The Solutions of the Boussinesq and Generalized Fifth-Order KdV Equations by Using the Direct Algebraic Method

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Abstract

In the present study, the dark and bright solitary wave solutions for the three kinds of Boussinesq equations and the generalized fifth-order KdV equations are presented. By implementing the direct algebraic method, the new exact solutions of the Boussinesq, the sixth-order generalized Boussinesq, Kawahara, Sawada-Kotera and Kaup-Kupershmidt equations are obtained. Also, the stability of these solutions and the movement role of the waves by making the graphs of the exact solutions are analyzed. All solutions are obtained precisely and so the efficiency of the method can be demonstrated.

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1 Introduction

Theoretical models of shallow water waves are often derived under application driven assumptions facilitating analysis and numerical computation. The family of KdV equations describes the uni-directional propagation of shallow water waves, while the family of Boussinesq equations describes the bi-directional propagation of such waves [1-3].

It is well known that the velocity field in shallow water is actually more complicated than these models would seem to indicate. This is not so surprising as these models are valid when the "small-amplitude" and "long-wavelength" parameters bear a certain relationship as they approach zero. These restrictions are too rigid and very unlikely to hold in general in shallow water for arbitrary values of these parameters. More appropriate models for shallow water waves that can be more accurately predict actual velocity field and other associated quantities can be obtained...
by incorporating the effect of each that two parameters. In fact, this strategy can
be used in a straight-forward manner to derive the shallow water wave models that
would lead to the family of KdV and Boussinesq equations [3].

There are many classical methods proposed to solve the generalized Boussinesq
and generalized fifth order KdV equations, including direct integration, Lyapunov
approach, Hirota’s dependent variable transformation, the inverse scattering trans-
form, and the Bäcklund transformation [4-6]. A direct algebraic method has also
been developed by Parkes and Duffy in which the solutions to the particular equation
are represented by an automated tanh-function method [7-8].

This paper is organized as follows: An introduction is given in section one. In
section 2, an analysis of the Direct algebraic method is formulated. The exact
solutions for the generalized Boussinesq and generalized fifth order KdV equations
are obtained in section 3.

2 An analysis of the direct algebraic method

The following gives a detail of the direct algebraic method [8] for the nonlinear partial
differential equations (Boussinesq and generalized fifth-order KdV equations) with
two variables \(x\) and \(t\)

\[
F(u, u_t, u_{tt}, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}) = 0 ,
\]

where \(F\) is a polynomial function with respect to the indicated variables or some
functions which can be reduced to a polynomial function by using some transforma-
tions.

**Step 1:** Assume that equation (1) has the following formal solution:

\[
u(x, t) = u(\xi) = \sum_{i=0}^{m} a_i \varphi^i(\xi),
\]

where \(\varphi^2 = \alpha \varphi^2(\xi) + \beta \varphi^3(\xi)\) and \(\xi = kx + \omega t,\)

**Step 2:** Balancing the highest order derivative term and the highest order non-
linear term of equation(1), and the coefficients of series \(\alpha, \beta, a_0, a_1, \ldots, a_m, k, \omega\)
are parameters can be determined.

**Step 3:** Substituting from equations (2) and (3) into equation(1) and col-
lecting coefficients of \(\varphi^i\) and \(\varphi^i \varphi^{(i)}\), then setting coefficients equal zero. a set
of algebraic equations will be obtained. By solving the system, the parameters
\(\alpha, \beta, a_0, a_1, \ldots, a_m, k, \omega\) can be determined.

**Step 4:** Substituting the parameters \(\alpha, \beta, a_0, a_1, \ldots, a_m, k, \omega\) and \(\varphi(\xi)\)
obtained in step 3 into equation (2). The solutions of equation (1) can be derived.
3 The applications of the method

1. The Boussinesq equation The Boussinesq equation was first derived to describe the propagation of long waves in shallow water [9]. It also arises in many other applications of physical interest including one-dimensional nonlinear lattice waves [10], vibrations in a nonlinear string [11], and ion sound waves in a plasma [12]. The Boussinesq equation has the form as the following [13-14]

\[ u_{tt} + \frac{1}{2}(u^2)_{xx} - u_{xxxx} = 0. \]  

(4)

Consider the traveling wave solutions \( u(x, t) = u(\xi) \) and \( \varphi^2 = \alpha \varphi^2(\xi) + \beta \varphi^3(\xi) \), then equation (4) becomes as

\[ \omega^2 u'' + k^2 (u')^2 + k^2 uu'' - k^4 u^{(4)} = 0. \]  

(5)

Balancing the nonlinear term \( uu'' \) and the highest order derivative \( u^{(4)} \) gives \( m = 1 \). Suppose the solution of equation (4) is in the form

\[ u(\xi) = a_0 + a_1 \varphi. \]  

(6)

Substituting (6) into equation (5) yields a set of algebraic equations for \( a_0, a_1, \alpha, \beta, k, \omega \). The solution of the system of this algebraic equations, can be found as

\[ a_0 = \frac{-\omega^2}{k^2}, \quad a_1 = 3k^2 \beta, \quad \alpha = 0; \quad a_0 = \frac{k^4 \alpha - \omega^2}{k^2}, \quad a_1 = 3k^2 \beta. \]  

(7)

Substituting from equation (7) into (6), the following solution of equation (4) can be obtained as

\[ u(x, t) = \frac{-\omega^2}{k^2} + \frac{12k^2 \beta}{(\sqrt{\beta (kx + \omega t)} \pm 1)^2}, \]

\[ u(x, t) = \frac{k^4 \alpha - \omega^2}{k^2} - 3\alpha k^2 \left( \operatorname{Sech} \left[ \frac{\sqrt{\alpha}}{2} (kx + \omega t) \right] \right)^2. \]  

(8)

The exact solitary wave solutions (8) are represented in figures (1a-1b) for the Boussinesq equation (4), with \( k = 1 \), \( \omega = 1 \), \( \beta = 4 \) and \( \alpha = 0 \); \( k = 2 \), \( \omega = 1 \) and \( \alpha = 1 \) in the interval \([ -10, 10 \])

2. The Boussinesq equation The usual form of the Boussinesq equation is the following as [15-16]

\[ u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0. \]  

(9)

Let us consider the traveling wave solutions \( u(x, t) = u(\xi) \), then equation (9) becomes as

\[ \omega^2 u'' - 12k^2 (u')^2 - k^2 u'' - 12k^2 uu'' - k^4 u^{(4)} = 0. \]  

(10)
Balancing the nonlinear term $uu''$ and the highest order derivative $u^{(4)}$ gives $m = 1$. It can be supposed the solution of equation (9) is in the form

$$u(\xi) = a_0 + a_1 \varphi.$$  \hspace{1cm} (11)

By substituting from (11) into equation (10) yields a set of algebraic equations for $a_0, a_1, \alpha, \beta, k, \omega$. The solution of this system of equations, we can be found as

$$a_0 = \frac{\omega^2 - k^2}{12k^2}, \quad a_1 = -\frac{k^2\beta}{4}, \quad \alpha = 0; \quad a_0 = \frac{\omega^2 - k^2 - k^4\alpha}{k^2}, \quad a_1 = -\frac{k^2\beta}{4}. \hspace{1cm} (12)$$

By substituting from equation (12) into (11), the following solution of Boussinesq equation (9) can be obtained as

$$u(x, t) = \frac{\omega^2 - k^2}{12k^2} - \frac{k^2\beta}{4(\sqrt{\beta(kx + \omega t)} + 1)^2},$$

$$u(x, t) = \frac{\omega^2 - k^2 - k^4\alpha}{k^2} + \frac{k^2\alpha}{4} \left( \text{Sech} \left[ \frac{\sqrt{\alpha}}{2}(kx + \omega t) \right] \right)^2. \hspace{1cm} (13)$$

Figures (2a-2b), represent the both dark and bright solitary wave solutions (13) of the Boussinesq equation (9), with $k = 1, \omega = 2, \beta = 4$ and $\alpha = 0$; $k = -1, \omega = 2$ and $\alpha = 1$ in the interval $[-10, 10]$. 
3. Generalized Boussinesq equation

In modeling the nonlinear lattice dynamics in elastic crystals, Maugin \cite{17} proposed the nonlinear generalized Boussinesq equation by considering a sixth-order term in the approximating expansion as the following \cite{17-18}
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 6(u^2)_{xx} - \frac{2}{5}u_{xxxxxx} = 0. \tag{14}
\]

Feng et al \cite{19} studied the solitary waves and their interactions for the above equation (14). Let us consider the traveling wave solutions \( u(x, t) = u(\xi) \), then equation (14) becomes
\[
\omega^2 u'' - 12k^2(u')^2 - k^2 u'' - 12k^2 uu'' - k^4 u^{(4)} - \frac{2}{5}k^6 u^{(6)} = 0. \tag{15}
\]

By balancing the nonlinear term \( uu'' \) and the highest order derivative \( u^{(6)} \) gives \( m = 2 \). It can be supposed the solution of equation (14) is in the form
\[
u(\xi) = a_0 + a_1 \varphi + a_2 \varphi^2. \tag{16}
\]

Substituting from (16) into equation (15) yields a set of algebraic equations for \( a_0, a_1, a_2, \alpha, \beta, k, \omega \). These equations have solutions as
\[
a_0 = \frac{169\omega^2 - 79k^2}{2028k^2}, \quad a_1 = 0, \quad a_2 = -\frac{7}{2}k^4, \quad \alpha = -\frac{5}{26k^2}. \tag{17}
\]

Substituting from equation (17) into (16), the following solution of the generalized Boussinesq equation (14) can be obtained as
\[
u(x, t) = \frac{169\omega^2 - 79k^2}{2028k^2} + \frac{7}{2}k^4 \beta \left( Sech \left[ \frac{\sqrt{\alpha}}{2}(kx + \omega t) \right] \right)^4. \tag{18}
\]

If \( \alpha = -\frac{5}{26k^2}, \beta = -\frac{1}{k^2} \), then
\[
u(x, t) = \frac{169\omega^2 - 79k^2}{2028k^2} + \frac{35}{52} \left( Sech \left[ \frac{\sqrt{5}}{2k\sqrt{26}}(kx + \omega t) \right] \right)^4. \tag{19}
\]

The periodic wave solutions of equations (19) are shown in figures (3a-3b), with \( k = 1 \) and \( \omega = 0.5 \); \( k = -1 \) and \( \omega = 1 \) in the interval \([-10, 10] \).

4. The Kawahara equation

Consider the Kawahara equation \cite{20}, which occurs in the theory of magneto-acoustic waves in a plasma \cite{21} and in the theory of shallow water waves with surface tension \cite{22-23} as
\[
u_t + \nu u_x + u_{xxx} - u_{xxxxx} = 0. \tag{20}
\]

Consider the traveling wave solution \( u(x, t) = u(\xi) \), then equation (20) becomes as
\[
\omega u' + kuu' + k^3 u^{(3)} - k^5 u^{(5)} = 0 \tag{21}.
\]
By balancing the nonlinear term $uu'$ and the highest order derivative $u^{(5)}$ gives $m = 2$. Suppose the solution of equation (21) is in the form

$$u(\xi) = a_0 + a_1 \varphi + a_2 \varphi^2 .$$

(22)

By substituting from (22) into equation (21) yields a set of algebraic equations for $a_0, a_1, a_2, \alpha, \beta, k, \omega$. These system of equations have the solution as

$$a_0 = -\frac{36k + 169\omega}{169k}, \quad a_1 = 0, \quad a_2 = \frac{21k^2}{5}, \quad \alpha = \frac{1}{13k^2}, \quad \beta = \frac{1}{5k} .$$

(23)

By substituting from equation (23) into (22), the following solution of the Kawahara equation (20) can be obtained as

$$u(x, t) = -\frac{36k + 169\omega}{169k} - \frac{21k}{13} \left( Sech \left[ \frac{kx + \omega t}{2\sqrt{13}k} \right] \right)^4 .$$

(24)

Fig.(4a-4b), the plots represent the dark and bright solitary wave solutions of equation (24), with $k = 1$ and $\omega = 1$ ; $k = -1$ and $\omega = 1$ in the interval $[-10, 10]$. 5. **The generalized fifth-order KdV equation** Consider the generalized fifth order KdV equation as

$$u_t + 45u^2u_x - \gamma u_xu_{xx} - 15uu_{xxx} + uu_{xxxx} = 0 .$$

(25)
Now, two important unidirectional nonlinear evolution equations that have been studied extensively over the last two decades are the Sawada-Kotera (SK) (or Caudrey-Dodd-Gibbon) equation, if \( \gamma = 15 \) [24-25]

\[
 u_t + 45u^2u_x - 15u_x u_{xx} - 15uu_{xxx} + u_{xxxx} = 0 ,
\]

and the Kaup-Kupershmidt (KK) equation, if \( \gamma = \frac{75}{2} \) [26-27]

\[
 u_t + 45u^2u_x - \frac{75}{2} u_x u_{xx} - 15uu_{xxx} + u_{xxxxx} = 0 .
\]

Consider the traveling wave solutions \( u(x, t) = u(\xi) \), then equation (25) becomes as

\[
 \omega u' + 45ku^2u' - 15k^3u(3) + k^5u(5) = 0 .
\]

Balancing the nonlinear term \( uu(3) \) and the highest order derivative \( u(5) \) gives \( m = 1 \). It can be supposed that the solution of equation (28) is in the form

\[
 u(\xi) = a_0 + a_1 \varphi .
\]

By substituting from (29) into equation (28) yields a set of algebraic equations for \( a_0, a_1, \gamma, \alpha, \beta, k, \omega \). These equations are found as

\[
 k^5\alpha^2a_1 + \omega a_1 - 15k^3\alpha a_0a_1 + 45ka_0^2a_1 = 0 ,
\]

\[
 15k^5\beta a_1 - 45k^3\beta a_0a_1 - 15k^3\alpha a_1^2 - k^3\gamma a_1^2 + 90ka_0a_1^2 = 0 ,
\]

\[
 \frac{45}{2}k^5\beta^2a_1 - 45k^3\beta a_1^2 - \frac{3}{2}k^3\beta\gamma a_1^2 + 45ka_1^3 = 0 .
\]

**Case I**, if \( \gamma = 15 \), the solution of the system of equations (30), can be found as

\[
 a_0 = \frac{5k^3\alpha \pm \sqrt{5k^6\alpha^2 - 20k\omega}}{30k} , \quad a_1 = \frac{k^2\beta}{2} ; \quad a_0 = \frac{k^2\alpha}{3} , \quad a_1 = k^2\beta , \quad \omega = -k^5\alpha^2 .
\]

Substituting from equation (31) into (29), the following solution of the Sawada-Kotera equation (26) can be obtained as

\[
 u(x, t) = \frac{5k^3\alpha \pm \sqrt{5k^6\alpha^2 - 20k\omega}}{30k} - \frac{k^2\alpha}{2} \left( \text{Sech} \left[ \frac{\sqrt{\alpha}}{2}(kx + \omega t) \right] \right)^2 .
\]

\[
 u(x, t) = \frac{k^2\alpha}{3} - k^2\alpha \left( \text{Sech} \left[ \frac{\sqrt{\alpha}}{2}(kx - k^5\alpha^2t) \right] \right)^2 .
\]

**Case II**, if \( \gamma = \frac{75}{2} \), the solution of the system of equations (30), can be found as

\[
 a_0 = \frac{2k^2\alpha}{3} , \quad a_1 = 2k^2\beta , \quad \omega = -11k^5\alpha^2 , \quad a_0 = \frac{k^2\alpha}{12} , \quad a_1 = \frac{k^2\beta}{4} , \quad \omega = -\frac{k^5\alpha^2}{16} .
\]
By substituting from equation (34) into (29), we have obtained the following solution of the Kaup-Kupershmidt equation (27) as

$$u(x, t) = \frac{2k^2\alpha}{3} - 2k^2\alpha \left( Sech \left( \frac{\sqrt{\alpha}}{2} (kx - 11k^5\alpha^2t) \right) \right)^2. \quad (35)$$

$$u(x, t) = \frac{k^2\alpha}{12} - \frac{k^2\alpha}{4} \left( Sech \left( \frac{\sqrt{\alpha}}{2} (kx - k^5\alpha^2t) \right) \right)^2. \quad (36)$$

Figs.(5a-5d) represent the dark solitary wave solutions $u(x, t)$ of equations (32-33) and (35-36), with $k = 1$, $\omega = -1$ and $\alpha = 4$ in the interval $[-10, 10]$; with $k = 2$ and $\alpha = 1$ in the interval $[-1, 1]$; with $k = 0.5$ and $\alpha = 4$ in the interval $[-1, 1]$; with $k = -0.5$ and $\alpha = 1$ in the interval $[-10, 10]$.

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