller than the precision such as \( 10^{-20} \). The eight-point cube is then represented by a quadratic or a cubic equation [3,4]. The vertical coordinate in the cube, the z-coordinate in the equation, is assigned as \(-1\). This assignment draws attention to the bottom plane of the cube where the trial numbers are located. The equation representing the bottom plane of the prism is evaluated in still lower precision, say ten digits. The result is a quadratic or a cubic equation for the four-point rectangular data array.

The preceding paragraphs summarize the generation of quadratic and cubic equations for the four-point rectangular data array [2]. A method for the four-
Quadratic and cubic equations

point diamond array, denoted by vertices B,D,F,H in Fig. 1 above, follows the same approach. The method is flexible: it permits adjustments and refinements.

Assign the numbers at vertices B,D,F,H as \( u(2), u(4), u(6), u(8) \), respectively. Rotate the diamond \( 45^\circ \) counterclockwise to form a rectangle. The numbers at vertices A,C,G,I are now \( u(4), u(2), u(8), u(6) \), respectively. See Fig. 1 above. They also lie at A,B,C,D of the cube in Fig. 1 of [2,3]. The top entries on the cube are

\[
\begin{align*}
&u(4+t), u(2+t), u(8+t), u(6+t),
\end{align*}
\]

at F,G,I,H, respectively. Find the quadratic or the cubic equation of the rectangle on the bottom plane of the cube as in Fig. 1 of [2].

A quadratic or a cubic equation for the diamond array BFHD is obtained from the quadratic or cubic equation for the four-point rectangular array ABDC as in the cube in Fig. 1 of [2,3]. The change is partly a rotation and partly a contraction. Put \( xt \) for every \( x \) and \( yt \) for every \( y \) in the equation for the four-point rectangle ABDC. Then put \( (x-y) \) for every \( xt \) and \( (x+y) \) for every \( yt \) in the equation and expand the result. It thereby becomes a quadratic or a cubic equation for the four-point diamond array. It reproduces the data at vertices B,D,F,H in Fig. 1 above. Their \((x,y)\) coordinates are \((0,-1),(-1,0),(1,0),(0,1)\), respectively.

Suppose data at B,D,F,H are \( 2^3, 4^3, 6^3, 8^3 \), respectively, in Fig. 1 above. By a \( 45^\circ \) counterclockwise rotation, the four-point rectangle then has \( 4^3, 2^3, 8^3, 6^3 \) at A,C,G, I, respectively. The quadratic equation for this rectangle is Eq. (1). It is the analog of Eq. (A) in [2]. By means of the transformation, it becomes Eq. (2), an equation for the four-point diamond array. The rotated data are put at vertices A,B,C,D of the cube in Fig. 1 of [2]. The analog of Eq. (B) in [2] is then Eq. (3). It applies to data at vertices A,C,G,I at \((-1,-1), (1,-1), (-1,1), (1,1)\), respectively, in Fig. 1 above. The transformation yields Eq. (4). It is a second equation for the diamond array. It also reproduces the data at diamond points B,D,F,H. The same method applies to the generation of cubic equations for the four-point diamond array.

\[
R = 16.10x^2 - 60.00xy + 55.91y^2 - 88.00x + 164.0y + 128.0
\]  

(1)

\[
R = 12.01x^2 + 79.62xy + 132.0y^2 + 76.00x + 252.0y + 128.0
\]  

(2)

\[
R = 15.00x^2 - 60.00xy + 60.00y^2 - 88.00x + 164.0y + 125.0
\]  

(3)

\[
R = 15.00x^2 + 90.00xy + 135.0y^2 + 76.00x + 252.0y + 125.0
\]  

(4)
3. Trigonometric equation for the four-point diamond array

The trigonometric equation for the four-point diamond array is prepared from the trigonometric equation for the eight-point cube described in [5]. Here, only the hyperbolic sine-and-cosine method is illustrated. In trigonometric cases, better results are usually obtained by enhancing the precision of the calculations to 35 digits. Let the incrementing term be $10^{-25}$ and the data at B,D,F,H be $2^1,4^2,6^3,8^4$, respectively. The equation is evaluated with 16-digit precision. With coefficients rounded, the trigonometric equation for the four-point diamond array is Eq. (5).

$$R = (138.120)\cosh(0.453854x+1.18295y) - (38.2432)\cosh(1.18295x+0.453854y) + (171.368)\sinh(0.453854x+1.18295y) - (3.02508)\sinh(1.18295x+0.453854y) + 55.68$$  \hspace{1cm} (5)

If the data are $2^2+100,2^4+100,2^6+100,2^8+100$ at B,D,F,H, respectively, the trigonometric interpolating equation is Eq. (6).

$$R = (32)\cosh(2.07944y + 0.693147x) + (32)\sinh(2.07944y + 0.693147x) + 100$$  \hspace{1cm} (6)

4. Discussion

Table 1 illustrates quadratic equations for the four-point diamond array. Table 2 illustrates cubic equations for the same array. In both tables, B=u(2), D=u(4), F=u(6), H=u(8). The five-point diamond array is described in [6].

Table 1. Quadratic equations for the four-point diamond BDHF in Fig. 1. The notations (A) and (B) refer to the four-point-rectangle equations with the same designations in Ref. [2]. They are converted into equations for the diamond array by transformations. All coefficients are rounded.

<table>
<thead>
<tr>
<th>Function, P</th>
<th>Quadratic Equation for Diamond Array</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100/P$, Eq. (A)</td>
<td>$R = 0.541x^2+4.87xy+11.0y^2-4.17x-18.8y+20.3$</td>
</tr>
<tr>
<td>$100/P$, Eq. (B)</td>
<td>$R = 0.234x^2+3.16xy+10.7y^2-4.17x-18.8y+20.6$</td>
</tr>
<tr>
<td>$(P^3)^{1/2}$, Eq. (A)</td>
<td>$R = 0.178x^2+1.05xy+1.56y^2+3.35x+9.90y+11.2$</td>
</tr>
<tr>
<td>$(P^3)^{1/2}$, Eq. (B)</td>
<td>$R = 0.155x^2+0.975xy+1.53y^2+3.35x+9.90y+11.2$</td>
</tr>
<tr>
<td>$\ln(2^y/P)$, Eq. (A)</td>
<td>$R = 0.03x^2+0.164xy+0.232y^2+0.49x+1.39y+1.85$</td>
</tr>
<tr>
<td>$\ln(2^y/P)$, Eq. (B)</td>
<td>$R = 0.01x^2+0.095xy+0.213y^2+0.49x+1.39y+1.87$</td>
</tr>
</tbody>
</table>
Table 2. Some cubic equations for four-point diamond arrays. If the function is $P^4$ then $u(8)$ is $8^4$. All coefficients are rounded.

<table>
<thead>
<tr>
<th>Function, $P$</th>
<th>$R = -0.018x^3 - 0.358x^2y - 2.41xy^2 - 5.43y^3 + 0.234x^2 + 3.16xy + 10.7y^2 - 4.15x - 13.3y + 20.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100/P$</td>
<td>$R = 0.032x^3 + 0.502x^2y + 2.64xy^2 + 4.61y^3 + 0.339x^2 + 3.56xy + 9.34y^2 + 2.37x + 7.99y + 3.66$</td>
</tr>
<tr>
<td>$2^9/10$</td>
<td>$R = 13.5x^3 + 134x^2y + 445xy^2 + 492y^3 + 128x^2 + 849xy + 1408y^2 + 507x + 1548y + 648$</td>
</tr>
</tbody>
</table>

References


Received: March, 2012