The First Integral Method and its Applications to Nonlinear Equations

M. F. El - Sabbagh\textsuperscript{a} and S. I. El - Ganaini\textsuperscript{b}

\textsuperscript{a} Mathematics Department, Faculty of Science, Minia University, Egypt
\textsuperscript{b} Mathematics Department, Faculty of Science, Damanhour University, Egypt

Faculty of Science at Dawadmi, Beside Abu Zeid Hospital, Dawadmi 11911, Box No. 1040, Saudi Arabia

Abstract

In this paper, the first integral method is used to construct exact solutions of the Sharma-Tasso-Olver equation and the (2+1)-dimensional Konopelchenko-Dubrovsky equation. The first integral method is an efficient method for obtaining exact solutions of nonlinear equations. The power of this manageable method is shown as conjectured.

Mathematics Subjects Classification: 58, 35, 53

Keywords: First integral method; Sharma-Tasso-Olver (STO) equation;(2+1)-dimensional Konopelchenko-Dubrovsky (KD) equation

1. Introduction

In recent years, the investigation of exact solutions to nonlinear equations has played an important role in nonlinear phenomena. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics and fluid dynamics. In order to better understand these nonlinear phenomena, many mathematicians and physical scientists make efforts to seek more exact solutions to them. Several powerful methods have been proposed to obtain exact solutions of
nonlinear equations, such as inverse scattering method[1], Backlund transformation method [2], Hirota direct method[3,4], tanh-sech method [5-8], extended tanh method [9-12], hyperbolic function method [13], sine-cosine method [14-16], homogeneous balance method [17-20], Jacobi elliptic function expansion method [21], F-expansion method [22], the transformed rational function method[23], exp function method [24-26] and etc...

The first integral method was first proposed by Feng in [27] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method has been widely used by many such as in [28-31] and by the references therein. Raslan [29] proposed the first integral method to solve the Fisher equation. Abbasbandy and Shirzadi[30] have been solved the modified Benjamin-Bona-Mahony equation by the first integral method. Tascan et al., [31] have been obtained some exact solutions for the modified Zakharov-Kuznetsov equation and the ZK-MEW equation using the first integral method. Therefore, our aim in this paper is the use of the first integral method to obtain some explicit exact solutions to the Sharma-Tasso-Olver equation and the (2+1) - dimensional Konopelchenko-Dubrovsky equation.

2. The first integral method

Consider a general nonlinear partial differential equation in the form

\[ P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0, \]  

where \( u = u(x,t) \) is the solution of the nonlinear partial differential equation (1). We use the transformation

\[ u(x,t) = u(\xi), \]  

where \( \xi = x - ct + \zeta \), where \( \zeta \) is an arbitrary constant. This enables us to use the following changes:

\[ \frac{\partial}{\partial t}(\bullet) = -c \frac{\partial}{\partial \xi}(\bullet), \quad \frac{\partial}{\partial x}(\bullet) = \frac{\partial}{\partial \xi}(\bullet), \quad \frac{\partial^2}{\partial x^2}(\bullet) = \frac{\partial^2}{\partial \xi^2}(\bullet), \ldots \]

Using Eq.(3) to transfer the nonlinear partial differential equation (1) to nonlinear ordinary differential equation

\[ G(u(\xi), \frac{\partial u(\xi)}{\partial \xi}, \frac{\partial^2 u(\xi)}{\partial \xi^2}, \ldots) = 0. \]

Next, we introduce new independent variables

\[ X(\xi) = u(\xi), \quad Y(\xi) = \frac{\partial u(\xi)}{\partial \xi} \]

which leads to a system of nonlinear ordinary differential equations
The first integral method

\[
\begin{align*}
\frac{\partial X(\xi)}{\partial \xi} &= Y(\xi) , \\
\frac{\partial Y(\xi)}{\partial \xi} &= F ( X(\xi), Y(\xi) )
\end{align*}
\]

(6)

According to the qualitative theory of ordinary differential equations\[32\] , if we can find two first integrals to system (6) under the same conditions , then analytic solutions to Eq. (6) can be solved directly . However , in general , it is difficult to realize this even for one first integral , because for a given plane autonomous system , there is no systematic theory that can tell us how to find its first integrals , nor is there a logical way for telling us what these first integrals are?.

We will apply the Division Theorem to obtain one first integral to Eq.(6) which reduces Eq.(4) to a first order integrable ordinary differential equation . An exact solution to Eq.(1) is then obtained by solving this equation . For convenience , first let us recall the Division Theorem :

Suppose that \( P(w,z) , Q(w,z) \) are polynomials in \( C[w,z] \) and \( P(w,z) \) is irreducible in \( C[w,z] \) . If \( Q(w,z) \) vanishes at all zero points of \( P(w,z) \) , then there exists a polynomial \( G[w,z] \) in \( C[w,z] \) such that \( Q(w,z) = P(w,z) G(w,z) \).

3. The Sharma – Tasso – Olver (STO) equation

Consider the Sharma – Tasso – Olver equation \[33\]

\[
\begin{align*}
\alpha ( u^3)_x + \frac{3}{2} \alpha ( u^2 )_{xx} + \alpha u_{xxx} &= 0 ,
\end{align*}
\]

(7)

which is a prominent double nonlinear dispersive model and comprises of the linear dispersive term \( \alpha u_{xxx} \) and the double nonlinear terms \( \alpha ( u^3)_x \) and \( \frac{3}{2} \alpha ( u^2 )_{xx} \) . Although , so many methods are successfully applied to obtain exact solutions of the STO equation , some solutions could be still lost \[34\] and the references therein . Our aim here is to obtain some exact solutions of the STO equation using the first integral method .

We apply the first integral method presented above on the STO equation

\[
\begin{align*}
-c u + \alpha u^3 + 3 \alpha u u' + \alpha u'' &= 0 ,
\end{align*}
\]

(8)

obtained upon using the wave variable \( \xi = x - ct + \xi \) and integrating once, where prime denotes the derivative with respect to the same variable \( \xi \).

Next, we introduce new independent variables \( X = u \), \( Y = u_\xi \), which change (8) to a system of ordinary differential equations
According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (9) and (10), and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i = 0 ,$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi)) Y^i(\xi) = 0 ,$$

where $a_i(X)$, ($i = 0, 1, 2, ..., m$) are polynomials of $X$ and $a_m(X) \neq 0$.

Eq.(11) is called the first integral to (9),(10). Due to the Division Theorem, there exists a polynomial $h(X) + g(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX} \frac{dX}{d\xi} + \frac{dQ}{dY} \frac{dY}{d\xi} ,$$

$$= (h(X) + g(X)Y) \left( \sum_{i=0}^{m} a_i(X) Y^i \right) ,$$

(12)

Here, we take two different cases, by assuming that $m = 1$ and $m = 2$ in Eq.(11).

**Case I**: Suppose that $m = 1$, by equating the coefficients of $Y^i$ ($i = 2, 1, 0$) on both sides of (12), we have

$$a'_1(X) = g(X) a_1(X) ,$$

(13)

$$a'_0(X) - 3X a_1(X) = h(X) a_1(X) + g(X) a_0(X) ,$$

(14)

$$a_1(X) \left[ - X^3 + \left( \frac{c}{\alpha} \right) X \right] = h(X) a_0(X) .$$

(15)

Since, $a_i(X)$ ($i = 0, 1$) are polynomials, then from(13) we conclude that $a_1(X)$ is a constant and $g(X) = 0$. For simplicity, we take $a_1(X) = 1$, and balancing the degrees of $h(X)$ and $a_0(X)$ we conclude that deg($h(X)$) = 1, only.

Now suppose that $h(X) = AX + B$, and $A \neq 0$, then we find $a_0(X)$

$$a_0(X) = \left( \frac{A+3}{2} \right) X^2 + B X + D ,$$

(16)

where $D$ is an arbitrary integration constant.
Substituting $a_0(X)$, $a_1(X)$ and $h(X)$ in Eq. (15), and setting all the coefficients of powers $X$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

\[ c = -2\, D\, \alpha, \quad A = -2, \quad B = 0, \quad (17) \]
\[ c = 0, \quad A = -2, \quad D = 0, \quad B = 0, \quad (18) \]
\[ c = 9\, \alpha, \quad D = 0, \quad A = -1, \quad B = 3. \quad (19) \]

Setting (17) in (11), we obtain

\[ Y(\xi) = -\frac{1}{2} X^2(\xi) - D. \quad (20) \]

Combining (20) with (9), we obtain the exact solution to equation (8) and then an exact solution to the Sharma-Tasso-Olver equation can be written as

\[ u_1(x,t) = \sqrt{2} \, \sqrt{D} \, \text{Tan} \left( \frac{1}{2} \left( -\sqrt{2} \, \sqrt{D} \left( x + 2\, D\, \alpha \, t + \xi \right) + 2 \sqrt{2} \, \sqrt{D} \, \xi_0 \right) \right), \quad (21) \]

where $\xi_0$ is an arbitrary constant.

Similarly, in the case of (18), from (11), we obtain

\[ Y(\xi) = -\frac{1}{2} X^2(\xi), \quad (22) \]

Combining (22) with (9), we obtain the exact solution to (8) and then an exact solution to the Sharma-Tasso-Olver equation can be written as

\[ u_2(x,t) = \frac{2}{x + \xi - \frac{2}{2} \xi_0}, \quad (23) \]

Also, in the case of (19), from (11), we obtain

\[ Y(\xi) = -X^3(\xi) - 3X(\xi), \quad (24) \]

Combining (24) with (9), we obtain the exact solution to (8) and an exact solution to the Sharma-Tasso-Olver equation can be written as

\[ u_3(x,t) = \frac{3}{-1 + \exp \left[ 3 \left( x - 9\, \alpha \, t + \xi - \xi_0 \right) \right]}, \quad (25) \]

**Case II**: Suppose that $m = 2$,

by equating the coefficients of $Y^i$ ($i = 3$, 2, 1, 0) on both sides of (12), we have
\[ a'_{2}(X) = g(X) a_{2}(X), \]  \hspace{1cm} (26) \\
\[ a'_{1}(X) - 6X a_{2}(X) = h(X) a_{2}(X) + g(X) a_{1}(X), \]  \hspace{1cm} (27) \\
\[ a'_{0}(X) - 3X a_{1}(X) + 2 a_{2}(X) [-X^3 + \left( \frac{c}{\alpha} \right) X] = h(X) a_{1}(X) + g(X) a_{0}(X), \]  \hspace{1cm} (28) \\
\[ a_{1}(X) [-X^3 + \left( \frac{c}{\alpha} \right) X] = h(X) a_{0}(X). \]  \hspace{1cm} (29)

Since, \( a_{i}(X) \) \((i = 0, 1, 2)\) are polynomials, then from (26) we deduce that \( a_{2}(X) \) is a constant and \( g(X) = 0 \). For simplicity, we take \( a_{2}(X) = 1 \), and balancing the degrees of \( h(X) \) and \( a_{0}(X) \) we conclude that \( \deg(h(X)) = 1 \), only. In this case, we assume that \( h(X) = AX + B \), and \( A \neq 0 \), then we find \( a_{1}(X) \) and \( a_{0}(X) \) as follows
\[ a_{1}(X) = \left( \frac{A}{2} + \frac{6}{2} \right) X^2 + BX + D, \]  \hspace{1cm} (30)
\[ a_{0}(X) = \left( \frac{(A + 6) (A + 3)}{2} \right) X^4 + \frac{1}{2} B (A + 4) X^3 + \left( \frac{B^2 + D(A + 3)}{2} \right) X^2 + BDX + F, \]  \hspace{1cm} (31)

where \( D \) and \( F \) is an arbitrary integration constants.

Substituting \( a_{0}(X), a_{1}(X), a_{2}(X) \) and \( h(X) \) in Eq. (29), and setting all the coefficients of powers \( X \) to be zero, then we obtain a system of nonlinear algebraic equations and by solving it using Mathematica 8, we get
\[ A = -2, \quad F = 0, \quad D = 0, \quad B = 0. \]  \hspace{1cm} (32)
\[ c = 0, \quad A = -2, \quad F = 0, \quad B = 0. \]  \hspace{1cm} (33)
\[ c = \frac{B^2 \alpha}{4}, \quad F = 0, \quad A = -2, \quad D = 0. \]  \hspace{1cm} (34)
\[ c = B^2 \alpha, \quad F = 0, \quad A = -3, \quad D = -\frac{B^2}{2}. \]  \hspace{1cm} (35)
\[ c = -\frac{D \alpha}{3}, \quad F = \frac{D^2}{6}, \quad A = -2, \quad B = 0. \]  \hspace{1cm} (36)

Using the conditions (32) in (11), we obtain
The first integral method

\[ Y_1(\xi) = -\frac{\sqrt{c} \; X(\xi) - X^2(\xi)}{\sqrt{\alpha}}. \]  
(37)

\[ Y_2(\xi) = \frac{\sqrt{c} \; X(\xi) - X^2(\xi)}{\sqrt{\alpha}}. \]  
(38)

Combining (37) and (38) with (9), then we obtain the exact solution to Eq.(8) and exact solutions to the Sharma-Tasso-Olver equation can be written as

\[ u(x,t) = \frac{\sqrt{c}}{\exp \left( \sqrt{c} \left( \frac{x - c \; t + \xi}{\sqrt{\alpha}} - \xi_0 \right) \right) - \sqrt{\alpha}}, \]  
(39)

\[ u(x,t) = -\frac{\sqrt{c}}{-\exp \left( \sqrt{c} \left( - \frac{x - c \; t + \xi}{\sqrt{\alpha}} + \xi_0 \right) \right) + \sqrt{\alpha}}, \]  
(40)

Similarly, in the case of (33), from (11), we have

\[ Y(\xi) = -\frac{D}{2} - X^2(\xi) + \frac{1}{2} \sqrt{D(D + 2 X^2(\xi))}, \]  
(41)

and then exact solutions of the Sharma-Tasso – Olver equation can be written as

\[ u_4(x,t) = \frac{2 D \left( x + \xi - 2 \xi_0 \right)}{-2 + D \left( x + \xi - 2 \xi_0 \right)^2}, \]  
(42)

\[ u_5(x,t) = \frac{2 D \left( x + \xi + 2 \xi_0 \right)}{-2 + D \left( x + \xi + 2 \xi_0 \right)^2}, \]  
(43)

Similarly, for Eq.(34), we have

\[ Y(\xi) = \frac{1}{2} (-B \; X(\xi) - 2 \; X^2(\xi)) \]  
(44)

and then an exact solution of the Sharma-Tasso – Olver equation can be written as

\[ u_6(x,t) = \frac{B \exp[B \xi_0]}{\exp[B \left( x - \frac{B^2 \alpha}{4} \; t + \xi \right)] - 2 \exp[\xi_0]} \]  
(45)

For Eq.(35), we obtain

\[ Y(\xi) = \frac{1}{2} (-B \; X(\xi) - 2 \; X^2(\xi)) \]  
(46)
and then an exact solution of the Sharma-Tasso–Olver equation can be written as
\[ u_7(x,t) = B \tanh \left[ \frac{B}{2} \left( x - B^2 \alpha t + \varsigma + 2 \xi_0 \right) \right]. \] (47)

Similarly, for Eq.(36) , we have
\[ Y(\xi) = \frac{1}{6} \left\{ -3D - 6X^2(\xi) \mp \sqrt{3} \sqrt{D^2 + 2DX^2(\xi)} \right\} \] (48)

and then exact solutions of the Sharma-Tasso–Olver equation can be written as
\[ u_8(x,t) = -\frac{\sqrt{D}}{1 + \sqrt{3}} \left( 1 + \sqrt{3} \cos \left( \frac{\sqrt{D}}{\sqrt{3}} \left( x + \frac{D}{3} \alpha t + \varsigma - 6 \xi_0 \right) \right) \right), \] (49)

\[ u_9(x,t) = -\frac{\sqrt{D}}{1 + \sqrt{3}} \left( 1 + \sqrt{3} \cos \left( \frac{\sqrt{D}}{\sqrt{3}} \left( x + \frac{D}{3} \alpha t + \varsigma - 6 \xi_0 \right) \right) \right), \] (50)

\[ u_{10}(x,t) = -\frac{\sqrt{D}}{1 + \sqrt{3}} \left( 1 + \sqrt{3} \cos \left( \frac{\sqrt{D}}{\sqrt{3}} \left( x + \frac{D}{3} \alpha t + \varsigma + 6 \xi_0 \right) \right) \right), \] (51)
The first integral method

\[
\begin{align*}
\sqrt{D} \sin \left( \frac{\sqrt{D} \left( x + \frac{D}{3} t + \xi + 6 \xi_0 \right)}{\sqrt{3}} \right)
= - \frac{u_{11}(x,t)}{1 + \sqrt{3} \cos \left( \frac{\sqrt{D} \left( x + \frac{D}{3} t + \xi + 6 \xi_0 \right)}{\sqrt{3}} \right)}.
\end{align*}
\] (52)

where \( \xi_0 \) is an arbitrary constant.

These solutions are all new exact solutions.

4. The (2+1)-dimensional Konopelchenko-Dubrovsky (KD) equation

The (2+1)-dimensional Konopelchenko-Dubrovsky (KD) equation presented by Konopelchenko and Dubrovsky in [35]

\[
u_t - u_{xxx} - 6b u_x u + \frac{3}{2} a^2 u_x u - 3 v_y + 3 a u_x v = 0,
\]

\[
\begin{align*}
u_y &= v_x,
\end{align*}
\] (53)

where \( u = u(x,y,t) \) is a sufficiently often differentiable function, \( a \) and \( b \) are arbitrary parameters.

Introducing the wave variable \( \xi = x + y - ct + \zeta \) and proceeding as before, we find

\[- cu' - u'' - 6buu' + \frac{3}{2} a^2 u_x^2 u' - 3v' + 3au'v = 0,\]

\[
\begin{align*}
u' &= v'.
\end{align*}
\] (54)

Integrating the last equation gives \( u = v \) where the constants of integration are considered to be zeros. The first equation in (54) becomes

\[- (c + 3)u' - u'' + 3(a - 2b)uu' + \frac{3}{2} a^2 u_x^2 u' = 0,\] (55)

Now by integrating both sides once, we obtain

\[- (c + 3) u - u'' + \frac{3}{2} (a - 2b) u_x^2 + \frac{1}{2} a^2 u_x^3 = 0.\] (56)

Using (5) and (6), we get
\[
\begin{aligned}
X' (\xi) & = Y (\xi) \\
Y' (\xi) & = \frac{a^2}{2} X^3 (\xi) + \frac{3}{2} (a - 2b) X^2 (\xi) - (c + 3) X (\xi)
\end{aligned}
\]

(57) \quad (58)

According to the first integral method, we suppose that \( X(\xi) \) and \( Y(\xi) \) are the nontrivial solutions of (57) and (58) and

\[
Q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i,
\]

is an irreducible polynomial in the complex domain \( C[X,Y] \) such that

\[
Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi)) Y'(\xi) = 0,
\]

(59)

where \( a_i(X), \ (i = 0,1,2,\ldots,m) \) are polynomials of \( X \) and \( a_m(X) \neq 0 \).

Eq.(59) is called the first integral to (57) and (58). Due to the Division Theorem, there exists a polynomial \( h(X) + g(X) Y \) in the complex domain \( C[X,Y] \) such that

\[
\frac{dQ}{d\xi} = \frac{dQ}{dX} \frac{dX}{d\xi} + \frac{dQ}{dY} \frac{dY}{d\xi},
\]

\[
= [h(X) + g(X) Y] \left\{ \sum_{i=0}^{m} a_i(X) Y^i \right\}.
\]

(60)

Suppose that \( m = 1 \), by equating the coefficients of \( Y^i (i = 2, 1, 0) \) on both sides of Eq. (60), we have

\[
\begin{aligned}
a'_1(X) & = g(X) a_1(X), \\
a'_0(X) & = h(X) a_1(X) + g(X) a_0(X), \\
a_1(X) \left[ \frac{a^2}{2} X^3 + \frac{3}{2} (a - 2b) X^2 - (c + 3) X \right] & = h(X) a_0(X).
\end{aligned}
\]

(61) \quad (62) \quad (63)

Since, \( a_i(X), \ (i = 0,1) \) are polynomials, then from (61) we deduce that \( a_1(X) \) is a constant and \( g(X) = 0 \). For simplicity, we take \( a_1(X) = 1 \), and balancing the degrees of \( h(X) \) and \( a_0(X) \), we conclude that \( \deg(h(X)) = 1 \), only.

Now suppose that \( h(X) = A X + B \), and \( A \neq 0 \), then we find \( a_0(X) \)

\[
a_0(X) = \left( \frac{A}{2} \right) X^2 + B X + D.
\]

(64)

where \( D \) is an arbitrary integration constant.
Substituting \( a_0(X), a_1(X) \) and \( h(X) \) to Eq.(63), and setting all the coefficients of powers \( X \) to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

\[
c = - \frac{4(a^2 - ab + b^2)}{a^2}, \quad D = 0, \quad A = -a, \quad B = -\frac{a + 2b}{a}, \quad (65)
\]

\[
c = - \frac{4(a^2 - ab + b^2)}{a^2}, \quad D = 0, \quad A = a, \quad B = \frac{a - 2b}{a}, \quad (66)
\]

Using the condition (65) in (59), we obtain

\[
Y(\xi) = \left( \frac{a}{2} \right) X^2(\xi) - \left( \frac{a + 2b}{a} \right) X(\xi).
\]

Combining (67) with (57), then we obtain the exact solution to Eq.(56) and then an exact solution to the \((2+1)\)-dimensional Konopelchenko-Dubrovsky equation can be written as

\[
u_1(x, y, t) = \frac{2(a - 2b) \exp\left[ x + y + \frac{4(a^2 - ab + b^2)}{a^2} t + \xi + 2a\xi_0 \right]}{a^2}
\]

\[-a^2 \exp\left[ x + y + \frac{4(a^2 - ab + b^2)}{a^2} t + \xi + 2a\xi_0 \right] + \exp\left[ -\frac{a^2}{a} \right] (68)
\]

Similarly, in the case of (66), from (59) we obtain

\[
Y(\xi) = \left( -\frac{a}{2} \right) X^2(\xi) - \left( \frac{a - 2b}{a} \right) X(\xi).
\]

and then another exact solution to the \((2+1)\)-dimensional Konopelchenko-Dubrovsky equation can be written as

\[
u_2(x, y, t) = \frac{2(a - 2b)}{a^2} \left( a - 2b \right) \left( x + y + \frac{4(a^2 - ab + b^2)}{a^2} t + \xi - 2a\xi_0 \right), \quad (70)
\]

where \( \xi_0 \) is an arbitrary constant.

These solutions are all new exact solutions.

Notice that the results in this paper are based on the assumption of \( m = 1,2 \) for the considered equations. For the cases of \( m = 3,4 \) for these equations, the discussions become more complicated and involves the irregular singular point theory and the
elliptic integrals of the second kind and hyperelliptic integrals. Some solutions in the functional form can not be expressed explicitly. One does not need to consider the cases \( m \geq 5 \) because it is well known that an algebraic equation with the degree greater than or equal to 5 is generally not solvable.

5. Conclusion

In the present work, we have succeeded in implementing and applying the first integral method for finding new explicit exact solutions to the Sharma-Tasso-Olver equation as well as new explicit exact solutions for the (2+1)-dimensional Konopelchenko-Dubrovsky equation. These new solutions may be important for the explanations of some practical physical problems. The first integral method described herein is not only efficient but also has the merit of being widely applicable. Therefore, this method can applied to other nonlinear evolution equations and this will be done elsewhere.

References


