Permanence and Extinction of a Periodic Multi-Delays Competitive System with Toxicant and Birth Pulses

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Abstract

In this article, a periodic multi-delays two-species competitive system with toxicant and birth pulses is proposed. Easily verifiable criteria for permanence and extinction of system are obtained, respectively. The effects of toxicant, birth pulse and delays on the dynamics of system are also discussed.

Keywords: Delay; birth pulse; competitive system; toxicant; permanence; extinction

1. Introduction

In [1], Chattopadhyay incorporated the effect of toxic substances in a classical two-species Lotka-Volterra competitive system by considering that each species produces a toxic substance to the other but only when the other is present, and established the following model

\[
\begin{align*}
x_1'(t) &= x_1(t)[r_1 - a_1 x_1(t) - b_1 x_2(t) - c_1 x_1(t)x_2(t)], \\
x_2'(t) &= x_2(t)[r_2 - a_2 x_2(t) - b_2 x_1(t) - c_2 x_1(t)x_2(t)],
\end{align*}
\]

where \(x_1, x_2\) are the population densities of two competing species; \(r_1, r_2\) are the intrinsic growth rates; \(a_1, a_2\) are the rates of intra-specific competition; \(b_1, b_2\) are the rates of inter-specific competition; \(c_1\) and \(c_2\) are, respectively, the rates of toxic inhibition of the first species by the second and vice versa.
Considering the effect of delays, Mukhopadhyay et al. [2] modified system (1.1), and obtained
\[
\begin{align*}
    x_1'(t) &= x_1(t)[r_1 - a_1 x_1(t) - b_1 x_2(t) - c_1 x_1(t) x_2(t - \tau_2)], \\
    x_2'(t) &= x_2(t)[r_2 - a_2 x_2(t) - b_2 x_1(t) - c_2 x_2(t) x_1(t - \tau_1)].
\end{align*}
\]
(1.2)

However, for the system with single discrete delay, many researchers have argued that finite number of discrete delays are more appropriate than a single discrete delay (see [3, 4]). According to the idea, a revised version of system (1.2) leads to
\[
\begin{align*}
    x_1'(t) &= x_1(t) \left[ r_1 - a_1 x_1(t) - b_1 x_2(t) - \sum_{l=1}^{n} c_{1l} x_1(t) x_2(t - \tau_{2l}) \right], \\
    x_2'(t) &= x_2(t) \left[ r_2 - a_2 x_2(t) - b_2 x_1(t) - \sum_{l=1}^{n} c_{2l} x_2(t) x_1(t - \tau_{1l}) \right].
\end{align*}
\]
(1.3)

It is well known that any biological and environmental parameters are nationally subject to the fluctuation in time. It motivates us to extend system (1.3) to a periodic form
\[
\begin{align*}
    x_1'(t) &= x_1(t) \left[ r_1(t) - a_1(t) x_1(t) - b_1(t) x_2(t) - \sum_{l=1}^{n} c_{1l}(t) x_1(t) x_2(t - \tau_{2l}(t)) \right], \\
    x_2'(t) &= x_2(t) \left[ r_2(t) - a_2(t) x_2(t) - b_2(t) x_1(t) - \sum_{l=1}^{n} c_{2l}(t) x_2(t) x_1(t - \tau_{1l}(t)) \right].
\end{align*}
\]
(1.4)

Here the intrinsic growth rates \( r_i(t) = \beta_i(t) - d_i(t), \beta_i(t) \) and \( d_i(t) \) are the birth and death rates of the species \( x_i \), respectively. Obviously, in system (1.4), it is assumed that each species reproduces throughout its whole life history, that is, the births of both species are continuous. However, owing to the enormous diversity of plant, insect and animal life histories, the births of some species occur in regular pulse. So we further revise system (1.4) to the following impulsive form
\[
\begin{align*}
    \begin{cases}
        x_1'(t) = x_1(t) \left[ -d_1(t) - a_1(t) x_1(t) - b_1(t) x_2(t) - \sum_{l=1}^{n} c_{1l}(t) x_1(t) x_2(t - \tau_{2l}(t)) \right], \\
        x_2'(t) = x_2(t) \left[ -d_2(t) - a_2(t) x_2(t) - b_2(t) x_1(t) - \sum_{l=1}^{n} c_{2l}(t) x_2(t) x_1(t - \tau_{1l}(t)) \right], \\
        x_1(t_k^+) = (1 + \beta_1(t_k)) x_1(t_k), \\
        x_2(t_k^+) = (1 + \beta_2(t_k)) x_2(t_k),
    \end{cases} \quad t \neq t_k, \\
    k \in \mathbb{Z}^+
\end{align*}
\]
(1.5)

where \( \beta_{i,k} \) are the birth rates of the species \( x_i(t) \) at \( t = t_k \) and \( \mathbb{Z}^+ \) is the set of positive integers.
In the whole paper, we need to use the assumptions (A1)-(A3).

(A1) $0 < t_1 < t_2 < \cdots$ are fixed impulsive points with $\lim_{k \to \infty} t_k = \infty$.

(A2) $\{\beta_{i,k}\}$ are real non-negative sequences and there exists a positive integer $q$ such that $t_{k+q} = t_k + \omega, \beta_{i,k+q} = \beta_{i,k}$.

(A3) $a_i(t), b_i(t), c_{il}(t) \in ([0, \infty), (0, \infty)), d_i(t), \tau_{il}(t) \in ([0, \infty), [0, \infty))$ are locally summable $\omega$-periodic functions.

2. Main results

In order to discuss the permanence and extinction of system (1.5), we first introduce two lemmas. The following generalized result is based on Lemma 2.1 in Ref.[6].

Lemma 2.1 [5]. Consider a single species model with impulsive perturbations

$$
\begin{cases}
    z'(t) = z(t)\left[g(t) - h(t)z(t)\right], & t \neq t_k, k \in \mathbb{Z}^+,
    \\
    z(t_k^+) = (1 + d_k)z(t_k),
\end{cases}
$$

where $g(t) \in C([0, \infty), (-\infty, \infty)), h(t) \in C([0, \infty), (0, \infty))$ are $\omega$-periodic functions. $\{d_k\}$ is a real sequence and $d_k > -1$. If there exists a positive integer $q$ such that $t_{k+q} = t_k + \omega, d_{k+q} = d_k$ and

$$
\prod_{k=1}^{q}(1 + d_k)e^{\int_0^\omega g(t)dt} > 1,
$$

then system (2.1) has a unique positive $\omega$-periodic solution $z^*(t, z_0^*)$ for which $z^*(0, z_0^*) = z_0^*$ and $z^*(t, z_0^*) > 0, t \geq 0$, and $z^*(t, z_0^*)$ is globally asymptotically stable in the sense that $\lim_{t \to \infty}|z(t, z_0) - z^*(t, z_0^*)| = 0$, where $z(t, z_0)$ is any solution of system (2.1) with $z(0, z_0) = z_0 > 0$.

From Theorem 1.4.3 in [8], the following comparison theorem for impulsive differential equation is obvious (see [7]).

Lemma 2.2 [7]. Suppose that $z(t)$ is the solution of system (2.1) with initial value $z_0 > 0$, $s(t)$ satisfies the following inequalities

$$
\begin{cases}
    s'(t) \leq s(t)\left[g(t) - h(t)s(t)\right], & t \neq t_k, k \in \mathbb{Z}^+,
    \\
    s(t_k^+) = (1 + d_k)s(t_k), & t = t_k,
    \\
    s(0) \leq z_0,
\end{cases}
$$

and $m(t)$ satisfies the above reverse inequalities. Then we have

$$
s(t) \leq z(t) \leq m(t).
$$
From Lemma 2.1, it is easy to see that if

$$\prod_{k=1}^{q} (1 + \beta_{1,k}) e^{-\int_{0}^{t} d_1(t) dt} > 1 \text{ (or } \prod_{k=1}^{q} (1 + \beta_{2,k}) e^{-\int_{0}^{t} d_2(t) dt} > 1),$$

then system (1.5) has a semi-trivial $\omega$-periodic solution $(x_1^*(t), 0)$ (or $(0, x_2^*(t))$), where $x_1^*(t)$ and $x_2^*(t)$ are, respectively, the unique positive $\omega$-periodic solution of the following subsystems (2.6) and (2.7)

$$\begin{align*}
x_1'(t) &= x_1(t) \left[-d_1(t) - a_1(t)x_1(t)\right], \quad t \neq t_k, k \in \mathbb{Z}^+, \\
x_1(t_{k}^+) &= (1 + \beta_{1,k})x_1(t_k), \quad t = t_k.
\end{align*}$$

$$\begin{align*}
x_2'(t) &= x_2(t) \left[-d_2(t) - a_2(t)x_2(t)\right], \quad t \neq t_k, k \in \mathbb{Z}^+, \\
x_2(t_{k}^+) &= (1 + \beta_{2,k})x_2(t_k), \quad t = t_k.
\end{align*}$$

**Theorem 2.1.** Under the assumptions (A1)-(A3), if both of two inequalities

$$\prod_{k=1}^{q} (1 + \beta_{1,k}) e^{-\int_{0}^{t} (d_1(t) + b_1(t)x_2^*(t)) dt} > 1, \quad \prod_{k=1}^{q} (1 + \beta_{2,k}) e^{-\int_{0}^{t} (d_2(t) + b_2(t)x_1^*(t)) dt} > 1$$

hold. Then system (1.5) is permanent.

*Proof.* It is easy to prove that when $t \geq 0$, if $x_1(0^+) \geq 0$ and $x_2(0^+) \geq 0$, then we have $x_1(t) \geq 0, x_2(t) \geq 0$, and further if $x_1(0^+) > 0, x_2(0^+) > 0$, then $x_1(t) > 0, x_2(t) > 0$. Suppose that $(x_1(t), x_2(t))$ is a positive solution of system (1.5) with positive initial value $(x_1(0), x_2(0))$, we consider the following impulsive system

$$\begin{align*}
m'(t) &= m(t) \left[-d_1(t) - a_1(t)m(t)\right], \quad t \neq t_k, k \in \mathbb{Z}^+, \\
m(t_{k}^+) &= (1 + \beta_{1,k})m(t_k), \quad m(0) = x_1(0) > 0, \quad t = t_k,
\end{align*}$$

and denote its solution by $m(t)$. Note that the first inequality in (2.8) implies that the corresponding first inequality in (2.5) holds. So by Lemma 2.1, we have $\lim_{t \to \infty} (m(t) - x_1^*(t)) = 0$, which implies that there exists a $T_1 > 0$ such that $m(t) \leq x_1^*(t) + \delta_1$ for $t \geq T_1$, where $\delta_1 > 0$ is a sufficient small constant. Thus, by Lemma 2.2 we have $x_1(t) \leq m(t) \leq x_1^*(t) + \delta_1$ for $t \geq T_1$.

Similarly, for $x_2(t)$, we can get that there exists a $T_2 > 0$ such that $x_2(t) \leq x_2^*(t) + \delta_2$ for $t \geq T_2$, where $\delta_2 > 0$ is also a sufficient small constant. Thus, we obtain that there exists $T_3 = \max\{T_1, T_2\}$ such that

$$x_1(t) \leq x_1^*(t) + \delta_1, \quad x_2(t) \leq x_2^*(t) + \delta_2, \quad \text{for } t \geq T_3.$$
Now, let us consider the impulsive equation

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\ y'(t) = y(t) \left[ -d(t) - b_1(t)(x_2(t) + \delta) - \left( a_1(t) + \sum_{i=1}^n c_{i1}(t)(x_2(t) + \delta) \right) s(t) \right], \quad t \neq t_k, \\
\ y(t_k) = (1 + \beta_{1,k})y(t_k), \quad t = t_k, k \in \mathbb{Z}^+.
\end{array} \right.
\end{aligned}
\]

Since the constant \(\delta_2 > 0\) is sufficient small, it follows from the first inequality in (2.8) that we can choose the constant \(\delta_2\) such that

\[
\prod_{k=1}^q (1 + \beta_{1,k})e^{\int_0^\infty [-d(t) - b_1(t)(x_2(t) + \delta_2)]dt} > 1.
\]

It is obvious that \(a_1(t) + \sum_{i=1}^n c_{i1}(t)(x_2(t) + \delta) > 0\). As a consequence, it follows from Lemma 2.1 that system (2.11) has a unique positive \(\omega\)-periodic solution denoted by \(x_{1*}(t)\). We denote the solution of (2.11) satisfying \(s(T_3) = x_1(T_3)\) by \(s(t)\), and continue to choose a positive constant \(\eta_1 > 0\) such that \(\eta_1 < \min_{t \in [0, \omega]} x_{1*}(t)\). By Lemma 2.1 and Lemma 2.2, and the asymptotic property of \(x_{1*}(t)\), there exists \(T_4 > T_3\) such that

\[
x_{1*}(t) - \eta_1 \leq s(t) \leq x_1(t), \quad \text{for} \quad t > T_4.
\]

Similarly, for \(x_2(t)\), it follows from the second inequality in (2.8) that we can choose the sufficient small constant \(\delta_1 > 0\) such that

\[
\prod_{k=1}^q (1 + \beta_{2,k})e^{\int_0^\infty [-d_2(t) - b_2(t)(x_1(t) + \delta_1)]dt} > 1.
\]

Therefore, there exist \(x_{2*}(t), \eta_2\) and \(T_5\) which correspond to \(x_{1*}(t), \eta_1\) and \(T_4\), respectively, such that

\[
0 < x_{2*}(t) - \eta_2 \leq x_2(t), \quad \text{for} \quad t > T_5,
\]

where \(\eta_2 < \min_{t \in [0, \omega]} x_{2*}(t)\). Let

\[
\eta = \min_{t \in [0, \omega]} \{ x_{1*}(t) - \eta_1, x_{2*}(t) - \eta_2 \}, \quad \delta = \max_{t \in [0, \omega]} \{ x_{1*}(t) + \delta_1, x_{2*}(t) + \delta_2 \}.
\]

So we get \(\eta \leq x_1(t), x_2(t) \leq \delta\) for \(t > T = \max\{T_4, T_5\}\). The proof is complete.

Next, we will discuss the extinction of system (1.5).

**Theorem 2.2.** Under the assumptions (A1)-(A3), if

\[
\prod_{k=1}^q (1 + \beta_{1,k})e^{-\int_0^\infty d(t)dt} < 1, \quad \prod_{k=1}^q (1 + \beta_{2,k})e^{-\int_0^\infty d_2(t)dt} < 1.
\]


Then system (1.5) is extinct.

Proof. Consider the impulsive differential equations

\[
\begin{align*}
\begin{cases}
  u_1'(t) = -u_1(t)d_1(t) & t \neq t_k, \\
  u_2'(t) = -u_2(t)d_2(t) & t \neq t_k, \\
  u_1(t_k^+) = (1 + \beta_{1,k})u_1(t_k) & t = t_k, \\
  u_2(t_k^+) = (1 + \beta_{2,k})u_2(t_k) & t = t_k,
\end{cases}
\end{align*}
\]  

(2.18)

It is easy to prove that when \( t \geq 0 \), if \( u_1(0^+) > 0, u_2(0^+) > 0 \), then \( u_1(t) > 0, u_2(t) > 0 \). By the first and third equations in (2.18), we suppose \( t \in (n\omega, (n+1)\omega] \) and let \( \mathfrak{B}_1 = \max \prod_{t \in [0, \omega]} (1 + \beta_{1,k}) \), a straightforward calculation shows that

\[
\begin{align*}
  u_1(t) &= \prod_{0 < t_k < t} (1 + \beta_{1,k}) e^{-\int_0^t d_1(s)ds} u_1(0^+) \\
  &= \prod_{0 < t_k < n\omega} (1 + \beta_{1,k}) \prod_{n\omega \leq t_k < t} (1 + \beta_{1,k}) e^{-\int_0^t d_1(s)ds} u_1(0^+) \\
  &= \left( \prod_{0 < t_k < \omega} (1 + \beta_{1,k}) e^{-\int_0^\omega d_1(s)ds} \right)^n \prod_{0 \leq t_k < t - n\omega} (1 + \beta_{1,k}) e^{-\int_0^t d_1(s)ds} u_1(0^+) \\
  &= \left( \prod_{k=1}^q (1 + \beta_{1,k}) e^{-\int_0^\omega d_1(s)ds} \right)^n \prod_{0 \leq t_k < t - n\omega} (1 + \beta_{1,k}) e^{-\int_0^t d_1(s)ds} u_1(0^+) \\
  &\leq \left( \prod_{k=1}^q (1 + \beta_{1,k}) e^{-\int_0^\omega d_1(s)ds} \right)^n \mathfrak{B}_1 u_1(0^+).
\end{align*}
\]

(2.19)

By the first inequality in (2.17) we know that \( \lim_{n \to \infty} \left( \prod_{k=1}^q (1 + \beta_{1,k}) e^{-\int_0^\omega d_1(s)ds} \right)^n = 0 \). As a consequence, it follows from (2.19) that we have

\[
\lim_{t \to \infty} u_1(t) = 0.
\]

(2.20)

Similarly, for \( u_2(t) \), it follows from the second inequality in (2.17), the second and fourth equations in (2.18) that we obtain

\[
\lim_{t \to \infty} u_2(t) = 0.
\]

(2.21)

Furthermore, by (1.5), (2.18), and the comparison theorem for the impulsive equation, we have

\[
0 < x_1(t) \leq u_1(t), \quad 0 < x_2(t) \leq u_2(t).
\]

(2.22)

So by (2.20)-(2.22) we can see that every solution \((x_1(t), x_2(t))\) of system (1.5) satisfies

\[
\lim_{t \to \infty} x_1(t) = 0, \quad \lim_{t \to \infty} x_2(t) = 0.
\]

(2.23)
The proof is complete. □

To discuss the effect of toxicant on the permanence and extinction, let us consider the following two-species Lotka-Volterra competitive system with birth pulses

\[
\begin{aligned}
&x_1'(t) = x_1(t) \left[-d_1(t) - a_1(t)x_1(t) - b_1(t)x_2(t)\right], t \neq t_k, \\
&x_2'(t) = x_2(t) \left[-d_2(t) - a_2(t)x_2(t) - b_2(t)x_1(t)\right], \\
&x_1(t_k^+) = (1 + \beta_1,k)x_1(t_k), \\
&x_2(t_k^+) = (1 + \beta_2,k)x_2(t_k),
\end{aligned}
\]

(2.24)

Applying Theorems 2.1 and 2.2 to system (2.24), we easily obtain the following two corresponding corollaries.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, system (2.24) is permanent.

**Corollary 2.2.** Under the assumptions of Theorem 2.2, system (2.24) is extinct.

**Remark 2.1.** By comparing Theorems 2.1 and 2.2 with Corollaries 2.1 and 2.2, we can see that their assumptions are the same, respectively. The fact implies that for the impulsive two-species competitive system (2.24), both toxic inhibition and delays are harmless for the permanence and extinction under some appropriate conditions, respectively. However, the pulses have influences on the permanence and extinction of systems (1.5) and (2.24).

**References**


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