Digital Redesign of Infinite-Dimensional Controllers Based on Numerical Integration

Nobuko Kosugi and Koichi Suyama

Tokyo University of Marine Science and Technology
2-1-6 Etchujima, Koto-ku, Tokyo 135-8533, Japan
{kosugi, suyama}@kaiyodai.ac.jp

Abstract

Continuous-time infinite-dimensional controllers that include Laplace transforms of time functions with compact support are indispensable for the advanced control of delay systems. However, no study has yet been conducted on digital redesign for obtaining digital controllers that are used in sampled-data control systems from predesigned continuous-time infinite-dimensional controllers. We introduce a new form to represent the continuous-time input-output relation of linear time-invariant multi-input multi-output systems, called the finite-interval integral form. Using numerical integration, we approximate the continuous-time input-output relation of a predesigned infinite-dimensional controller in finite-interval integral form to obtain a digital controller.

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1 Introduction

Owing to recent advances in digital technology, the sampled-data control of continuous-time plants has provided various advantages over continuous-time control, such as better performance and flexibility. However, in sampled-data control systems, continuous-time controllers cannot be implemented as they are, and thus we require digital controllers.

Digital redesign [6, 17] is one of the most important methodologies for designing digital controllers that are used in sampled-data control systems. This method derives a digital controller from a predesigned continuous-time controller so that the performance of the continuous-time control system with
the predesigned controller is maintained in the resulting sampled-data control system. Moreover, this method has an advantage of using many well-established methods for obtaining a predesigned continuous-time controller, thereby achieving high performance. Many studies have been conducted on digital redesign, especially on state/output feedback control for linear time-invariant plants [1, 14, 18]. Recently, the digital redesign of nonlinear control has also attracted widespread attention [4, 7, 10, 11].

However, only a few studies have been performed till date on the digital redesign of infinite-dimensional controllers for infinite-dimensional plants, such as delay systems. For example, [13] addressed the digital redesign of $H_\infty$ controllers based on state-delayed observers. However, because the inclusion of pure delays in such controllers can be reduced to shift operators by several steps, the infinite-dimensionality of predesigned continuous-time controllers is not required in digital redesign.

There is a need for more complicated infinite-dimensional controllers including Laplace transforms of time functions with compact support on $[0, \infty)$ for the advanced control of delay systems, such as finite spectrum assignment [8, 9, 15], and for repetitive control [16]. The digital redesign of such complicated infinite-dimensional controllers has never been examined until now. The resulting sampled-data control system consists of an obtained digital controller of finite steps, i.e., finite dimension, and a continuous-time infinite-dimensional plant. Thus, it is difficult to compare the resulting sampled-data control system with the continuous-time one with a predesigned controller in the sense of the performance.

In this paper, we focus on the desirable performance of the continuous-time control system with a predesigned continuous-time infinite-dimensional controller, not on the performance of the resulting sampled-data control system. We obtain a digital controller that approximates the continuous-time input-output relation of a predesigned infinite-dimensional controller with high accuracy in order to maintain the desired performance.

The proposed digital redesign consists of the following two steps:

**Step 1:** We represent the continuous-time input-output relation of a predesigned infinite-dimensional controller in finite-interval integral form.

We introduce a new form to represent the continuous-time input-output relation of linear time-invariant multi-input multi-output systems, which is the finite-interval integral form. We can represent complicated continuous-time infinite-dimensional systems in this form, which is also well-matched with numerical integration.

**Step 2:** Using numerical integration, we approximate the continuous-time input-output relation in the finite-interval integral form obtained in Step 1 to get a digital controller.
The step size of the numerical integration is determined by the sampling period of a sampled-data control system. If it is sufficiently short, the obtained digital controller approximates the continuous-time input-output relation of the predesigned infinite-dimensional controller with high accuracy. Thus, the proposed digital redesign is potentially useful for applications of advanced control using infinite-dimensional controllers, such as finite spectrum assignment.

The following notations are used in this paper. \(\mathbb{R}\) and \(\mathbb{C}\): the fields of real and complex numbers, respectively, \(\mathbb{R}_+\): the set of nonnegative real numbers, \(\mathbb{C}_+\): the closed right-half plane, i.e., \(\{s \in \mathbb{C} | \text{Re} \ s \geq 0\}\), \(\mathbb{Z}_+\): the set of non-negative integers, \(\mathcal{Y}^{\ell \times m}\): the set of \(\ell \times m\) matrices with elements in \(\mathcal{Y}\), \(E\): an identity matrix of appropriate dimensions, \(^tA\): the transposed matrix of \(A\), \(\deg x\): the degree in \(x\) of a polynomial, and \(r\)-deg\(_{x,k}\): the degree in \(x\) of the \(k\)-th row of a polynomial matrix.

2 Finite-interval integral form

2.1 Sets of Laplace transforms of time functions with compact support

Consider the following set of absolutely (Lebesgue) integrable time functions with compact support on \([0, \infty)\):

\[
\mathcal{T} = \{ f(t) \in L^1(\mathbb{R}_+, \mathbb{R}) \mid \text{supp}(f) \subseteq [0, \infty) \text{ and compact}\}. \]

The set \(\mathcal{T}\) is closed with regard to both point-wise addition and the convolution product as multiplication. Let \(\delta_L(t)\) denote the Dirac’s delta function at \(t = L\). Then, we define a ring \(\tilde{\mathcal{T}}\) generated by elements in \(\mathcal{T} \cup \{\delta_L(t) \mid 0 \leq L < \infty\}\) with the convolution product as multiplication. The identity element of this ring is \(\delta_0(t)\). Note that \(\mathcal{T} \subset \tilde{\mathcal{T}}\).

Let \(\mathcal{S}\) and \(\tilde{\mathcal{S}}\) denote the sets of Laplace transforms of all elements in \(\mathcal{T}\) and \(\tilde{\mathcal{T}}\), respectively. Here, \(\mathcal{S}\) is closed with regard to point-wise addition and multiplication. On the other hand, \(\tilde{\mathcal{S}}\) forms a commutative ring with the identity element \(\mathcal{L}[\delta_0(t)] = 1\). Note that \(\mathbb{R} \subset \tilde{\mathcal{S}}\) and \(\mathcal{S} \subset \tilde{\mathcal{S}}\).

The elements belonging to \(\tilde{\mathcal{S}}\) are used for the description of plants with delays and infinite-dimensional controllers that are designed for such plants. On the other hand, the elements belonging to \(\mathcal{S}\) are used in the finite-interval integral form, which will be introduced in Section 2.2 for the digital redesign of such infinite-dimensional controllers.

**Example 2.1** (Distributed delay) Consider the following function belonging to \(\mathcal{T}\):

\[
\theta_{L,a}(t) = \begin{cases} 
e^a t, & 0 \leq t \leq L \\ 0, & t > L \end{cases} \quad (2)
\]
where \( a \in \mathbb{R} \), and \( 0 < L < \infty \). It becomes a distributed delay operator

\[
\begin{align*}
  f(t) & \mapsto \int_0^L e^{a\tau} f(t - \tau) d\tau, \quad (3)
\end{align*}
\]

where \( f(t) \) belongs to \( \mathcal{L}_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R}) \), which denotes the set of locally and absolutely integrable time functions. Its Laplace transform is given by

\[
\Theta_{L, a}(s) = \frac{1 - e^{-L(s-a)}}{s - a} \quad (\in \mathcal{S}),
\]

which is an entire function in \( s \).

We can easily prove the following proposition, which will be used to represent the continuous-time input-output relation of a predesigned infinite-dimensional controller in a new representational form.

**Proposition 2.2** For any \( F(s) \in \mathcal{S} \) and \( \tilde{F}(s) \in \tilde{\mathcal{S}} \), it holds that \( F(s)\tilde{F}(s) = \tilde{F}(s)F(s) \) belongs to \( \mathcal{S} \).

**Proof** Omitted.

\[\square\]

### 2.2 Definition

We now introduce a new representational form for the continuous-time input-output relation, which will play an essential role in the proposed digital redesign of infinite-dimensional controllers.

**Definition 2.3** The following representational form for the continuous-time input-output relation of linear time-invariant multi-input multi-output systems is called a finite-interval integral form:

\[
y(t) = \int_0^\infty V(\tau)y(t - \tau)d\tau + \int_0^\infty W(\tau)u(t - \tau)d\tau, \quad t \geq 0,
\]

where \( y(t) \in \mathbb{R}^\ell \) is the output, \( u(t) \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R}^m) \) is the input, \( V(t) \in \mathcal{T}^{\ell \times \ell} \), and \( W(t) \in \mathcal{T}^{\ell \times m} \).

The two integrals on the right-hand side of (5) are performed on the finite intervals corresponding to their supports. We assume that an initial function \( y(t) \in \mathcal{C}^\infty([-T_v, 0], \mathbb{R}^\ell) \) satisfies

\[
y(0) = \int_0^\infty V(\tau)y(-\tau)d\tau,
\]

where \( y(t) \in \mathcal{C}^\infty([-T_v, 0], \mathbb{R}^\ell) \) satisfies
where $\mathcal{L}_V$ ($0 < \mathcal{L}_V < \infty$) is derived from the support of $V(t)$, as follows:

$$\mathcal{L}_V = \min \left\{ t_0 \in \mathbb{R}_+ \mid V(t) = 0, t > t_0 \right\}. \tag{7}$$

The transfer function matrix of the system represented as (5) is given by

$$\{E - \hat{V}(s)\}^{-1}\hat{W}(s),$$

where $\hat{V}(s) = \mathcal{L}[V(t)] \in S^{t \times \ell}$ and $\hat{W}(s) = \mathcal{L}[W(t)] \in S^{t \times m}$.

The following example shows that this representational form has a high level of description capability:

**Example 2.4** Consider a linear time-invariant system with its state-space equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{8}$$

where $x(t) \in \mathbb{R}^{ms}$ is the state variable vector, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^\ell$ is the output, $A \in \mathbb{R}^{ms \times ms}$, $B \in \mathbb{R}^{ms \times m}$, and $C \in \mathbb{R}^{\ell \times ms}$. If this system is canonical, i.e., controllable and observable, the observability gramian

$$G_0(\tau) = \int_\tau^L \exp(-Ah)^tCC\exp(-Ah)dh \tag{9}$$

is regular for any $\tau$ ($0 \leq \tau < L < \infty$). Now, we can represent the input-output relation of this system in (5) by considering $V(t)$ and $W(t)$ as follows:

$$V(\tau) = \left\{ \begin{array}{ll} CG_0^{-1}(0)\exp(-Ah)^tC, & 0 \leq \tau \leq L \\ 0, & \tau > L \end{array} \right. \tag{10}$$

$$W(\tau) = \left\{ \begin{array}{ll} CG_0^{-1}(0)G_0(\tau)\exp(-A\tau)B, & 0 \leq \tau \leq L \\ 0, & \tau > L \end{array} \right. \tag{11}$$

It can be easily proved as follows. From (10) and (11), we have

$$\hat{V}(s) = CG_0^{-1}(0)\left[E - \exp\{-sE + \text{tr}(A)L\}\right](sE + \text{tr}(A))^{-1}C \tag{12}$$

$$\hat{W}(s) = C(sE - A)^{-1}B - CG_0^{-1}(0)\left[E - \exp\{-sE + \text{tr}(A)L\}\right] \times (sE + \text{tr}(A))^{-1}CC(sE - A)^{-1}B. \tag{13}$$

Thus, the transfer function is given by

$$\{E - \hat{V}(s)\}^{-1}\hat{W}(s) = C(sE - A)^{-1}B, \tag{14}$$

which is equal to the transfer function matrix of (8).
2.3 Properties

2.3.1 Spectrum and stability

We begin by stating the spectrum (i.e., the set of eigenvalues) of the system represented as (5).

**Theorem 2.5** The system represented as (5) has only the point spectrum

\[ \Lambda = \{ s \in \mathbb{C} \mid \det \{ E - \hat{V}(s) \} = 0 \} \tag{15} \]

For any \( \lambda \in \Lambda \), the corresponding generalized eigenspace is finite dimensional.

**Proof** We can prove this theorem by applying the discussion in [5] to the following autonomous system:

\[ y(t) = \int_{0}^{\infty} V(\tau)y(t-\tau)d\tau. \tag{16} \]

That is, we consider the solution operator \( \mathcal{S}(t) \) of (16) and the infinitesimal generator \( \mathcal{A} \) of \( \mathcal{S}(t) \). Then, we obtain the characteristic equation of \( \mathcal{A} \) by

\[ \det \{ E - \hat{V}(s) \} = 0. \]

See Section 7 in [5].

We define the \( L^1 \) input-output stability, as in [3].

**Definition 2.6** The system represented as (5) is said to be \( L^1 \) stable if any \( u(t) \in L^1(\mathbb{R}^+, \mathbb{R}^m) \) produces \( y(t) \in L^1(\mathbb{R}^+, \mathbb{R}^l) \).

Note that \( L^1 \) stability implies \( L^p \) stability (\( 1 < p \leq \infty \)) for linear time-invariant systems as shown in [3]. Applying the discussion in [5], we can easily obtain the following theorem:

**Theorem 2.7** The system represented as (5) is \( L^1 \) stable if and only if it does not have any eigenvalues in \( \mathbb{C}^+ \).

**Proof** Omitted.

The following lemma gives a simple sufficient condition for stability:

**Proposition 2.8** The system represented as (5) is stable if

\[ \| \hat{V}(s) \|_{\infty} < 1, \tag{17} \]

where \( \| \hat{V}(s) \|_{\infty} \) denotes the \( H_{\infty} \) norm of \( \hat{V}(s) \).

**Proof** By the small gain theorem in [3], we can prove that the system represented as (5) is stable.

We can easily derive the following proposition from Theorem 2.7 and Proposition 2.8:

**Proposition 2.9** If the system represented as (5) satisfies (17), then

\[ \det \{ E - \hat{V}(s) \} = 0 \] does not have any zeros in \( \mathbb{C}^+ \).

**Proof** Omitted.

We will use this proposition in the proof of Theorem 2.11.
2.3.2 Canonicity

Next, we consider the canonicity [12] (i.e., the controllability and observability) of each eigenvalue.

**Theorem 2.10** In the system represented as (5), $\lambda \in \Lambda$ is canonical if and only if

$$\text{rank} \left[ E - \hat{V}(\lambda) \hat{W}(\lambda) \right] = \ell. \quad (18)$$

**Proof** Let $\mathcal{N}_\lambda$ denote the generalized eigenspace of the infinitesimal operator $\mathcal{A}$ corresponding to an eigenvalue $\lambda$, and $d(\lambda)$ denotes its dimension. Using a method that is very similar to the spectral decomposition in [5], we can obtain the projection of the system represented as (5) onto $\mathcal{N}_\lambda$ as follows:

$$\begin{align*}
\begin{cases}
\frac{\zeta(t)}{dt} &= A_\lambda \zeta(t) + Q_\lambda \int_0^\infty W(\tau)u(t-\tau)d\tau, \\
y_\lambda(t) &= R_\lambda \zeta(t),
\end{cases}
\end{align*} \quad (19)$$

where $y_\lambda(t)$ is the output in $\mathcal{N}_\lambda$, $A_\lambda \in \mathbb{C}^{d(\lambda) \times d(\lambda)}$ has all eigenvalues at $\lambda$, $Q_\lambda \in \mathbb{C}^{d(\lambda) \times \ell}$, and $R_\lambda \in \mathbb{C}^{\ell \times d(\lambda)}$. In addition, $Q_\lambda$ satisfies

$$\int_0^\infty \int_\eta^\infty \exp[-A_\lambda(\xi - \eta)] Q_\lambda V(\eta) R_\lambda \exp(A_\lambda \xi)d\xi = E, \quad (20)$$

and the elements of $R_\lambda \exp(A_\lambda \xi) \in \mathbb{C}^{\ell \times d(\lambda)}$ span $\mathcal{N}_\lambda$ as its basis. In a manner that is very similar to [12], it can be shown that the system in (19) is canonical if and only if the rank condition of (18) is satisfied. \qed

2.3.3 Relation used in digital redesign

The following theorem presents an important relation that will be used to represent the continuous-time input-output relation of a pre-designed infinite-dimensional controller in finite-interval integral form.

**Theorem 2.11** For any $a \in \mathbb{R}$ and $L (0 < L < \infty)$, define $\Phi_{L,a}(s)$, $\Psi_{L,a}(s) \in \mathcal{S}$ by

$$\Phi_{L,a}(s) = \frac{1}{1 - e^{-2La}} \Theta_{L,-a}(s) - \frac{e^{-2La}}{1 - e^{-2La}} \Theta_{L,a}(s) \quad (21)$$

$$\Psi_{L,a}(s) = \frac{a}{1 - e^{-2La}} \Theta_{L,-a}(s) + \frac{ae^{-2La}}{1 - e^{-2La}} \Theta_{L,a}(s), \quad (22)$$

where $\Theta_{L,-a}(s)$ and $\Theta_{L,a}(s)$ are defined by (4). Then, it holds that

$$s \Phi_{L,a}(s) = 1 - \Psi_{L,a}(s). \quad (23)$$

Furthermore, if $a < 0$, then $\Phi_{L,a}(s)$ does not have any zeros in $\mathbb{C}_+$. 
Proof Substituting (21) and (22) into (23), we can easily prove (23).
From (23), we have
\[
\frac{1}{s - a} = \frac{\Phi_{L,a}(s)}{1 - a\Phi_{L,a}(s) - \Psi_{L,a}(s)}.
\]
(24)
The right-hand side of (24) is the Laplace transform of the scalar system represented as (5) with
\[
\begin{align*}
V(t) &= \frac{2a}{1 - e^{-2La}} \theta_{L,-a}(t), \\
\theta_{L,a}(t) &= \text{defined by (2)}. \quad \text{For this scalar } V(t), \quad \|\hat{V}(s)\|_\infty \leq \int_0^\infty |V(t)|dt
\end{align*}
\]
(25)
Thus, if \(a < 0\), then \(\|\hat{V}(s)\|_\infty < 1\). Hence, by Proposition 2.9, \(1 - a\Phi_{L,a}(s) - \Psi_{L,a}(s)\) does not have any zeros in \(\mathbb{C}_+\). By considering \((s - a)\Phi_{L,a}(s) = 1 - a\Phi_{L,a}(s) - \Psi_{L,a}(s)\), we can prove that \(\Phi_{L,a}(s)\) does not have any zeros in \(\mathbb{C}_+\) if \(a < 0\). \(\square\)

For the proposed digital redesign in Section 3, let us define \(\phi_{L,a}(t) = \mathcal{L}^{-1}[\Phi_{L,a}(s)], \psi_{L,a}(t) = \mathcal{L}^{-1}[\Psi_{L,a}(s)] \in \mathcal{T}\) by
\[
\phi_{L,a}(t) = \begin{cases} 
\frac{1}{1 - e^{-2La}} e^{-at} - \frac{e^{-2La}}{1 - e^{-2La}} e^{at}, & 0 \leq t \leq L \\
0, & t > L
\end{cases}
\]
(27)
\[
\psi_{L,a}(t) = \begin{cases} 
\frac{a}{1 - e^{-2La}} e^{-at} + \frac{ae^{-2La}}{1 - e^{-2La}} e^{at}, & 0 \leq t \leq L \\
0, & t > L
\end{cases}
\]
(28)

3 Digital redesign of infinite-dimensional controllers
Suppose that we have the following continuous-time infinite-dimensional controller:
\[
C(s) = D^{-1}(s)N(s),
\]
(29)
where \(D(s) \in (\tilde{S}[s])^{\ell \times \ell}\) and \(N(s) \in (\tilde{S}[s])^{\ell \times m}\). This is presdesigned for a continuous-time multi-input multi-output plant \(P(s)\) to compose a continuous-time output-feedback control system as shown in Figure 1 (a). Let \(U(s)\) and
$E(s)$ denote the Laplace transforms of $u(t)$ and $e(t)$, respectively. Then, we have

$$D(s)U(s) = N(s)E(s),$$

which describes the input-output relation of the predesigned continuous-time infinite-dimensional controller (29).

Let $d_{ij}(s)$ and $n_{ij}(s)$ denote the $(i, j)$-element of $D(s)$ and $N(s)$, respectively. We assume the following:

**Assumption 3.1**

(a) $d_{ii}(s)$ is monic in $s$, and $\deg_s d_{ii}(s) = \nu_i$, where $1 \leq i \leq \ell$.

(b) $\deg_s d_{ij}(s) < \nu_i$, where $1 \leq i, j \leq \ell$ and $i \neq j$.

(c) $\deg_s n_{ij} < \nu_i$, where $1 \leq i \leq \ell$ and $1 \leq j \leq m$.

This assumption implies the following:

(i) $\det D(s) \in \tilde{S}[s]$ is monic in $s$.

(ii) $D(s)$ is row reduced in $s$.

(iii) $r-deg_{s,i}N(s) < r-deg_{s,i}D(s)$, where $1 \leq i \leq \ell$.

Then, Assumption 3.1 confirms that the continuous-time infinite-dimensional controller (29) is retarded-type and proper in $s$.

The continuous-time control system in Figure 1 (a) is theoretical because $C(s)$ itself cannot be realized by an analog computing device in its present state. Therefore, by digital redesign, we obtain a digital controller $C_d(z)$ from $C(s)$, where $z$ is the $Z$-transform operator corresponding to an appropriately given sampling period $T$. The $C_d(z)$ can be realized by a digital computing device to compose a sampled-data output-feedback control system as shown in Figure 1 (b).

The proposed digital redesign consists of the following two steps:

**Step 1:** We represent the continuous-time input-output relation of the predesigned infinite-dimensional controller (29) in finite-interval integral form.

**Step 2:** Using numerical integration, we approximate the continuous-time input-output relation in the finite-interval integral form obtained in Step 1 to get a digital controller $C_d(z)$. 
3.1 Step 1: An infinite-dimensional controller in finite-interval integral form

Left-multiplying both sides of (30) by
\[ \Gamma(s) = \text{diag}\{\Phi_{L,a}(s)^{\nu_1}, \Phi_{L,a}(s)^{\nu_2}, \ldots, \Phi_{L,a}(s)^{\nu_i}\} \]  
and using (23), we have
\[ \{E - \hat{V}(s)\}U(s) = \hat{W}(s)E(s), \]  
where \( \hat{V}(s) \in S^{\ell \times \ell} \) and \( \hat{W}(s) \in S^{\ell \times m} \). It can be seen that there exist \( \hat{V}(s) \) and \( \hat{W}(s) \), as given in the (i)–(iii) below:

(i) Under Assumption 3.1 (a), we can suppose that \( d_{ii}(s) \) is of the form
\[ d_{ii}(s) = s^{\nu_i} + \gamma_{\nu_i-1}s^{\nu_i-1} + \cdots + \gamma_0, \]  
where \( \gamma_k \in \hat{S}, k = 0, 1, \ldots, \nu_i - 1 \). Then, the \((i,i)\)-element of \( \Gamma(s)D(s) \) is given by
\[ \{\Phi_{L,a}(s)^{\nu_i}\}d_{ii}(s) = \left\{1 - \Psi_{L,a}(s)^{\nu_i}\right\} + \gamma_{\nu_i-1}\left\{1 - \Psi_{L,a}(s)^{\nu_i-1}\Phi_{L,a}(s)\right\} + \cdots + \gamma_0\left\{\Phi_{L,a}(s)^{\nu_i}\right\} 
= 1 - \hat{v}_{ii}(s), \]  
where
\[ \hat{v}_{ii}(s) = 1 - \left\{1 - \Psi_{L,a}(s)^{\nu_i}\right\} - \gamma_{\nu_i-1}\left\{1 - \Psi_{L,a}(s)^{\nu_i-1}\Phi_{L,a}(s)\right\} - \cdots - \gamma_0\left\{\Phi_{L,a}(s)^{\nu_i}\right\}. \]
By Proposition 2.2, \( \hat{v}_{ii}(s) \) belongs to \( S \). It is the \((i, i)\)-element of \( \hat{V}(s) \) in (32).

(ii) Under Assumption 3.1 (b), we can suppose that \( d_{ij}(s) \) \((i \neq j)\) is of the form
\[
d_{ij}(s) = \gamma_{\nu_i - 1} s^{\nu_i - 1} + \cdots + \gamma_0,
\]
where \( \gamma_k \in \hat{\mathcal{S}}, \ k = 0, 1, \ldots, \nu_i - 1 \). Then, the \((i, j)\)-element of \( \Gamma(s) D(s) \) is given by
\[
\{\Phi_{L, a}(s)\}^\nu d_{ij}(s) = \gamma_{\nu_i - 1} \{1 - \Psi_{L, a}(s)\}^\nu - 1 \Phi_{L, a}(s) + \cdots + \gamma_0 \{\Phi_{L, a}(s)\}^\nu,
\]
which belongs to \( S \) by Proposition 2.2. Thus, the \((i, j)\)-element of \( \hat{V}(s) \) in (32) is given by
\[
\hat{v}_{ij}(s) = -\{\Phi_{L, a}(s)\}^\nu d_{ij}(s) \in S.
\]

(iii) Under Assumption 3.1 (c), the \((i, j)\)-element \( \hat{w}_{ij}(s) \) of \( \hat{W}(s) \) in (32) can be obtained in a manner that is very similar to (ii).

Let \( V(t) = \mathcal{L}^{-1}[\hat{V}(s)] \in T^{\ell \times \ell} \) and \( W(t) = \mathcal{L}^{-1}[\hat{W}(s)] \in T^{\ell \times m} \). Then, from (32), we have the continuous-time input-output relation of the infinite-dimensional controller (29) in finite-interval integral form, as follows:
\[
u(t) = \int_0^\infty V(\tau) u(t - \tau) d\tau + \int_0^\infty W(\tau) e(t - \tau) d\tau.
\]
Because
\[
\det\{E - \hat{V}(s)\} = \det \Gamma(s) D(s) = \{\Phi_{L, a}(s)\}^n \det D(s),
\]
where \( n = \sum_{i=1}^\ell \nu_i \), the system represented as (39) has the zeros of \( \Phi_{L, a}(s) \) as redundant eigenvalues, while they are not present in the infinite-dimensional controller (29). However, according to Theorem 2.10, they are not canonical. Furthermore, if \( a < 0 \), they are stable according to Theorem 2.11, i.e., they are not located in \( \mathbb{C}_+ \).

### 3.2 Step 2: Numerical integration

We approximate the continuous-time input-output relation in finite-interval integral form obtained in Step 1 by using numerical integration with a step size \( T \), which is the given sampling period, as follows.

In the finite-interval integral form (39), define the following:
\[
\overline{\nu}_V = \max\left\{ k \in \mathbb{Z}_+ \mid V(t) = 0, \ t < kT \right\}
\]
\[
\underline{\nu}_V = \min\left\{ k \in \mathbb{Z}_+ \mid V(t) = 0, \ t > kT \right\}
\]
\[
\overline{\nu}_W = \max\left\{ k \in \mathbb{Z}_+ \mid W(t) = 0, \ t < kT \right\}
\]
\[
\underline{\nu}_W = \min\left\{ k \in \mathbb{Z}_+ \mid W(t) = 0, \ t > kT \right\}.
\]
Furthermore, the approximation error in this case can be evaluated as follows:

$$u(kT) = \int_0^{\infty} V(\tau)u(kT - \tau)d\tau + \int_0^{\infty} W(\tau)e(kT - \tau)d\tau$$

$$\approx \sum_{j=\underline{\nu}}^{\overline{\nu}} A_j u(kT - jT) + \sum_{j=\underline{\nu}}^{\overline{\nu}} B_j e(kT - jT),$$

where $A_j \in \mathbb{R}^{t \times t}$ ($j = \underline{\nu}, \ldots, \overline{\nu}$) and $B_j \in \mathbb{R}^{t \times t}$ ($j = \underline{\nu}, \ldots, \overline{\nu}$) are the weight coefficients matrices determined by $V(jT)$ and $W(jT)$, respectively, as follows. Let $a_{j,i_1i_2}$ and $v_{i_1i_2}(t)$ denote the $(i_1, i_2)$-elements of $A_j$ and $V(t)$, respectively. Then, we set

$$a_{j,i_1i_2} = \gamma_{j,i_1i_2} v_{i_1i_2}(jT), \quad j = \underline{\nu}, \ldots, \overline{\nu},$$

where $\gamma_{j,i_1i_2}$ is the weight coefficient in the numerical integration

$$\int_0^{\infty} v_{i_1i_2}(\tau)u_{i_2}(kT - \tau)d\tau \approx \sum_{j=\underline{\nu}}^{\overline{\nu}} \gamma_{j,i_1i_2} v_{i_1i_2}(jT)u_{i_2}(kT - jT),$$

where $u_{i_2}(t)$ is the $i_2$-th element of $u(t)$. This numerical integration is used to obtain the approximate value for $u_{i_1}(kT)$ in (45). For example, if $\overline{\nu} - \underline{\nu}$ is an even integer and $v_{i_1i_2}(t)(t)$ does not have any jump discontinuities in $(\underline{\nu}T, \overline{\nu}T)$, we can use the composite Simpson’s rule. Then, $\gamma_{j,i_1i_2}$ ($j = \underline{\nu}, \ldots, \overline{\nu}$) are given by $T/3$, $4T/3$, $2T/3$, $4T/3$, $2T/3$, $\ldots$, $4T/3$, and $T/3$. Furthermore, the approximation error in this case can be evaluated as follows:

$$\left| \int_0^{\infty} v_{i_1i_2}(\tau)u_{i_2}(kT - \tau)d\tau - \sum_{j=\underline{\nu}}^{\overline{\nu}} \gamma_{j,i_1i_2} v_{i_1i_2}(jT)u_{i_2}(kT - jT) \right|$$

$$< \frac{T^5(\overline{\nu} - \underline{\nu})}{180} \max_{\underline{\nu} < \tau < \overline{\nu}} \left| \frac{d^4}{d\tau^4}\{v_{i_1i_2}(\tau)u_{i_2}(kT - \tau)\} \right|,$$

where

$$L_V = \max \{t_0 \in \mathbb{R}_+ | V(t) = 0, \ t < t_0 \}$$

and $L_V$ is given by (7). We can have $B_j \in \mathbb{R}^{t \times t}$ ($j = \underline{\nu}, \ldots, \overline{\nu}$) from $W(t)$ in a very similar manner.
Let \( u[k] = u(kT) \) and \( e[k] = e(kT) \). Define their \( \mathcal{Z} \)-transforms by \( U(z) = \mathcal{Z}[u[k]] \) and \( E(z) = \mathcal{Z}[e[k]] \). Then, the approximation (45) gives the following:

\[
U(z) = \sum_{j=\mathbb{Z}_V} \pi_V A_j z^{-j} U(z) + \sum_{j=\mathbb{Z}_W} \pi_W B_j z^{-j} E(z).
\]

(50)

Thus, we have the following digital controller:

\[
C_d(z) = \left[ E - \sum_{j=\mathbb{Z}_V} \pi_V z^{-j} A_j \right]^{-1} \left[ \sum_{j=\mathbb{Z}_W} \pi_W z^{-j} B_j \right].
\]

(51)

**Remark 3.2** In the approximation (45), we should consider possible jump discontinuities in \( V(t) \) and \( W(t) \). By way of example, consider the numerical integration (47) included in (45). If \( v_{i_1,i_2}(t) \) has a finite number of jump discontinuities at \( t = t_1, t_2, \ldots, t_p \) \((L_V \leq t_1 < t_2 < \cdots < t_p \leq T_V; p < \infty)\), we have

\[
\int_0^\infty v_{i_1,i_2}(\tau)u_{i_2}(kT - \tau)d\tau = \int_{L_V}^{t_1} v_{i_1,i_2}(\tau)u_{i_2}(kT - \tau)d\tau
\]

\[
+ \int_{t_1}^{t_2} v_{i_1,i_2}(\tau)u_{i_2}(kT - \tau)d\tau + \cdots
\]

\[
+ \int_{t_p}^{T_V} v_{i_1,i_2}(\tau)u_{i_2}(kT - \tau)d\tau.
\]

(52)

We approximate each integral on the right-hand side of (52) by individually applying an appropriate numerical integration rule. Thus, from the perspective of the approximation accuracy, it is desirable that \( L_V, t_1, t_2, \ldots, t_p \) and \( T_V \) are integer multiples of the sampling period \( T \). To this end, we should consider \( L \) in \( \Phi_{L,a}(s) \) used in Step 1 by an integer multiple of \( T \). Furthermore, suppose \( t_1 = k_1 T \) and \( t_2 = k_2 T \), where \( k_1 \) and \( k_2 \) belong to \( \mathbb{Z}_+ \). In the numerical integration

\[
\int_{t_1}^{t_2} v_{i_1,i_2}(\tau)u_{i_2}(kT - \tau)d\tau \approx \sum_{j=k_1}^{k_2} \gamma_{j,i_1,i_2} v_{i_1,i_2}(jT)u_{i_2}(kT - jT),
\]

(53)

we take the values of \( v_{i_1,i_2}(k_1 T) \) and \( v_{i_1,i_2}(k_2 T) \) by \( v_{i_1,i_2}(t_1+) \) and \( v_{i_1,i_2}(t_2-) \), respectively, due to the jump discontinuities in \( v_{i_1,i_2}(t) \) at \( t = t_1 \) and \( t = t_2 \). The same applies to a finite number of jump discontinuities in \( W(t) \).
Remark 3.3 Consider a linear functional equation of the general form using the Stieltjes integral notation

$$\frac{d u(t)}{dt} = \int_{-r_1}^{0} dM_1(\theta) u(t + \theta) + \int_{-r_2}^{0} dM_2(\theta) e(t + \theta), \quad (54)$$

where $M_1$ and $M_2$ are $\ell \times \ell$ and $\ell \times m$ matrix functions of bounded variation on $[-r_1, 0]$ and $[-r_2, 0]$, respectively. This representational form has been widely used in the fields of retarded differential difference equations [5] and general delay systems [9, 12]. Using appropriate $\tilde{V}(t) \in \tilde{T}^{\ell \times \ell}$ and $\tilde{W}(t) \in \tilde{T}^{\ell \times m}$, we can rewrite (54) as

$$\frac{d u(t)}{dt} = \int_{0}^{\infty} \tilde{V}(\tau) u(t - \tau) d\tau + \int_{0}^{\infty} \tilde{W}(\tau) e(t - \tau) d\tau. \quad (55)$$

This general representational form is not suitable for digital redesign based on numerical integration because of the following points:

(a) Even if we apply an appropriate numerical integration rule to the right-hand side of (55), we can only have the approximation of $\left. \frac{d u(t)}{dt} \right|_{t = kT}$, which does not directly lead to a digital controller.

(b) If there exists an element of $\tilde{V}(t)$ belonging to $\tilde{T} \setminus T$, then $\tilde{V}(t)$ includes a finite number of the delta functions, $\delta_{h_1}(t), \delta_{h_2}(t), \ldots, \delta_{h_q}(t)$ ($1 \leq p < \infty$). Then, we need the values $u(kT - h_1), u(kT - h_2), \ldots, u(kT - h_q)$ for the approximation of the right-hand side of (55). If there exists $h_j$ that is not an integer multiple of $T$, we should obtain the approximate value of $u(kT - h_j)$ by appropriate numerical interpolation. The same can also be applied to $\tilde{W}(t)$.

(c) We do not have an established method for representing the continuous-time input-output relation of the infinite-dimensional controller (29) in the general representational form (55).

On the other hand, the finite-interval integral form (39) is well-matched with numerical integration as shown by the following points:

(a') If we apply an appropriate numerical integration rule to the right-hand side of (39), we can have the digital controller (51).

(b') Because $V(t)$ and $W(t)$ in (39) belong to $T^{\ell \times \ell}$ and $T^{\ell \times m}$, respectively, we need only numerical integration to obtain the digital controller (51).

(c') In Step 1, we presented a method for representing the continuous-time input-output relation of the infinite-dimensional controller (29) in the finite-interval integral form (39).
4 Example

Consider a two-input two-output plant

\[ P(s) = \begin{bmatrix} 0 & -e^{-s} \\ 1 & -e^{-s} \end{bmatrix} \begin{bmatrix} s & -e^{-s} \\ -1 & s + e^{-s} \end{bmatrix}^{-1}, \tag{56} \]

which includes a pure delay with a delay time of one. Its characteristic function is equal to \( s^2 + e^{-s}s - e^{-s} \). By finite spectrum assignment, we have a continuous-time infinite-dimensional controller

\[ C(s) = \begin{bmatrix} s + 2 & \Theta_{1,0} e^{-s} + e^{-s} \\ 1 & s - \Theta_{1,0} e^{-s} + \Theta_{1,0} - e^{-s} + 2 \end{bmatrix}^{-1} \times \begin{bmatrix} -e^{-s} - 2 & \Theta_{1,0} e^{-s} + e^{-s} + 1 \\ e^{-s} - 3 & -\Theta_{1,0} e^{-s} + \Theta_{1,0} - e^{-s} + 2 \end{bmatrix}, \tag{57} \]

where the characteristic function of the resulting closed-loop system is equal to \((s + 1)^4\). This satisfies Assumption 3.1 and is the object of the proposed digital redesign.

In Step 1, we have

\[ \begin{bmatrix} \Phi_{L,a} & 0 \\ 0 & \Phi_{L,a} \end{bmatrix} \begin{bmatrix} s + 2 & \Theta_{1,0} e^{-s} + e^{-s} \\ 1 & s - \Theta_{1,0} e^{-s} + \Theta_{1,0} - e^{-s} + 2 \end{bmatrix} = E - \begin{bmatrix} \Psi_{L,a} - 2\Phi_{L,a} \\ -\Phi_{L,a} \end{bmatrix} \begin{bmatrix} -\Phi_{L,a} \Theta_{1,0} e^{-s} - \Phi_{L,a} e^{-s} \\ \Psi_{L,a} + \Phi_{L,a} \Theta_{1,0} e^{-s} - \Phi_{L,a} \Theta_{1,0} + \Phi_{L,a} e^{-s} - 2\Phi_{L,a} \end{bmatrix} \] \tag{58} \]

and

\[ \begin{bmatrix} \Phi_{L,a} & 0 \\ 0 & \Phi_{L,a} \end{bmatrix} \begin{bmatrix} -e^{-s} - 2 & \Theta_{1,0} e^{-s} + e^{-s} + 1 \\ e^{-s} - 3 & -\Theta_{1,0} e^{-s} + \Theta_{1,0} - e^{-s} + 2 \end{bmatrix} = \begin{bmatrix} -\Phi_{L,a} e^{-s} - 2\Phi_{L,a} \\ \Phi_{L,a} e^{-s} - 3\Phi_{L,a} \end{bmatrix} \begin{bmatrix} \Phi_{L,a} \Theta_{1,0} e^{-s} + \Phi_{L,a} e^{-s} + \Phi_{L,a} \\ -\Phi_{L,a} \Theta_{1,0} e^{-s} + \Phi_{L,a} \Theta_{1,0} - \Phi_{L,a} e^{-s} + 2\Phi_{L,a} \end{bmatrix}. \tag{59} \]

Then, the continuous-time input-output relation of the infinite-dimensional controller (57) can be described in finite-interval integral form as follows:

\[ \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \int_0^\infty \begin{bmatrix} v_{11}(\tau) & v_{12}(\tau) \\ v_{21}(\tau) & v_{22}(\tau) \end{bmatrix} \begin{bmatrix} u_1(t - \tau) \\ u_2(t - \tau) \end{bmatrix} d\tau + \int_0^\infty \begin{bmatrix} w_{11}(\tau) & w_{12}(\tau) \\ w_{21}(\tau) & w_{22}(\tau) \end{bmatrix} \begin{bmatrix} e_1(t - \tau) \\ e_2(t - \tau) \end{bmatrix} d\tau \] \tag{60} \]
\[ v_{11} = \psi_{L,a} - 2\phi_{L,a} \]
\[ v_{12} = -\sigma(\phi_{L,a} \ast \theta_{1,0}) - \sigma(\phi_{L,a}) \]
\[ v_{21} = -\phi_{L,a} \]
\[ v_{22} = \psi_{L,a} + \sigma(\phi_{L,a} \ast \theta_{1,0}) - \phi_{L,a} \ast \theta_{1,0} + \sigma(\phi_{L,a}) - 2\phi_{L,a} \]
\[ w_{11} = -\sigma(\phi_{L,a}) - 2\phi_{L,a} \]
\[ w_{12} = \sigma(\phi_{L,a} \ast \theta_{1,0}) + \sigma(\phi_{L,a}) + \phi_{L,a} \]
\[ w_{21} = \sigma(\phi_{L,a}) - 3\phi_{L,a} \]
\[ w_{22} = -\sigma(\phi_{L,a} \ast \theta_{1,0}) + \phi_{L,a} \ast \theta_{1,0} - \sigma(\phi_{L,a}) + 2\phi_{L,a} \]

where, for brevity, we define the following operator:

\[ \sigma : f(t) \mapsto f(t-1), \quad (61) \]

where \( f(t) \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}) \); let \(*\) denote the convolution product.

In Step 2, we set \( L = 1 \) and \( a = -1 \) for \( \phi_{L,a} \) and \( \psi_{L,a} \). We suppose that the sampling period is \( T = 0.1 \). Although \( \bar{u}_V = \bar{u}_W = 0 \) and \( \bar{n}_V = \bar{n}_W = 30 \), by considering each support of \( v_{ij}(t) \) and \( w_{ij}(t) \) and possible jump discontinuities, we use the composite Simpson’s rule for the intervals \([0, 1], (1, 2], (2, 3]\). Because \( \sigma(\phi_{L,a} \ast \theta_{1,0}) \big|_{t=3} = 0 \), we have \( A_{30} = 0 \) and \( B_{30} = 0 \). Then, we have a digital controller

\[ C_d(z) = \left[ E - \sum_{j=0}^{29} z^{-j} A_j \right]^{-1} \left[ \sum_{j=0}^{29} z^{-j} B_j \right]. \quad (62) \]

Figure 2 shows the response in the output \( y_1(t) \) of the plant against a unit-step input in the reference signal \( r_1(t) \) for the resulting sampled-data control system shown in Figure 1 (b), where \( y(t) = [y_1(t) \ y_2(t)] \) and \( r(t) = [r_1(t) \ r_2(t)] \). In the continuous-time control system with the controller (57) shown in Figure 1 (a), the response can be analytically obtained by

\[ y_1(t) = \begin{cases} 
0, & 0 \leq t \leq 1 \\
3 - 3te^{-(t-1)}, & 1 < t \leq 2 \\
2 - 3te^{-(t-1)} + (t-1)e^{-(t-2)}, & t > 2,
\end{cases} \quad (63) \]

which is the theoretical response shown in Figure 2. There is only a small difference between the response of the resulting sampled-data control system and the theoretical response. This implies that the digital controller obtained by the proposed digital redesign can achieve almost the same performance as achieved theoretically by the continuous-time infinite-dimensional controller (57).
5 Conclusions

In this paper, we proposed the digital redesign of infinite-dimensional controllers. It is based on numerical integration of the continuous-time input-output relation of a predesigned controller in finite-interval integral form. This representational form was introduced to describe complicated infinite-dimensional systems. In addition, it is well-matched with numerical integration.

An important future work on the proposed digital redesign is to clarify the relationship that exists between the approximation accuracy of numerical integration and the performance of the resulting sampled-data control systems.

References


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