Global Existence for a Model of Nickel-Iron Alloy Electrodeposition on Rotating Disk Electrode in the Quadratic Case

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Abstract

To better understand the nickel-iron electrodeposition process, we are interested in the one-dimensional model. This model addresses dissociation, diffusion, electromigration, convection and deposition of multiple ion species. We study the global existence of solutions that are here different ion concentrations in the mixture as well as the electric potential. We present the new techniques to obtain global existence and positivity of classical solution for our model in the quadratic case, without no restriction of growth on the non linear terms.

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1 Introduction

In this work we are interested in the following problem
\[
\begin{align*}
\frac{\partial w_i}{\partial t} - d_i \frac{\partial^2 w_i}{\partial x^2} + b(x) \frac{\partial w_i}{\partial x} - m_i \frac{\partial}{\partial x} (w_i \frac{\partial \Phi}{\partial x}) &= S_i(w) & \text{on } Q_T \\
-\frac{\partial^2 \Phi}{\partial x^2} &= \frac{1}{\varepsilon_0} \sum_{i=1}^{5} z_i w_i & \text{on } Q_T \\
-d_i \frac{\partial w_i}{\partial x}(t, 0) - m_i w_i(t, 0) \frac{\partial \Phi}{\partial x}(t, 0) &= -\gamma_i(t) w_i(t, 0) & \text{for } t \in [0, T] \\
-d_i \frac{\partial w_i}{\partial x}(t, 3\delta) - m_i w_i(t, 3\delta) \frac{\partial \Phi}{\partial x}(t, 3\delta) &= -b(3\delta) w_i(t, 3\delta) & \text{for } t \in [0, T] \\
\Phi(t, 0) &= V(t), \Phi(t, 3\delta) = 0 & \text{for } t \in [0, T] \\
w_i(0, x) &= w_{i,0}(x) & \text{for } x \in [0, 3\delta]
\end{align*}
\]

(1)

such systems involve one-dimensional steady-state transport of the various species to a rotating disk electrode in the electrodeposition of NiFe (for more detail see [1]). Where \( Q_T = [0, T] \times [0, L] \), \( w_i \) is the concentration of species \( A_i \) in the reactor, \( S_i \) denotes the production rate of \( A_i \) due to all the homogeneous reactions, \( d_i \) is the diffusion coefficient of species \( A_i \), \( v \) is the fluid velocity vector, \( \Phi \) is the electric potential and \( u_i \) is the electrical mobility of species \( A_i \) (see [17]). The mobility and diffusion coefficient are related through the Einstein equation

\[
m_i = \frac{d_i z_i F}{RT}
\]

where \( z_i F \) is the charge carried by a mole of species \( A_i \), \( R \) is the universal gas constant and \( T \) is the local temperature. The permittivity of the solvent is denoted by \( \varepsilon_0 \) and \( \gamma_i(t) = \beta_i \exp(-\frac{\alpha_i z_i F}{RT}(V_n(t) - V(t))) \), \( \gamma_i(t) = 0 \) for \( i = 4, 5 \).

The initial concentration is given by \( w_i(0, x) = w_{i,0}(x) \), where \( w_{i,0} \) represents the total amount of component \( A_i \) added to the bulk solution.

The homogeneous reaction term in (1) takes the form

\[
\begin{align*}
S_i &= 0 & \text{for } 1 \leq i \leq 2 \\
S_3 &= S_4 = -S_5 = -S(w_3, w_4, w_5) \\
S(r, p, q) &= k_1 rp - k_{-1}q
\end{align*}
\]

(3)

The goal of this work is to prove global existence and positivity of classical solution for the model (1). This result is a generalization of the one obtained in [1]. With effect, we were interested to nonlinearities which are more general.

We have organized this paper as follows. In section 2, we give the precise setting of the problem. The main result is presented in section 3, we then propose an approximate problem and we give suitable estimates to prove that (1) has a global and classical solution in the quadratic case.
2 Mathematical analysis of the model

Throughout this paper we make the following assumptions:

\[ V \in L^\infty(0,T) \]  

(4)

for all \( i = 1, \ldots, 5 \), \( w_i, 0 \in L^\infty(\Omega) \) such that \( w_{i,0} \geq 0 \)  

(5)

Now, we have to clarify in which sense we want to solve (1).

**Definition 1** \((w, \Phi) = (w_1, \ldots, w_5, \Phi)\) is said to be a solution of (1) if for all \( 1 \leq i \leq 5 \)

\[
\begin{cases}
    w \in C([0,T]; L^2(\Omega)^5) \cap L^2(0,T; H^1(\Omega)^5), & \text{and} \\
    \Phi \in L^\infty(0,T; H^1(\Omega)), & S_i(w) \in L^1(\Omega_T) \text{ and} \\
    \Phi \in L^\infty(0,T; H^1(\Omega)) \text{ and} \quad \Phi(t, 0) = V(t), \quad \Phi(t, 3\delta) = 0 \quad & \text{for all } \Phi \in D(\Omega) \text{ and } t \in [0,T[ \\
    \Phi \in L^\infty(0,T; H^1(\Omega)) \text{ and} \quad \Phi(t, 0) = V(t), \quad \Phi(t, 3\delta) = 0 \quad & \text{for all } \Phi \in D(\Omega) \text{ and } t \in [0,T[ \\
    w_i(0,.) = w_{i,0}(.) \quad & \text{for all } w_{i,0} \in L^\infty(\Omega) \text{ such that } w_{i,0} \geq 0
\end{cases}
\]

(6)

If in addition \( w \in L^\infty(Q_T)^5 \) then this solution is said to be classical.

3 Main Results

**Theorem 1** Assume that (3), (4) and (5) hold. Then the problem (1) has a classical solution \((w, \Phi)\) such that \( w \geq 0 \) in \( Q_T \).
3.1 Proof of theorem 1

3.1.1 Approximating scheme

For every $S_i$ we associate $\hat{S}_i$ such that $\hat{S}_i(w) = S_i(w^+)$ and we consider the truncated $\eta_n \in C^\infty_0(\mathbb{R}^5)$ which satisfied

$$0 \leq \eta_n \leq 1$$
$$\eta_n(r) = 1 \text{ if } |r| \leq n$$
$$\eta_n(r) = 0 \text{ if } |r| \geq n + 1$$

then we define for all $w \in \mathbb{R}^5$

$$S^n_i(w) = \eta_n(|w|)\hat{S}_i(w)$$

We remark that with the hypothesis (3), $S^n_i$ so definite is globally Lipschitz.

We consider also for $\Psi \in C([0,T]; H^1(\Omega))$ and $t \in [0,T]$, a bilinear form $a_i^\Psi(.,.,.)$ defined on $H^1(\Omega) \times H^1(\Omega)$ by

$$a_i^\Psi(t,u,\hat{u}) = d_i \int_\Omega \frac{\partial u}{\partial x} \frac{\partial \hat{u}}{\partial x} + \int_\Omega b(x) \frac{\partial u}{\partial x} \hat{u} + m_i \int_\Omega u \frac{\partial \Psi}{\partial x} \frac{\partial \hat{u}}{\partial x} + \lambda \int_\Omega u \hat{u} + \gamma_i(t)u(t,0)\hat{u}(t,0) - b(3\delta)u(t,3\delta)\hat{u}(t,3\delta)$$

where $\lambda > 0$ whose value will be fixed in the continuation.

Finally, we introduce for $v \in L^2(Q_T)$ the time regularizing

$$v^{(n)}(t,x) = \int_0^t nv(s,x) \exp(n(s-t))ds,$$

it is well known that (see [5])

$$v^{(n)} \in C([0,T]; L^2(\Omega)), v^{(n)} \to v \text{ in } L^2(Q_T) \text{ and } \sup_{0 \leq t \leq T} \|v^{(n)}(t,.)\|_{L^1(\Omega)} \leq \sup_{0 \leq t \leq T} \|v(t,.)\|_{L^1(\Omega)}$$

Let us now consider the truncated system

$$\begin{cases}
  w_n \in W(H^1), \Phi_n \in L^\infty(0,T; H^1(\Omega)) \text{ and} \\
  \bullet \text{ for all } \varphi \in H^1(\Omega), \int_\Omega \frac{\partial w_{i,n}}{\partial t} \varphi + a_i^\varphi_n(t,w_{i,n},\varphi) = \int_\Omega S^n_i(w_n)\varphi \text{ in } \mathcal{D}'(0,T) \\
  \bullet \text{ for all } \theta \in \mathcal{D}(\Omega), \text{ and a.e. } t \in ]0,T[, \int_\Omega \frac{\partial \Phi_n}{\partial x} \frac{\partial \theta}{\partial x} dx = \frac{4\pi F}{\varepsilon_0} \int_\Omega \sum_{i=1}^5 z_i w_{i,n}^{(n)} \theta dx \\
  \bullet \Phi_n(t,0) = V(t), \Phi_n(t,3\delta) = 0 \text{ for all } t \in ]0,T[. \\
  \bullet w_{i,n}(0,.) = w_{i,0}(.)
\end{cases}$$

(8)
where the Hilbert space $W(H^1)$ is defined by

$$W(H^1) = \left\{ u \in L^2(0, T, H^1(\Omega)^5); \frac{\partial u}{\partial t} \in L^2(0, T, H^{-1}(\Omega)^5) \right\}.$$ 

We note that all $u \in W(H^1)$ is nearly everywhere equal to a continuous function from $[0, T]$ to $L^2(\Omega)^5$. Besides, $W(H^1) \subset C([0, T]; L^2(\Omega)^5)$ and the injection is continuous (see Dautray and Lions [6]).

We have the following result

**Theorem 2** Assume that (3), (4) and (5) hold, then the problem (8) has a solution $(w_n, \Phi_n) \in W(H^1) \times L^\infty(0, T; W^{1,\infty}(\Omega))$ such that $w_n \geq 0$ in $Q_T$.

**Proof.** The proof of this theorem is in [1]. ■

**Remark 1** To prove the existence of classical solution on $Q_T$ with the nonlinearity $S$ itself, the game consists in proving $L^\infty$-estimates on the solution of the approximate system (8) which do not depend on $n$. Then choosing $n$ large enough, a solution of (8) will also be a solution of (1).

### 3.1.2 A priori estimates

In all the continuation of this paper, $C$ with or without subscripts, denote generic positive constants. Their values may be different on different occasions, but are independent of $n$.

**Theorem 3** 1) There exists a constant $C = C(T, \|w_0\|_{L^\infty(\Omega)})$ such that

$$\sup_{0 < t < T} \int_\Omega \sum_{i=1}^5 |w_{i,n}(t, x)| \, dx \leq C \quad \text{and} \quad \|\Phi_n\|_{L^\infty(0, T; W^{1,\infty}(\Omega))} \leq C.$$ 

2) There exists a constant $C$ independent of $n$ such that for all $i = 1, \ldots, 5$:

$$\int_0^T \int_\Omega |S^n_i(w_n)| \leq C.$$ 

3) For all $i = 1, \ldots, 5$, there exists a constant $C = C(d_i, m_i, \|w_i,0\|_{L^\infty(\Omega)})$ such that

$$\|w_{i,n}\|_{L^2(0,T,H^1)} \leq C \quad \text{and} \quad \|w_{i,n}S^n_i(w_n)\|_{L^1(Q_T)} \leq C.$$ 

**Proof.** 1) i) We multiply the first equation in (8) by $\alpha_i$ (with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, \alpha_5 = 2$), add on $i$ and integrate over $\Omega = ]0; 3\delta[$. Using $\sum_{i=1}^5 \alpha_i S^n_i(w_n) = 0$, we obtain

$$\sum_{i=1}^5 \alpha_i \int_\Omega \frac{\partial w_{i,n}}{\partial t} + \sum_{i=1}^5 \alpha_i \gamma_i(t)w_{i,n}(t, 0) - \sum_{i=1}^5 \alpha_i b(3\delta)w_{i,n}(t, 3\delta) + \sum_{i=1}^5 \alpha_i \int_\Omega b(x) \frac{\partial w_{i,n}}{\partial x} = 0 \quad (9)$$
moreover by integrating by part, we have
\[
\sum_{i=1}^{5} \alpha_i \int_{\Omega} \frac{\partial w_{i,n}}{\partial t} + \sum_{i=1}^{5} \alpha_i \gamma_i(t)w_{i,n}(t,0) - \sum_{i=1}^{5} \alpha_i \int_{\Omega} w_{i,n}b'(x) = 0
\]
the positivity of \( w_{i,n}, \gamma_i \) and \( b'(x) \leq 0 \) yield \( \sum_{i=1}^{5} \alpha_i \int_{\Omega} \partial w_{i,n} \leq 0 \).

Integrating this inequality on \([0, t[, \) for all \( 0 < t < T \), we get
\[
\sum_{i=1}^{5} \alpha_i \int_{\Omega} w_{i,n}(t, x) dx \leq \sum_{i=1}^{5} \alpha_i \int_{\Omega} w_{i,0}(x) dx
\]
then
\[
\sum_{i=1}^{5} \int_{\Omega} |w_{i,n}(t, x)| dx \leq C(\sum_{i=1}^{5} \| w_{i,0} \|_{L^\infty(\Omega)})
\]
consequently \( \sup_{0 < t < T} \int_{\Omega} \sum_{i=1}^{5} |w_{i,n}(t, x)| dx \leq C. \)

\( ii) \) \( \Phi_n \) satisfies
\[
\Phi_n(t, x) = -\frac{V(t)}{3\delta}(x - 3\delta) + \frac{4\pi F}{\epsilon_0} \sum_{i=1}^{3} \int_{0}^{3\delta} G(x, s) z_i w_{i,n}(t, s) ds
\]
as \( \sup_{0 < t < T} \left\| w_{i,n}(t, .) \right\|_{L^1(\Omega)} \leq \sup_{0 < t < T} \left\| w_{i,n}(t, .) \right\|_{L^1(\Omega)} \), and using (4) we obtain
\[
\| \Phi_n \|_{L^\infty(0,T; W^{1,\infty}(\Omega))} \leq C.
\]

2) In the fact that \( S_i^n = S_{i+1}^n = 0, |S_3^n(w_n)| = |S_4^n(w_n)| = |S_5^n(w_n)| \), the proof is remains to show that \( \int_0^T \int_{\Omega} |S_i^n(w_n)| \leq C \) for some \( i = 3, 4, 5. \)

For this, we Consider the equation satisfied by \( w_{3,n} \). One can write
\[
\frac{\partial w_{3,n}}{\partial t} - d_3 \frac{\partial^2 w_{3,n}}{\partial x^2} + b(x) \frac{\partial w_{3,n}}{\partial x} - m_3 \frac{\partial}{\partial x}(w_{3,n} \frac{\partial \Phi_n}{\partial x}) = S_3^n(w_n)
\]
where \( S_3^n(w_n) = \eta_n(w_n)(-k_1 w_{3,n} w_{4,n} + k_{-1} w_{5,n}). \)

Integrating (10) on \( \Omega, \) we obtain
\[
\int_{\Omega} \frac{\partial w_{3,n}}{\partial t} - \int_{\Omega} b'(x) w_{3,n} + \gamma_3(t) w_{3,n}(t, 0) + \int_{\Omega} k_1 \eta_n(w_n) w_{3,n} w_{4,n} = \int_{\Omega} k_{-1} \eta_n(w_n) w_{5,n}
\]
using the first estimate of this theorem, the fact that \( b'(x) \leq 0 \) and \( 0 \leq \eta_n \leq 1 \), we have
\[
\int_\Omega \frac{\partial w_{3,n}}{\partial t} + \gamma_3(t)w_{3,n}(t,0) + \int_\Omega k_1\eta_n(w_n)w_{3,n}w_{4,n} \leq C
\]
and now we integrate on \((0,T)\). As \( \gamma_3(t)w_{3,n}(t,0) \geq 0 \) for all \( t \), we get
\[
\int_\Omega w_{3,n}(T,x)dx + \int_Q k_1\eta_n(w_n)w_{3,n}w_{4,n} \leq C + \int_\Omega w_{3,0}(x)dx
\]
by (5) and the positivity of \( w_n \), we obtain
\[
\int_Q k_1\eta_n(w_n)w_{3,n}w_{4,n} \leq C.
\]
Finally \( \int_0^T \int_\Omega |S_{3}^n(w_n)| \leq C \), then for all \( i = 1, \ldots, 5 : \int_0^T \int_\Omega |S_{i}^n(w_n)| \leq C. \)

3) For \( i = 1, \ldots, 4 \), we multiply the first equation in (8) by \( w_i,n \). By integrating on \( Q_t \), we have
\[
\int_\Omega \frac{1}{2}(w_{i,n}^2(t,x) - w_{i,0}^2(x))dx + d_i \int_{Q_t} \left| \frac{\partial w_{i,n}}{\partial x} \right|^2 + m_i \int_{Q_t} w_{i,n} \frac{\partial \Phi_n}{\partial x} \frac{\partial w_{i,n}}{\partial x} + \int_{Q_t} \gamma_i(s)w_{i,n}^2(s,0)ds - \frac{1}{2} \int_{Q_t} b'(x)w_{i,n}^2(s, x) + \int_{Q_t} w_{i,n} |S_{i}^n(w_n)| = 0
\]
then
\[
\frac{1}{2} \int_\Omega w_{i,n}^2(t, x)dx + d_i \int_{Q_t} \left| \frac{\partial w_{i,n}}{\partial x} \right|^2 \leq -m_i \int_{Q_t} w_{i,n} \frac{\partial \Phi_n}{\partial x} \frac{\partial w_{i,n}}{\partial x} + C \|w_{i,0}\|_{L^\infty(\Omega)}
\]
since \( \left\| \frac{\partial \Phi_n}{\partial x} \right\|_{L^\infty(Q_T)} \leq C \), and by Young’s inequality, we obtain
\[
\frac{1}{2} \int_\Omega w_{i,n}^2(t, x)dx + d_i \int_{Q_t} \left| \frac{\partial w_{i,n}}{\partial x} \right|^2 \leq \epsilon \int_{Q_t} \left| \frac{\partial w_{i,n}}{\partial x} \right|^2 + C\epsilon \int_{Q_t} w_{i,n}^2(x, s) + C \|w_{i,0}\|_{L^\infty(\Omega)}
\]
then
\[
\frac{1}{2} \int_\Omega w_{i,n}^2(t, x)dx + (d_i - \epsilon) \int_{Q_t} \left| \frac{\partial w_{i,n}}{\partial x} \right|^2 \leq C\epsilon \int_{Q_t} w_{i,n}^2(x, s)dxds + C \|w_{i,0}\|_{L^\infty(\Omega)}
\]
hence \( \|w_n\|_{L^2(0,T;H^1)} \leq C \).

On the other hand
\[
\int_{Q_T} w_{i,n} |S_1^n(w_n)| \leq -m_i \int_{Q_T} w_{i,n} \frac{\partial \Phi_n}{\partial x} \frac{\partial w_{i,n}}{\partial x} + C \|w_{i,0}\|_{L^\infty(\Omega)}
\]
we have in the same way \( \|w_{i,n} S_1^n(w_n)\|_{L^1(Q_T)} \leq C \).

Now treat case \( i = 5 \). Consider the equation satisfied by \( W_n = w_{3,n} + w_{5,n} \), and using \( S_3^n = -S_3^n \), we obtain
\[
\begin{align*}
\frac{\partial W_n}{\partial t} - d_5 \frac{\partial^2 W_n}{\partial x^2} + (d_5 - d_3) \frac{\partial^2 w_{3,n}}{\partial x^2} - m_5 \frac{\partial}{\partial x} \frac{W_n}{\partial x} \\
+ (m_5 - m_3) \frac{\partial}{\partial x} (w_{3,n} w_{5,n}) + b(x) \frac{W_n}{\partial x} = 0
\end{align*}
\]

We multiply by \( W_n \) and integrate on \( Q_T \). While following the same reasoning that before, we find
\[
\begin{align*}
\frac{1}{2} & \int_{Q_T} \frac{\partial}{\partial t} W_n^2 + d_5 \int_{Q_T} \left| \frac{\partial W_n}{\partial x} \right|^2 - \frac{1}{2} \int_{Q_T} b'(x) \frac{\partial}{\partial x} W_n^2 + \int_0^T (\gamma_3(s) w_{3,n}(s,0) + \gamma_5(s) w_{5,n}(s,0)) W_n(s,0) \\
\leq & \ (d_5 - d_3) \int_{Q_T} \frac{\partial w_{3,n}}{\partial x} \frac{\partial W_n}{\partial x} + \int_{Q_T} (m_3 w_{3,n} + m_5 w_{5,n}) \frac{\partial \Phi_n}{\partial x} \frac{\partial W_n}{\partial x}
\end{align*}
\]

hence
\[
\begin{align*}
\frac{1}{2} & \int_{\Omega} W_n^2(t,x)dx + d_5 \int_{Q_T} \left| \frac{\partial W_n}{\partial x} \right|^2 \leq \frac{1}{2} \int_{\Omega} W_n^2(0,x)dx + (d_5 - d_3) \int_{Q_T} \frac{\partial w_{3,n}}{\partial x} \frac{\partial W_n}{\partial x} \\
& + \int_{Q_T} (m_3 w_{3,n} + m_5 w_{5,n}) \frac{\partial \Phi_n}{\partial x} \frac{\partial W_n}{\partial x}
\end{align*}
\]

using Young and Hölder’s inequality, we deduce
\[
\begin{align*}
\frac{1}{2} & \int_{\Omega} W_n^2(t,x)dx + d_5 \int_{Q_T} \left| \frac{\partial W_n}{\partial x} \right|^2 \leq \frac{1}{2} \int_{\Omega} (w_{3,0}^2 + w_{5,0}^2)(x) + \epsilon_1 \int_{Q_T} \left| \frac{\partial W_n}{\partial x} \right|^2 + C_{\epsilon_1} \int_{Q_T} \left| \frac{\partial w_{3,n}}{\partial x} \right|^2 \\
& + \epsilon_2 \int_{Q_T} \left| \frac{\partial W_n}{\partial x} \right|^2 + C_{\epsilon_2} \int_{Q_T} (m_3 w_{3,n} + m_5 w_{5,n})^2
\end{align*}
\]
Concerning the term \( w_{5,n}S_5^w(w_n) \), we have

\[
\int_{Q_T} w_{5,n}S_5^w(w_n) = \int_\Omega \frac{1}{2}(w_{5,n}^2 - w_{5,0}(x)) + d_5 \int_{Q_T} \left| \frac{\partial w_{5,n}}{\partial x} \right|^2 + m_5 \int_{Q_T} w_{5,n} \frac{\partial \Phi_n}{\partial x} \frac{\partial w_{5,n}}{\partial x} \\
+ \int_0^T \gamma_5(s) w_{5,n}(s,0)ds - \frac{1}{2} \int_{Q_T} b'(x) w_{5,n}(s,x)
\]

using \( \|w_{5,n}\|_{L^2(0,T,H^1)} \leq C, \gamma_5, b' \) bounded in \( L^\infty \) and \( \|\frac{\partial \Phi_n}{\partial x}\|_{L^\infty(Q_T)} \leq C \), we then obtain the result. \( \Box \)

Now we recall classical estimates on the heat operator in one space dimension (see \cite{10}):

**Lemma 4** Let \( \Psi \) be a solution on \( Q_T \) of

\[
\left\{ \begin{array}{l}
\frac{\partial \Psi}{\partial t} - d \frac{\partial^2 \Psi}{\partial x^2} + \nu \frac{\partial \Psi}{\partial x} - \frac{\partial}{\partial x}(\alpha \Psi) = F \quad \text{on } Q_T \\
- d \frac{\partial \Psi}{\partial x}(t,x) - \nu \Psi(t,x) \alpha(x) = f \quad \text{for } t \in [0,T[ \text{ and } x \in \{0,3\delta\} \\
\Psi(0,x) = \Psi_0(x) \quad \text{for } x \in [0,3\delta[ \nabla
\end{array} \right.
\]

where \( d > 0, \nu \in C^1, \alpha \in L^\infty(Q_T), f \in L^\infty(\Omega) \text{ and } \Psi_0 \in L^\infty(\Omega) \). Then there exists \( C = C(T,d,\nu,\alpha,\Psi_0,f) \) such that

1) For all \( 1 \leq r \leq 3 \), \( \|\Psi\|_{L^r(Q_T)} \leq C(1 + \|F\|_{L^r(Q_T)}) \)

2) For all \( 1 \leq p < \frac{3}{2}, q = \frac{3p}{3 - 2p}, \|\Psi\|_{L^q(Q_T)} \leq C(1 + \|F\|_{L^p(Q_T)}) \)

3) For all \( p > \frac{3}{2}, \|\Psi\|_{L^\infty(Q_T)} \leq C(1 + \|F\|_{L^p(Q_T)}) \).

Using the last lemma, we can prove the \( L^\infty \)-estimates on the solution of the approximate system (8) which do not depend on \( n \).

**Lemma 5** Let \( (w_n, \Phi_n) \) be a solution of the truncated system (8). Then for all \( i = 1, \ldots, 5 \), \( w_{i,n} \in L^\infty(Q_T) \) and

\[ \|w_{i,n}\|_{L^\infty(Q_T)} \leq C. \]

**Proof.** We start with the \( L^2 \)-estimates of theorem 3, namely \( \|w_{i,n}\|_{L^2(0,T,H^1)} \leq C \).

Indeed, we have by theorem 3; \( S_t^w(w_n) \in L^1(Q_T) \) and \( \|S_t^w(w_n)\|_{L^1(Q_T)} \leq C \).

By lemma 4, we deduce that for all \( i = 1, \ldots, 5 \), \( w_{i,n} \in L^r(Q_T) \) for all \( 1 \leq r \leq 3 \).
Using the quadratic hypothesis of $S^n_i(w_n)$, we get $S^n_i(w_n) \in L^p(Q_T)$ for all $1 \leq p < \frac{3}{2}$.

By the second statement of lemma 4, $w_{i,n} \in L^q(Q_T)$ for all $1 \leq q < \infty$, hence $S^n_i(w_n) \in L^p(Q_T)$ for all $p > \frac{3}{2}$.

Finally, by the last statement of lemma 4, $\|w_{i,n}\|_{L^\infty(Q_T)} \leq C$, which finish this proof.

As explained in the remark 2, we apply the uniform estimate of lemma 5 to the solution of the truncated system (8). We choose $n$ larger than the constant $C$ of lemma 5. Then the solution of the truncated system (8) is also a solution of the system (1).

References


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