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On Nonlinear Volterra Integrodifferential Equations with Analytic Semigroups

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Abstract

The aim of the present paper is to establish the existence, uniqueness, continuation and continuous dependence on initial data of solutions of Volterra integrodifferential equations in an arbitrary Banach space by using the theory of analytic semigroup and fractional powers of operators.

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1 Introduction

Let X be a general Banach space with norm $\|.\|$. In the present paper we study nonlinear Volterra integrodifferential equation in a general Banach space X of the type

$$x'(t) + Ax(t) = f\left(t, x(t), \int_{t_0}^{t} k(t,s)x(s)ds\right), \quad t > t_0; \tag{1}$$

$$x(t_0) = x_0 \in X; \tag{2}$$

where $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in X, the kernel $k : [t_0, \infty) \times [t_0, \infty) \to R$, the function $f : [t_0, \infty) \times X \times X \to X$ is nonlinear, $t_0 \geq 0$ and x_0 is a given element of X.

In [1], Bahuguna obtained some sufficient conditions for the existence, uniqueness, regularity and continuation of solutions to integrodifferential equations of the type

$$\begin{aligned} x'(t) + Ax(t) &= g(t, x(t)) + \int_{t_0}^{t} a(t-s)h(s, x(s))ds, \\ x(t_0) &= x_0 \in X, \end{aligned}$$

where -A is the infinitesimal generator of an analytic semigroup, g and $h : [t_0, \infty) \times X \to X$ are nonlinear. For more results on existence, uniqueness and other properties of solutions of these equations (1)-(2) or their special forms by using various techniques, see [2, 5, 6, 7, 10, 11] and we also refer the reader to [3, 4, 8, 12, 13, 14, 15] for applications.

In the present paper, we prove the existence of local mild solution, local classical solution, uniqueness, continuation of solution and continuous dependence on initial data of nonlinear Volterra integrodifferential equations (1)-(2). The main tool employed in our analysis is based on the theory of analytic semigroups, the theory of fractional powers of operators and strict contraction mapping.

The paper is organized as follows: In section 2, we present the preliminaries and statements of our main results. Section 3 deals with the proofs of the main results. Finally, in section 4, we discuss an example to illustrate the theory.

2 Preliminaries and Statements of Results

Before proceeding to the statement of our main results, we setforth preliminaries and hypotheses that will be used in our subsequent discussion.

Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in X and let $J = [t_0, b]$, $0 \leq t_0 < b < \infty$. It is to be noted that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A + \alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. Therefore, we reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup and the generator is invertiable. For convenience, we suppose that $\|T(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$. For $\alpha > 0$ we define the fractional power $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{(\alpha-1)} T(t) dt$$

where $\Gamma(.)$ is the gamma function. Since $A^{-\alpha}$ is one to one, $A^{\alpha} = (A^{-\alpha})^{-1}$. For $0 < \alpha \leq 1$, A^{α} is a closed linear operator whose domain $D(A^{\alpha}) \supset D(A)$ is dense in X. The closedness of A^{α} implies that $D(A^{\alpha})$ endowed with the graph norm $\|x\|_{A^{\alpha}} = \|x\| + \|A^{\alpha}x\|$, $x \in D(A^{\alpha})$ is a Banach space. Since $0 \in \rho(-A)$, A^{α} is invertiable, and its graph norm is equivalent to the norm $\|x\|_{\alpha} = \|A^{\alpha}x\|$. Thus $D(A^{\alpha})$ equipped with the norm $\|.\|_{\alpha}$ is a Banach space which we denote by X_{α}.

Definition 1 *A mild solution of equations (1)-(2) on J is a continuous function $x : J \to X$ satisfying the integral equation*

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)f\left(s, x(s), \int_{t_0}^{s} k(s,\tau)x(\tau)d\tau\right) ds. \quad (3)$$

Definition 2 *A mild solution x of equations (1)-(2) on $J_0 = [t_0, b_0]$, where b_0 is such that $t_0 < b_0 < b$, is called a local mild solution of equations (1)–(2) on J.*

Definition 3 *A classical solution of equations (1)-(2) on J is a function $x \in C(J;X) \cap C^1(J - \{t_0\}; X)$ satisfying (1)-(2) on J.*

Definition 4 *A classical solution x of equations (1)-(2) on $J_0 = [t_0, b_0]$ where b_0 is such that $t_0 < b_0 < b$ is called a local classical solution of equations (1)-(2) on J.*

We need the following results in our subsequent discussion.

Lemma 2.1 *([14], p74) Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$. If $0 \in \rho(A)$ then*

1. *$T(t) : X \to D(A^{\alpha})$ for every $t > 0$ and $\alpha \geq 0$.*
2. *For every $x \in D(A^{\alpha})$ we have $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.*
3. *For every $t > 0$ the operator $A^{\alpha}T(t)$ is bounded and $\|A^{\alpha}T(t)\| \leq C_{\alpha}t^{-\alpha}e^{-\delta t}$.*
4. *Let $0 < \alpha \leq 1$ and $x \in D(A^{\alpha})$ then $\|T(t)x - x\| \leq C_{\alpha}t^{\alpha}\|A^{\alpha}x\|$.*

Lemma 2.2 *([14], p113) Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$. If $f \in L^1(0, b; X)$ is locally Holder continuous on $(0, b]$ then for every $x_0 \in X$ the initial value problem $x'(t) + Ax(t) = f(t)$, $x(0) = x_0$ has a unique solution.*

Lemma 2.3 *([9], p198) Let $v(.), w(.) : [0, b] \to [0, \infty)$ be continuous functions. If $w(.)$ is nondecreasing and there are constants $p > 0,\ 0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + p \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0, b],$$

then

$$v(t) \leq exp\left\{ \frac{[p\Gamma(\alpha)b]^n}{\Gamma(n\alpha)} \right\} \sum_{j=0}^{n-1} \left(\frac{pb^\alpha}{\alpha} \right)^j w(t)$$

for every $t \in [0, b]$ and every $n \in N$ such that $n\alpha > 1$, $\Gamma(.)$ is the gamma function.

For convenience, we list the following hypotheses used in our further discussion.

(H_1) Let U be an open subset of $[0, \infty) \times X_\alpha \times X_\alpha$ and for every $(t, x, y) \in U$ there exist a neighbourhood $V \subset U$ of (t, x, y) and a constant $L_0 > 0$ such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_0(\|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_\alpha) \tag{4}$$

for all (t, x_1, y_1) and $(t, x_2, y_2) \in V$.

(H_2) Let U be an open subset of $[0, \infty) \times X_\alpha \times X_\alpha$ and for every $(t, x, y) \in U$ there exist a neighbourhood $V \subset U$ of (t, x, y) and constants $L_1 > 0,\ 0 < \theta < 1$ such that

$$\begin{aligned} \|f(t_1, x_1, y_1) \ &- \ f(t_2, x_2, y_2)\| \\ &\leq \ L_1(|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_\alpha) \end{aligned} \tag{5}$$

for all (t_1, x_1, y_1) and $(t_1, x_2, y_2) \in V$.

(H_3) There exist constants $L_2 \geq 0$ and $0 < \beta_1, \beta_2 \leq 1$ such that

$$|k(t_1, s_1) - k(t_2, s_2)| \leq L_2(|t_1 - t_2|^{\beta_1} + |s_1 - s_2|^{\beta_2}) \tag{6}$$

for all $t_1, s_1, t_2, s_2 \in J$.

(H_4) For each $t \in J$, $k(t, s)$ is measurable on $[t_0, t]$ and

$$k_1(t) = \text{ess sup}\{|k(t, s)|, t_0 \leq s \leq t\}$$

is bounded on J.

(H_5) There exists a continuous nondecreasing function $p : [0, \infty) \to R^+$ such that

$$\|f(t, x, y)\| \leq p(t)(\|x\|_\alpha + \|y\|_\alpha) \tag{7}$$

for every $t \in [0, \infty)$ and $x, y \in X_\alpha$.

We are now in position to state our main results to be proved in this paper.

Theorem 2.4 *Suppose that the operator* $-A$ *generates the analytic semigroup* $T(t)$ *with* $\|T(t)\| \leq M,\ t \geq 0$ *and* $0 \in \rho(-A)$. *Let the hypotheses* (H_1) *and* (H_4) *be satisfied. Then, for each* $x_0 \in X_\alpha$, *the initial value problem (1)-(2) has a unique local mild solution.*

Remark 1 *We note that for* $A \in BL(X)(BL(X)$*-the Banach space of bounded linear operators on* X*),* $0 \in \rho(A)$ *iff* $0 \in \rho(-A)$.

Theorem 2.5 *Suppose that* $-A$ *generates the analytic semigroup* $T(t)$ *such that* $\|T(t)\| \leq M,\ t \geq 0$ *and* $0 \in \rho(-A)$. *Let the hypotheses* $(H_2) - (H_4)$ *be satisfied. Then, for each* $x_0 \in X_\alpha$, *the initial value problem (1)-(2) has a unique local classical solution.*

Theorem 2.6 *Let* $0 \in D(A)$ *and* $-A$ *be the infinitesimal generator of an analytic semigroup* $T(t)$ *satisfying* $\|T(t)\| \leq M$ *for* $t \geq t_0$. *Let the hypotheses* $(H_2) - (H_5)$ *be satisfied. Then, for each* $x_0 \in X_\alpha$, *the initial value problem (1)-(2) has a unique classical solution on* $[0, \infty)$.

Theorem 2.7 *Suppose that the hypotheses* $(H_2) - (H_5)$ *hold and* $x_0 \in X_\alpha$. *Suppose that the functions* x_1 *and* x_2 *satisfy the equation (1) for* $n \in N$ *and* $0 < \alpha < 1$, $t_0 \leq t \leq b < \infty$ *with* $x_1(t_0) = x_0^*$ *and* $x_2(t_0) = x_0^{**}$ *respectively and* $x_1(t),\ \ x_2(t) \in X_\alpha$ *then*

$$\begin{aligned} \|x_1(t) \ &- \ x_2(t)\|_\alpha \\ &\leq \ \|x_0^* - x_0^{**}\|_\alpha (M + Mb) exp\left\{ \frac{[C_0\Gamma(1-\alpha)b]^n}{\Gamma(n(1-\alpha))} \right\} \sum_{j=0}^{n-1} \left(\frac{C_0 b^{1-\alpha}}{1-\alpha} \right)^j \end{aligned} \tag{8}$$

where $C_0 = \max\{C_\alpha L_1[1 + \frac{b}{1-\alpha}], C_\alpha L_1[1 + \frac{b}{1-\alpha}]k_b\}$.

Remark 2 *The problems of existence, uniqueness and other properties of solutions of equations (1)-(2) with functional arguments are studied by M. B. Dhakne and B. G. Pachpatte [6] by using method of successive approximation, comparison theorems and the integral inequalities established by Pachpatte. Here we note that our method employed to these equations (1)-(2) is different from the method used by authors in [6].*

3 Proofs of Theorems

Proof of Theorem 2.4 : Let U be an open subset of $[0, \infty) \times X_\alpha \times X_\alpha$ and fix a point $(t_0, x_0, 0) \in U$. Choose $t_1^{'} > t_0$ and $\delta_1, \delta_2 > 0$, such that the condition (4) for the function f is satisfied on the set

$$V = \{(t, x, y) \in U : t_0 \leq t \leq t_1^{'}, \quad \|x - x_0\|_\alpha \leq \delta_1, \|y\|_\alpha \leq \delta_2\} \tag{9}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and $B = \sup_{t_0 \le t \le t_1'} \|f(t, x_0, 0)\|$. Choose $t_1 > t_0$ such that

$$\|T(t-t_0) - I\| \, \|A^\alpha x_0\| \le \frac{\delta}{2}, \tag{10}$$

for $t_0 \le t \le t_1$ and

$$t_1 - t_0 < \min\left\{t_1' - t_0, [\frac{\delta(1-\alpha)}{2C_\alpha[B + L_0(\delta + k_b(\delta + \|A^\alpha x_0\|)b)]}]^{\frac{1}{1-\alpha}}\right\} \tag{11}$$

where C_α is a positive constant depending on α such that

$$\|A^\alpha T(t)\| \le C_\alpha t^{-\alpha} \quad for \quad t > t_0 \tag{12}$$

and for $t_0 \le t \le b$

$$k_b = \sup_{t \in J} k_1(t). \tag{13}$$

Let $Y = C([t_0, t_1]; X)$ be the Banach space with supremum norm $\|y\|_Y = \sup_{t_0 \le t \le t_1} \|y(t)\|$. Consider a map F on Y defined by

$$\begin{aligned} Fy(t) &= T(t-t_0)A^\alpha x_0 \\ &+ \int_{t_0}^{t} A^\alpha T(t-s) f(s, A^{-\alpha}y(s), \int_{t_0}^{s} k(s,\tau)A^{-\alpha}y(\tau)d\tau)ds \end{aligned} \tag{14}$$

Clearly for every $y \in Y$, $Fy(t_0) = A^\alpha x_0$ and for $t_0 \le s \le t \le t_1$, we have

$$\begin{aligned} & \|(Fy)(t) - (Fy)(s)\| \\ \le & \|T(t-t_0) - T(s-t_0)\| \|A^\alpha x_0\| \\ + & \int_{t_0}^{s} \|A^\alpha[T(t-\tau) - T(s-\tau)]\| \|f(\tau, A^{-\alpha}y(\tau), \int_{t_0}^{\tau} k(\tau,\sigma)A^{-\alpha}y(\sigma)d\sigma)\| d\tau \\ + & \int_{s}^{t} \|A^\alpha T(t-\tau)\| \|f(\tau, A^{-\alpha}y(\tau), \int_{t_0}^{\tau} k(\tau,\sigma)A^{-\alpha}y(\sigma)d\sigma)\| d\tau \end{aligned} \tag{15}$$

Using hypotheses (H_1), (H_4) and condition (13), we have

$$\begin{aligned} & \|f(t, \ A^{-\alpha}y(t), \int_{t_0}^{t} k(t,s)A^{-\alpha}y(s)ds)\| \\ & \qquad \le L_0(\|y\|_Y + \|A^\alpha x_0\| + k_b b \|y\|_Y) + B := B_1 \end{aligned} \tag{16}$$

By making use of (12) and (16) in (15), we get

$$\|(Fy)(t) - (Fy)(s)\| \to \ 0 \ \text{ as } \ |t-s| \to 0.$$

It follows that $F : Y \to Y$ is continuous.

Consider the nonempty closed and bounded set W defined by

$$W = \{y \in Y : y(t_0) = A^\alpha x_0, \quad \|y(t) - A^\alpha x_0\| \leq \delta\}.$$

Using hypotheses $(H1)$, $(H4)$ and (9)–(13), for $y \in W$ we have

$$\|Fy(t) \quad - \quad A^\alpha x_0\| \leq \delta$$

This yields that $F : W \to W$. Now, we prove that F is a strict contraction on W. From (14) and using hypotheses (H_1) and (H_4), for $y, z \in W$, we have

$$\|(Fy)(t) \quad - \quad (Fz)(t)\| \leq \frac{1}{2}\|y - z\|_Y.$$

Thus F is a strict contraction map from W into W and therefore by the Banach contraction principle there exists a unique $y \in W$ such that $Fy = y$.
Let $x = A^{-\alpha}y$. Then for $t \in [t_0, t_1]$, we have

$$x(t) = A^{-\alpha}y(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)f(s, x(s), \int_{t_0}^{s} k(s, \tau)x(\tau)d\tau)ds.$$

Hence, x is a unique local mild solution of the initial value problem $(1) - (2)$.
Proof of Theorem 2.5 : By Theorem 2.4, there exist b_0, $t_0 < b_0 < b$ and a function x such that x is a unique mild solution of initial value problem (1)–(2) on $J_0 = [t_0, b_0]$ given by equation (3). Define $u(t) = A^\alpha x(t)$. Then, we have

$$\begin{aligned} u(t) &= T(t - t_0)A^\alpha x_0 \\ &+ \int_{t_0}^{t} A^\alpha T(t - s)f(s, A^{-\alpha}u(s), \int_{t_0}^{s} k(s, \tau)A^{-\alpha}u(\tau)d\tau)ds. \end{aligned} \quad (17)$$

Put

$$g(t) = f(t, A^{-\alpha}u(t), \int_{t_0}^{t} k(t, s)A^{-\alpha}u(s)ds) \quad (18)$$

From equations (17) and (18), we get

$$u(t) = T(t - t_0)A^\alpha x_0 + \int_{t_0}^{t} A^\alpha T(t - s)g(s)ds. \quad (19)$$

Using hypotheses $(H_2) - (H_4)$, (13) and continuitity of $x(t)$ on J_0, we get

$$\|g(t) \quad - \quad g(s)\| \to 0 \ \text{ as } \ |t - s| \to 0$$

This shows that g is continuous on J_0 and consequently it is bounded on J_0, i.e. there exist some positive constant N_0 such that $\|g(t)\| \leq N_0$ for $t \in J_0$.

To prove g is locally Holder continuous on J_0 first we prove that u is locally Holder continuous on J_0. By Lemma 2.1, it follows that for every $0 < \beta < 1-\alpha$ and every $0 < h < 1$, we have

$$\|(T(h)-I)A^{\alpha}T(t-t_0)\| \leq h^{\beta}C_{\beta}C_{\alpha+\beta}(t-t_0)^{-(\alpha+\beta)} \leq r_1 h^{\beta} \tag{20}$$

where r_1 depends on t and blows up as t decreases to t_0. Now,

$$\begin{aligned} \|u(t+h)-u(t)\| &\leq \|(T(h)-I)A^{\alpha}T(t-t_0)\|\|x_0\| \\ &+ \int_{t_0}^{t} \|(T(h)-I)A^{\alpha}T(t-s)\|\|g(s)\|ds \\ &+ \int_{t}^{t+h} \|A^{\alpha}T(t+h-s)\|\|g(s)\|ds \end{aligned} \tag{21}$$

Furthermore

$$\begin{aligned} &\int_{t_0}^{t} \|(T(h)-I)A^{\alpha}T(t-s)\|\|g(s)\|ds \\ &\qquad \leq C_{\beta}C_{\alpha+\beta}N_0 h^{\beta}\{1-(\alpha+\beta)\}^{-1}\{(t-t_0)^{1-(\alpha+\beta)}\} \\ &\qquad \leq r_2 h^{\beta} \end{aligned} \tag{22}$$

where r_2 is also depends on t. Also we have

$$\int_{t}^{t+h} \|A^{\alpha}T(t+h-s)\|\|g(s)\|ds \leq C_{\alpha}N_0(1-\alpha)^{-1}h^{1-\alpha} \leq r_3 h^{\beta} \tag{23}$$

where r_3 is independent of t. Using (20), (17) and (23) in (21), we get

$$\|u(t+h)-u(t)\| \leq (r_1\|x_0\| + r_2 + r_3)h^{\beta}$$

It follows that there exists a constant r such that for every $t_0^{'} > t_0$, we have

$$\|u(t)-u(s)\| \leq r\,|t-s|^{\beta}, \tag{24}$$

for all $t_0 < t_0^{'} < t, s < b_0$.

Using the result (24) and hypotheses $(H_2)-(H_5)$, for all s, t such that $t_0 < t_0^{'} < s < t < b_0$, we have

$$\|g(t) \quad - \quad g(s)\| \leq C_1\,|t-s|^{\delta} \tag{25}$$

for some positive constant $C_1 = L_1[1 + r + \|u\|_Y L_2(b_0) + \|u\|_Y k_b(b_0)^{1-\beta_1}]$ and $0 < \delta < 1$.

Consider the following initial value problem

$$\hat{u}'(t) + A\hat{u}(t) = g(t), \quad \hat{u}(t_0) \quad = \quad x_0 \quad t > t_0. \tag{26}$$

By Lemma 2.2, the initial value problem (26) has the unique solution $\hat{u} \in C^1((t_0, b_0]; X)$ given by

$$\hat{u}(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)g(s)ds. \tag{27}$$

For $t > t_0$, each term on the right hand side belongs to $D(A)$ and hence belongs to $D(A^\alpha)$. Applying A^α to both sides of (27) and using the uniqueness of $\hat{u}(t)$, we have $A^\alpha \hat{u}(t) = u(t)$ where $u(t)$ is as in (17). Thus, we have that $A^\alpha \hat{u}(t) = u(t) = A^\alpha x(t)$. Hence $\hat{u}(t) = A^{-\alpha}u(t) = x(t)$. This shows that $x(t)$ is the classical solution of the initial value problem (1)-(2) on J_0.

Proof of Theorem 2.6 : By Theorem 2.5 there exist a $b_0,\ \ t_0 < b_0 < b < \infty$ and a unique classical solution $x(t)$ of the initial value problem (1)-(2) on $J_0 = [t_0, b_0]$. We note that if $\|x(t)\|_\alpha \leq C$ for $t \in J_0$, where C is some positive constant, then the solution $x(t)$ may be continued further on the right of b_0. Therefore it is sufficient to prove that $\|x(t)\|_\alpha$ is bounded as $t \uparrow b$. Since $x(t)$ is a classical and also a mild solution, we have

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)f(s, x(s), \int_{t_0}^{s} k(s,\tau)x(\tau)d\tau)ds \tag{28}$$

Operating A^α on both sides, we have

$$A^\alpha x(t) = A^\alpha T(t-t_0)x_0 + \int_{t_0}^{t} A^\alpha T(t-s)f(s, x(s), \int_{t_0}^{s} k(s,\tau)x(\tau)d\tau)ds \tag{29}$$

Taking norm on both sides of equation (29) and using properties of $T(t)$ and A that they commute, $\|T(t)\| \leq M$, $\|A^\alpha T(t)\| \leq C_\alpha t^{-\alpha}$ for $t > t_0$ and hypotheses (H_4), $\ (H_5)$, we get

$$\|x(t)\|_\alpha \leq M\|A^\alpha x_0\| + C_\alpha p_{_b} \textstyle\int_{t_0}^{t} (t-s)^{-\alpha}(\|x(s)\|_\alpha + k_b \int_{t_0}^{s} \|x(\tau)\|_\alpha d\tau)ds \tag{30}$$

where $p_{_b} = \sup_{t\in J} p(t)$. Integrating (30) from t_0 to t, we have

$$\begin{aligned}\int_{t_0}^{t} \|x(\xi)\|_\alpha d\xi &\leq M\|A^\alpha x_0\|b \\ &+ \frac{C_\alpha p_{_b} b}{1-\alpha}\int_{t_0}^{t}(t-s)^{-\alpha}(\|x(s)\|_\alpha + k_b\int_{t_0}^{s}\|x(\tau)\|_\alpha d\tau)ds \end{aligned} \tag{31}$$

Adding corresponding sides of inequalities (30) and (31), we get

$$\begin{aligned}\|x(t)\|_\alpha &+ \int_{t_0}^{t}\|x(\xi)\|_\alpha d\xi \\ &\leq C_2 + C_3\int_{t_0}^{t}(t-s)^{-\alpha}\left(\|x(s)\|_\alpha + \int_{t_0}^{s}\|x(\tau)\|_\alpha d\tau\right)ds \end{aligned} \tag{32}$$

for some positive constants $C_2 = M\|A^\alpha x_0\|[1+b]$ and $C_3 = \max\{C_\alpha p_{_b}[1+\frac{b}{1-\alpha}], C_\alpha p_{_b}[1+\frac{b}{1-\alpha}]k_b\}$ depending on α and b. Define

$$z(t) = \|x(t)\|_\alpha + \int_{t_0}^{t} \|x(\xi)\|_\alpha d\xi \tag{33}$$

Using (33), the inequality (32) becomes

$$z(t) \leq C_2 + C_3 \int_{t_0}^{t} (t-s)^{-\alpha} z(s)ds. \tag{34}$$

Applying Lemma 2.3 to (34), we obtain

$$z(t) \leq exp\Big\{\frac{[C_3\Gamma(1-\alpha)b]^n}{\Gamma(n(1-\alpha))}\Big\} \sum_{j=0}^{n-1} \left(\frac{C_3 b^{1-\alpha}}{1-\alpha}\right)^j C_2 = C,$$

for every $t \in [t_0, b]$ and every $n \in N$ such that $n(1-\alpha) > 1$.

$$\|x(t)\|_\alpha + \int_{t_0}^{t} \|x(\xi)\|_\alpha d\xi = z(t) \leq C$$

which yields $\|x(t)\|_\alpha \leq C$. This proves the Theorem 2.6.

Proof of Theorem 2.7 : Let the functions $x_1(t)$ and $x_2(t)$ satisfy the equation (1) on $t_0 \leq t \leq b < \infty$ with $x_1(t_0) = x_0^*$ and $x_2(t_0) = x_0^{**}$ respectively. Using hypotheses (H_2), (H_4) and properties of $T(t)$ and A, we have

$$\begin{aligned} \|A^\alpha x_1(t) - A^\alpha x_2(t)\| &\leq M\|x_0^* - x_0^{**}\|_\alpha + \int_{t_0}^{t} C_\alpha (t-s)^{-\alpha} L_1(\|x_1(s) - x_2(s)\|_\alpha \\ &+ \int_{t_0}^{s} k_1(s)\|x_1(\tau) - x_2(\tau)\|_\alpha d\tau)ds \end{aligned}$$

By using the definition of α-norm, we have

$$\begin{aligned} \|x_1(t) - x_2(t)\|_\alpha &\leq M\|x_0^* - x_0^{**}\|_\alpha + C_\alpha L_1 \int_{t_0}^{t} (t-s)^{-\alpha}(\|x_1(s) - x_2(s)\|_\alpha \\ &+ k_b \int_{t_0}^{s} \|x_1(\tau) - x_2(\tau)\|_\alpha d\tau)ds \end{aligned} \tag{35}$$

Define $m(t) = \|x_1(t) - x_2(t)\|_\alpha$. Then from inequality (35), we get

$$m(t) \leq M\|x_0^* - x_0^{**}\|_\alpha + C_\alpha L_1 \int_{t_0}^{t} (t-s)^{-\alpha}(m(s) + k_b \int_{t_0}^{s} m(\tau)d\tau)ds$$

Repeating the same construction as in the proof of Theorem 2.6 we get desired bound as in (8). This proves the Theorem 2.7.

4 Application

Now, we give an example to illustrate the application of our results established in previous section. We consider the following boundary value problem

$$\frac{\partial u(t,\xi)}{\partial t} - \frac{\partial^2 u(t,\xi)}{\partial \xi^2} = F(t, u(t,\xi), \int_0^t k(t,s)u(s,\xi)ds), \quad (36)$$

$$u(0,\xi) = u_0(\xi), \quad 0 \le \xi \le \pi \quad (37)$$

$$u(t,0) = u(t,\pi) = 0, \quad t > 0, \quad (38)$$

where $k : R^+ \times R^+ \to R$ and $F : [0,\infty) \times R \times R \to R$.

Let $X = L^2([0,\pi]; R)$. We define an operator $A : X \to X$ by $Ax = -x''$ with domain $D(A) = \{x \in X : x'' \in X \text{ and } x(0) = x(\pi) = 0\}$. Then the operator A can be expressed as $Ax = \sum_{n=1}^{\infty} n^2 \langle x, x_n \rangle x_n$ where $x_n(t) = (\sqrt{2/\pi})\sin(nt), \quad n = 1, 2, ...$ is the orthogonal set of eigen vectors of A and A is the infinitesimal generator of an analytic semigroup $T(t),\ t \ge 0$ on X and is given by

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, x_n \rangle x_n, \qquad x \in X.$$

Define the function $f : [0,\infty) \times X \times X \to X$ by $f(t,x,y)(u) = F(t, x(u), y(u))$ with this choice of the function, the equations (36)-(38) can be formulated as an abstract Volterra integrodifferential equation in Banach space X of type (1)-(2). Further, for every $x \in X$,

$$A^{-\frac{1}{2}} x = \sum_{n=1}^{\infty} 1/n \langle x, x_n \rangle x_n$$

with $\|A^{-\frac{1}{2}}\| = 1$ and the operator $A^{\frac{1}{2}}$ is given by $A^{\frac{1}{2}} x = \sum_{n=1}^{\infty} n \langle x, x_n \rangle x_n$ on the space $D(A^{\frac{1}{2}}) = \{x \in X : \sum_{n=1}^{\infty} n \langle x, x_n \rangle x_n \in X\}$. Let X_α denote the space $D(A^\alpha)$ with $\alpha = 1/2$. Under the assumptions that hypotheses $(H_1)-(H_5)$ are satisfied then by Theorems 2.4 - 2.6 there exists a unique global classical solution of the equation (1)-(2) which guarantees the existence of a unique global classical solution of initial value problem (36)-(38).

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