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LS_r-valued Gauss Maps and Spacelike Surfaces of Revolution in $\mathbb{R}^4_1$

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Abstract

To study spacelike surfaces in the Lorentz-Minkowski space $\mathbb{R}^4_1$, we construct a pair of maps, called $\mathfrak{l}^{\pm}_r$-Gauss maps, whose values are in the lightcone. We can use these maps to study umbilical spacelike surfaces and find parametrizations of spacelike surfaces of revolution of hyperbolic and elliptic types in some particular cases.

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1 Introduction

It is well known that Gauss maps have been used as one of the most useful tools to study the behavior and geometric invariants of hypersurfaces. The Gauss maps associated with an arbitrary normal field ν are particularly important for the study of surfaces with codimension larger than one. This method provides a convenient way to study invariants and characters of surfaces that are dependent or independent on ν. For instance, Izumiya et al. [6] used the Gauss map associated with a normal field to study ν-umbilicity for spacelike surfaces of codimension two in the Lorentz-Minkowski space. Marek Kossowski [10], on the other hand, introduced two S^2-valued Gauss maps, whose values are in the lightcone, in order to study spacelike 2-surfaces in $\mathbb{R}^4_1$, and the method was continued by Izumiya et al. [4] to study properties of spacelike surfaces of codimension two. Introducing the HS_r-valued Gauss maps, called $\mathbf{n}^{\pm}_r$-Gauss

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maps, Cuong and Hieu [1] has studied the umbilical spacelike surfaces of codimension two in $\mathbb{R}^{n+1}_1$. By a similar way, it is possible to introduce LS_r-valued Gauss maps, called $\mathfrak{l}^{\pm}_r$ -Gauss maps, for the study of spacelike surfaces of codimension two.

As an application of the $\mathfrak{l}^{\pm}_r$-Gauss maps, in this paper we are going to use LS_r-valued Gauss maps in order to study umbilcal spacelike surfaces of codimension two, especially when the surfaces lie in de Sitter. We are able to calculate the curvatures of spacelike surfaces of revolution of hyperbolic and elliptic types by using $\mathfrak{l}^{\pm}_r$-Gauss maps. We then specifically describe the totally umbilical and maximal spacelike surfaces of revolution and give explicitly their parametrizations. The proofs of these results are mainly based on two useful lemmas that involve solving differential equations. From now on we assume that the surface M is a spacelike surface of codimension two.

2 Preliminary Notes

2.1 The Lorentz-Minkowski Space $\mathbb{R}^4_1$

The Lorentz-Minkowski space $\mathbb{R}^4_1$ is the 4-dimensional vector space

$$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbb{R}, i = 1, \dots, 4\},$$

endowed with the pseudo scalar product defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{3} x_i y_i - x_4 y_4,$$

where $\mathbf{x} = (x_1, \dots, x_4), \mathbf{y} = (y_1, \dots y_4) \in \mathbb{R}^4$. Since $\langle \cdot, \cdot \rangle$ is non-positively defined, $\langle \mathbf{x}, \mathbf{x} \rangle$ may receive negative value, , for a given $\mathbf{x} \in \mathbb{R}^4_1$. We say that a nonzero vector $\mathbf{x} \in \mathbb{R}^4_1$ is spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4_1$ are said to be pseudo-orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

The norm of a vector $\mathbf{x} \in \mathbb{R}^4_1$ is defined as follows

$$\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}.$$

For a nonzero vector $\mathbf{n} \in \mathbb{R}^4_1$ and a contant $c \in \mathbb{R}$, the hyperplane with pseudo normal $\mathbf{n}$ is defined by

$$HP(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}^4_1 : \langle \mathbf{x}, \mathbf{n} \rangle = c\}.$$

The hyperplane $HP(\mathbf{n}, c)$ is called spacelike, lightlike or timelike if $\mathbf{n}$ is timelike, lightlike or spacelike, respectively.

For a vector $\mathbf{a} \in \mathbb{R}^4_1$ and a positive constant R, we define:

1. The Hyperbolic with center $\mathbf{a}$ and radius R by
$$H^3(\mathbf{a}, R) = \{\mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = -R\}.$$
2. The de Sitter with center $\mathbf{a}$ and radius R by
$$S^3_1(\mathbf{a}, R) = \{\mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = R\}.$$
3. The Lightcone with vertex $\mathbf{a}$ by
$$LC^3(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^4_1 : \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\},$$
if $\mathbf{a} = 0$ we have Lightcone LC^3.

2.2 $\mathfrak{l}^{\pm}_r$-Gauss Maps

In this section, we will introduce the Gauss maps of a surface and its curvature.

Let M be a spacelike surface with parametrization $\mathbf{X} : U \to \mathbb{R}^4_1$, where $U \subset \mathbb{R}^2$ is open and $M = \mathbf{X}(U)$ is identified with $\mathbf{X}$. The normal plane of M at $p \in M$, denoted by N_pM, can be viewed as a timelike 2-plane passing the origin. It is easy to see that N_pM intersects with the lightcone LC^3 by two lines. Fix $r > 0$, the hyperplane $\{x_4 = r\}$ meets these lines exactly at two points, denoted by $\mathfrak{l}^{\pm}_r(p)$.

The maps
$$\begin{array}{rcl} \mathfrak{l}^{\pm}_r : M & \to & LS_r := LC^3 \cap \{x_4 = r\}, \\ p & \mapsto & \mathfrak{l}^{\pm}_r(p) \end{array}$$
are called $\mathfrak{l}^{\pm}_r$-Gauss maps.

Algebraically, $\mathfrak{l}^{\pm}_r$ are the solutions of the following system of equations
$$\begin{cases} \langle \mathfrak{l}, \mathbf{X}_{u_i} \rangle = 0, \ i = 1, 2 \\ \langle \mathfrak{l}, \mathfrak{l} \rangle = 0, \\ \mathfrak{l}_4 = r. \end{cases} \tag{1}$$

It is easy to show that $\mathfrak{l}^{\pm}_r$ are smooth vector fields, see [1]. Therefore, using the symbol " * " instead of " + " or " - ", we have the following notions and facts.

The derivative of $\mathfrak{l}^*_r$ at p,
$$d\mathfrak{l}^*_r(p) \ : \ T_pM \to T_{\mathfrak{l}^*_r(p)}LC^3 \subset T_pM \oplus N_pM,$$
can be descomposed as
$$d\mathfrak{l}^*_r(p) = d\mathfrak{l}^{*T}_r(p) + d\mathfrak{l}^{*N}_r(p),$$
where $d\mathfrak{l}^{*T}_r$ and $d\mathfrak{l}^{*N}_r$ are the tangent and normal components of $d\mathfrak{l}^*_r$, respectively.

So, we define:

1. The $\mathfrak{l}_r^*$-*Weingarten map* of M at p by
$$A_p^{\mathfrak{l}_r^*} := -d\mathfrak{l}_r^{*T}(p).$$

2. The $\mathfrak{l}_r^*$-Gauss-Kronecker curvature of M at p by
$$K_p^{\mathfrak{l}_r^*} := \det(A_p^{\mathfrak{l}_r^*}).$$

3. The $\mathfrak{l}_r^*$-mean curvature of M at p by
$$H_p^{\mathfrak{l}_r^*} := \frac{1}{2}\mathrm{trace}(A_p^{\mathfrak{l}_r^*}).$$

4. The $\mathfrak{l}_r^*$-principal curvatures of M at p by the eigenvalues of $A_p^{\mathfrak{l}_r^*}$,
$$k_1^{\mathfrak{l}_r^*}(p), k_2^{\mathfrak{l}_r^*}(p).$$

It is clear that
$$K_p^{\mathfrak{l}_r^*} = k_1^{\mathfrak{l}_r^*}(p)k_2^{\mathfrak{l}_r^*}(p),$$
and
$$H_p^{\mathfrak{l}_r^*} = \frac{1}{2}(k_1^{\mathfrak{l}_r^*}(p) + k_2^{\mathfrak{l}_r^*}(p)).$$

Furthermore, we have some well-known facts:

1. The $\mathfrak{l}_r^*$-Weingarten map is self-adjoint.

2. The $\mathfrak{l}_r^*$-principal curvatures $k_1^{\mathfrak{l}_r^*}(p)$ and $k_2^{\mathfrak{l}_r^*}(p)$ of M at p are the solutions of the following equation
$$\det(b_{ij}^{\mathfrak{l}_r^*}(p) - kg_{ij}(p)) = 0, \tag{2}$$
where $b_{ij}^{\mathfrak{l}_r^*}(p) := \langle \mathbf{X}_{u_iu_j}(p), \mathfrak{l}_r^*(p)\rangle,\ i,j = 1,2,$ are the coefficients of the $\mathfrak{l}_r^*$-second fundamental form of M at p.

3. $K_p^{\mathfrak{l}_r^*} = \det(b_{ij}^{\mathfrak{l}_r^*}(p)).\det(g_{ij}(p))^{-1}$.

Now, let $\{e_3, e_4\}$ be an orthonormal frame of the normal bundle of M, and let A^{e_3}, A^{e_4} be the shape operators associated to e_3, e_4, respectively, where $\langle e_3, e_3\rangle = 1, \langle e_4, e_4\rangle = -1$. The mean curvature vector H is given by
$$H = \frac{1}{2}(\mathrm{trace}A^{e_3})e_3 - \frac{1}{2}(\mathrm{trace}A^{e_4})e_4 = H^{e_3}e_3 - H^{e_4}e_4.$$
It is easily to see that $H = 0$ iff $H^{e_3} = 0$ and $H^{e_4} = 0$, which is equivalent to $H^{\mathfrak{l}_r^+} = 0$ and $H^{\mathfrak{l}_r^-} = 0$.

Definition 2.1.

1. *A point $p \in M$ is said to be $\mathfrak{l}_r^*$-umbilic if $k_1^{\mathfrak{l}_r^*}(p) = k_2^{\mathfrak{l}_r^*}(p) = k$. In the case $k = 0$ we said that p is $\mathfrak{l}_r^*$-flat.*

2. *M is said to be $\mathfrak{l}_r^*$-umbilic ($\mathfrak{l}_r^*$-flat) if each point $p \in M$ is $\mathfrak{l}_r^*$-umbilic ($\mathfrak{l}_r^*$-flat).*

3. *M is said to be totally umbilical (totally flat) if each point $p \in M$ is $\mathfrak{l}_r^*$-umbilic ($\mathfrak{l}_r^*$-flat) for every $r > 0$.*

4. *M is said to be maximal if $H = 0$.*

3 Totally Umbilical Spacelike Surfaces

In this section, we study the umbilicity of a surface by using the $\mathfrak{l}_r^*$-Gauss map with respect to a fixed r.
Following the approach developed in the proof of Theorem 3.2 in [1], we are able to obtain a similar result. Our method is based on the $\mathfrak{l}_r^*$-Gauss map on a lightcone instead of using the $\mathbf{n}_r^*$-Gauss map on a Hyperbolic.

Theorem 3.1. *Let M be a connected spacelike surface. The following statements are equivalent:*

1. *There exists a constant $r > 0$ such that M is $\mathfrak{l}_r^*$-flat.*

2. *There exists a constant $r > 0$ such that $\mathfrak{l}_r^*$ is constant.*

3. *There exist a lightlike vector $\mathbf{n} = (n_1, n_2, n_3, n_4)$ and a real number c such that $M \subset HP(\mathbf{n}, c)$.*

Using HS_r-valued Gauss maps, Cuong and Hieu [1] showed that a surface is totally umbilical iff it is ν-umbilic for every smooth normal vector field ν. Then the umbilicity of surface was studied by using $\mathbf{n}_r^*$-Gauss maps, especially when the surface lies in $H^n(0, R)$. Some of these results are based on the linear independence of the position vector $\mathbf{X}$ and the normal vector field $\mathbf{n}_r^*$. Unfortunately, when the surface is contained in $S_1^3(0, R)$, the pair of vectors $\mathbf{X}$ and $\mathfrak{l}_r^*$ may be linearly dependent. Therefore, we use $\mathfrak{l}_r^*$ instead of $\mathbf{n}_r^*$ in order to study the surface in de Sitter. The reader is referred to [1] for more details of the proof.

Corollary 3.2. *If M is contained in the intersection of a de Sitter and a hyperplane, then M is totally umbilic.*

Theorem 3.3. *Let M be a surface in $S_1^3(0, R)$. The following statements are equivalent:*

1. *There exists $r > 0$ such that M is $\mathfrak{l}_r^*$-umbilic.*

2. *M is totally umbilic.*

3. *M is contained in a hyperplane.*

Using $\mathfrak{l}_r^*$-Gauss maps, we can find some necessary and sufficient conditions for a surface lying in a de Sitter to be a part of a horizontal hyperplane (i.e. the hyperplane is pseudo orthogonal to the time-axis). The reader is referred to Theorem 4.7 in [1] for more details of the proof.

Theorem 3.4. *Let M be a surface contained in $S_1^3(\boldsymbol{a}, R)$. The following statements are equivalent:*

1. *M is contained in a hyperplane $HP(\mathbf{n}, c)$, where $\mathbf{n} = (0, 0, 0, 1)$.*

2. *$\mathfrak{l}_r^*$ is parallel for any $r > 0$.*

3. *There exist two different parallel normal fields $\mathfrak{l}_{r_1}^*, \mathfrak{l}_{r_2}^*$.*

4. *There exists $r > 0$ such that $A^{\mathfrak{l}_r^*} = -\lambda id$, where λ is a constant.*

The following theorem gives another necessary and sufficient conditions for a surface to be a part of the intersection of $S_1^3(\mathbf{a}, R)$ and a horizontal hyperplane without assuming that it lies in the de Sitter. The reader is referred to Theorem 4.8 in [1] for more details of the proof.

Theorem 3.5. *Let M be a surface in $\mathbb{R}_1^4$. The following statements are equivalent:*

1. *There exists $r > 0$ such that $\mathfrak{l}_r^*$ is parallel but not a constant vector, and M is $\mathfrak{l}_r^*$-umbilic.*

2. *M is contained in the intersection of $S_1^3(\boldsymbol{a}, R)$ and a horizontal hypeplane $HP(\mathbf{n}, c)$.*

We will now study totally umbilical surfaces in the general case. Let us recall some imporant notions and facts.

We denote by $\nabla^\perp$ the normal connection on the surface M. The normal curvature of M is defined by

$$\begin{aligned} R^\perp : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp &\to \mathfrak{X}(M)^\perp \\ (V, W, X) &\mapsto \nabla^\perp_{[V,W]} X - \left[\nabla^\perp_V, \nabla^\perp_W\right] X, \end{aligned}$$

where $\mathfrak{X}(M)$ is the set of smooth tangent vector fields and $\mathfrak{X}(M)^\perp$ is the set of smooth normal vector fields on M, see [12].

It can be shown, as a consequence of the Ricci equation [12, p.125], that if p is an umbilical point for some normal field ν, then $R^\perp(p) = 0$. Therefore, if M is totally umbilical, then $R^\perp = 0$ on M.

Definition 3.6. [13, p.6] *A connection is called flat if its curvature is zero.*

Proposition 1.1.5 in [13] gives us the following corollary.

Corollary 3.7. *If M is totally umbilic, then for any $p \in M$ there exist a local neighbourhood $U_p \subset M$ of p and a parallel normal frame, $\{\mathbf{u}, \mathbf{v}\}$ on U_p.*

Using the notion in the Corollary 3.7, we have the following result.

Theorem 3.8. *Let M be a connected surface, then the following statements are equivalent:*

1. *M is totally umbilic.*

2. *For any $p \in M$, U_p is contained in the intersection of a hyperplane and either a Hyperbolic or a de Sitter.*

Proof.
(1) $\Rightarrow$ (2). Sine $\mathbf{u}$ is parallel, its norm is constant, therefore it is either lightlike, spacelike or timelike on U_p. Similarly to the vector $\mathbf{v}$.

(i) If $\mathbf{u}$ is timelike then U_p is contained in the intersection of a hyperbolic and a hyperplane, by virtue Theorem 4.3 in [6] and Theorem 4.5 in [1].

(ii) If $\mathbf{u}$ is spacelike then U_p is contained in the intersection of a de Sitter and a hyperplane, by virtue Theorem 4.3 in [6] and Theorem 3.4.

(iii) Ne now consider the case when both $\mathbf{u}$ and $\mathbf{v}$ are lightlike. Set

$$\mathbf{Z} = \frac{\mathbf{u}}{\langle \mathbf{u}, \mathbf{v} \rangle} - \mathbf{v}.$$

Then $\mathbf{Z}$ is a smooth normal field on U_p and

$$\langle \mathbf{Z}, \mathbf{Z} \rangle = -2, \ \langle \mathbf{Z}, \mathbf{v} \rangle = 1.$$

Therefore,

$$\langle d\mathbf{Z}, \mathbf{Z} \rangle = 0, \ \langle d\mathbf{Z}, \mathbf{v} \rangle = -\langle \mathbf{Z}, d\mathbf{v} \rangle = 0.$$

That means $\mathbf{Z}$ is parallel on U_p. Similarly to (i) we imply that U_p is contained in the intersection of a hyperbolic and a hyperplane.

(2) $\Rightarrow$ (1). It follows from Theorem 4.5 in [1] and Theorem 3.4. $\square$

4 The Spacelike Surfaces of Revolution

In this section we will use $\mathfrak{l}^*_r$-Gauss map to study totally umbilical and maximal spacelike surfaces of revolution of hyperbolic and elliptic types in $\mathbb{R}^4_1$.

4.1 Useful Lemmas

We begin with two useful lemmas.

Lemma 4.1. *If $f' \neq 0$ and $g''f' - g'f'' = 0$, then $g = cf + k$, where c, k are constants.*

Proof. We have

$$\frac{g''f' - g'f''}{(f')^2} = 0 \Rightarrow \left(\frac{g'}{f'}\right)' = 0 \Rightarrow g = cf + k,$$

where c, k are constants. □

Lemma 4.2. *The solutions of the equation*

$$\rho(u).\rho''(u) - (\rho'(u))^2 - 1 = 0$$

are

$$\rho(u) = \pm\frac{1}{\sqrt{C}}\cosh(\sqrt{C}(u - C_1)),$$

where $C > 0, C_2$ are constants.

Proof. Set $t = \rho'(u)$. Since

$$\rho''(u) = \frac{(\rho'(u))^2}{\rho(u)} + \frac{1}{\rho(u)},$$

we have

$$t'(\rho) = (t(\rho))^{-1}\left(\frac{(t(\rho))^2}{\rho(u)} + \frac{1}{\rho(u)}\right).$$

That means

$$t'(\rho) - \frac{1}{\rho}t(\rho) = \frac{1}{\rho}(t(\rho))^{-1}. \tag{3}$$

Equation (3) is a Bernoulli equation and its solutions are

$$t = \pm\sqrt{-1 + C\rho^2},\ C = const > 0.$$

That means

$$\rho'(u) = \pm\sqrt{-1 + C(\rho(u))^2}.$$

The last equation gives us the result of the lemma. □

Remark 4.3. *Similarly, we can show that:*

1. *The solutions of the equation*

$$\rho(u).\rho''(u) + (\rho'(u))^2 + 1 = 0$$

are

$$\rho(u) = \pm\sqrt{C - (u - C_1)^2},$$

where C_1, C_2 *are constants.*

2. *The solutions of the equation*

$$\rho(u).\rho''(u) - (\rho'(u))^2 + 1 = 0$$

are

$$\rho(u) = \pm\frac{1}{\sqrt{C}}\sinh(\sqrt{C}(u - C_1)),$$

where $C_1, C_2 > 0$ *are constants.*

3. *The solutions of the equation*

$$\rho(u).\rho''(u) + (\rho'(u))^2 - 1 = 0$$

are

$$\rho(u) = \pm\sqrt{(u - C_1)^2 - C_2},$$

where C_1, C_2 *are constants.*

4.2 Spacelike Surface of Revolution of Hyperbolic Type

Let C be a spacelike curve in span$\{e_1, e_2, e_4\}$ parametrized by arc-length,

$$z(u) = (f(u), g(u), 0, \rho(u)), \ u \in I.$$

Consider the surface M in $\mathbb{R}^4_1$ given by

$$X(u,v) = (f(u), g(u), \rho(u)\sinh v, \rho(u)\cosh v), \ u \in I, \ v \in \mathbb{R}. \tag{4}$$

The first partial derivatives of $X(u,v)$ can be calculated by

$$X_u = (f'(u), g'(u), \rho'(u)\sinh v, \rho'(u)\cosh v), \ X_v = (0, 0, \rho(u)\cosh v, \rho(u)\sinh v),$$

and the coefficients of the first fundamental form of M are

$$g_{11} = (f'(u))^2 + (g'(u))^2 - (\rho'(u))^2 = 1, \ g_{12} = 0, \ g_{22} = (\rho(u))^2 > 0.$$

It follows that M is a spacelike surface, which is called the *spacelike surface of revolution of hyperbolic type* in $\mathbb{R}^4_1$. From now on we always assume that

$f' \neq 0, g' \neq 0$ and $\rho' \neq 0$.
The system of equations (1) yields $\mathfrak{l}_r^{\pm} = (\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, r)$, where

$$\mathfrak{l}_1 = \frac{r\rho'}{f' \cosh v} - \frac{g'}{f'}\mathfrak{l}_2,$$

$$\mathfrak{l}_2 = \frac{g'\rho' \pm rf'}{\cosh v[(f')^2 + (g')^2]},$$

$$\mathfrak{l}_3 = r \tanh v.$$

Then the coefficients of the second fundamental form of M associated to $\mathfrak{l}_r^{\pm}$-Gauss maps are defined below:

$$b_{11}^{\mathfrak{l}_r^{\pm}} = \langle \mathfrak{l}_r^{\pm}, X_{uu} \rangle = f''\mathfrak{l}_1 + g''\mathfrak{l}_2 - \frac{r\rho''}{\cosh v},$$

$$b_{12}^{\mathfrak{l}_r^{\pm}} = \langle \mathfrak{l}_r^{\pm}, X_{uv} \rangle = r\rho' \cosh v. \tanh v - r\rho' \sinh v = 0,$$

$$b_{22}^{\mathfrak{l}_r^{\pm}} = \langle \mathfrak{l}_r^{\pm}, X_{vv} \rangle = -\frac{r\rho}{\cosh v}.$$

Solving the equation $\det \left[(b_{ij}^{\mathfrak{l}_r^{\pm}}) - k(g_{ij}) \right] = 0$, we obtain the following principal curvatures of the surface in terms of the $\mathfrak{l}_r^{\pm}$-Gauss maps

$$k_1^{\mathfrak{l}_r^{+}} = \frac{rf''\rho'}{f' \cosh v} - r\frac{\rho''}{\cosh v} + r\frac{g''f' - g'f''}{f'\left[(f')^2 + (g')^2\right] \cosh v} \left(g'\rho' + f'\right), \tag{5}$$

$$k_1^{\mathfrak{l}_r^{-}} = \frac{rf''\rho'}{f' \cosh v} - \frac{r\rho''}{\cosh v} + r\frac{g''f' - g'f''}{f'\left[(f')^2 + (g')^2\right] \cosh v} \left(g'\rho' - f'\right), \tag{6}$$

$$k_2^{\mathfrak{l}_r^{+}} = k_2^{\mathfrak{l}_r^{-}} = -\frac{r}{\rho(\cosh v)}. \tag{7}$$

Theorem 4.4. *If the surface defined by (4) is totally umbilic, then it is contained in a timelike hyperplane and parametrized by*

$$f(u) = \pm\frac{1}{\sqrt{C}} \sinh\left(\sqrt{C}(u - C_1)\right) + m,$$

$$g(u) = \frac{C_2}{\sqrt{C}} \sinh\left(\sqrt{C}(u - C_1)\right) + k,$$

$$\rho(u) = \pm\frac{1}{\sqrt{C}} \cosh\left(\sqrt{C}(u - C_1)\right),$$

where $C > 0, C_1, C_2, k$ are constants.

Proof. From (5), (6) and (7), the umbilical condition for M gives us the following equations:

$$k_1^{\mathfrak{l}_r^+} = k_2^{\mathfrak{l}_r^+} = k_2^{\mathfrak{l}_r^-} = k_1^{\mathfrak{l}_r^-}.$$

That mean we have a system of equations

$$\begin{cases} \dfrac{g''f' - g'f''}{f'} = 0, \\ \dfrac{f''\rho' - f'\rho''}{f'} = -\dfrac{1}{\rho}, \\ (f')^2 + (g')^2 - (\rho')^2 = 1. \end{cases}$$

The first equation tells us that M lies in a timelike hyperplane. By calculating f and g in terms of ρ, we have an equivalent system of equations

$$\begin{cases} g = C_2 f + k \\ f' = \sqrt{\dfrac{1 + (\rho')^2}{1 + C^2}} \\ \rho.\rho'' - (\rho')^2 - 1 = 0. \end{cases}$$

The conclusion of the theorem follows from Lemma 4.1 and Lemma 4.2. □

Remark 4.5. *By changing the parameters*

$$t = \sqrt{C}(u - C_1), \ v = v,$$

we get a new parametrization of the totally umbilical spacelike surface of revolution defined by (4)

$$X(t, v) = (A \sinh t + m, B \sinh t + k, A \cosh t \sinh v, A \cosh t \cosh v),$$

where $A = \pm\frac{1}{\sqrt{C}}, B = \frac{C_2}{\sqrt{C}}, k, m$ *are constants.*

Theorem 4.6. *If the surface defined by (4) is maximal, then it is contained in a timelike hyperplane and parametrized by*

$$f(u) = \frac{\sqrt{C_3}}{1 + C_2^2} \arcsin\left(\frac{u - C_1}{\sqrt{C_3}}\right) + m,$$

$$g(u) = C_2 f(u) + k,$$

$$\rho(u) = \pm\sqrt{C_3 - (u - C_1)^2},$$

where $C_1, C_2, C_3 > 0, k$ *are constants.*

Proof. It follows from (5), (6) and (7) that the conditions for the maximality of M are described by the following equations

$$k_1^{\mathfrak{l}_r^+} = -k_2^{\mathfrak{l}_r^+} = -k_2^{\mathfrak{l}_r^-} = k_1^{\mathfrak{l}_r^-}.$$

Therefore, we have the following system of equations

$$\begin{cases} \dfrac{g''f' - g'f''}{f'} = 0, \\ \dfrac{f''\rho' - f'\rho''}{f'} = \dfrac{1}{\rho}, \\ (f')^2 + (g')^2 - (\rho')^2 = 1. \end{cases}$$

Similar to the proof of Theorem 4.4, the conclusion of the theorem follows from Remark 4.3. □

Remark 4.7. *By changing the parameters*

$$t = \sin \frac{u - C_1}{\sqrt{C_3}}, \ \ v = v,$$

we get a new parametrization of the maximal spacelike surface of revolution defined by (4)

$$X(t, v) = (At + m, Bt + k, C \cos t \sinh v, C \cos t \cosh v),$$

where $A = \frac{\sqrt{C_3}}{1+C_2^2}, B = C_2, C = \pm\frac{1}{\sqrt{C_3}}, m, k$ *are constants.*

4.3 Spacelike Surface of Revolution of Elliptic Type

Let C be a spacelike curve in $\operatorname{span}\{e_1, e_3, e_4\}$ parametrized by arc-length,

$$z(u) = (\rho(u), 0, f(u), g(u)), \ u \in I.$$

Let the surface M in $\mathbb{R}^4_1$ be given by

$$X(u, v) = (\rho(u) \cos v, \rho(u) \sin v, f(u), g(u)), \ v \in \mathbb{R}. \tag{8}$$

We have the following coefficients of the first fundamental form of surface:

$$g_{11} = (\rho')^2 + (f')^2 - (g')^2 = 1, \ g_{12} = 0, \ g_{22} = (\rho(u))^2.$$

It follows that M is a spacelike surface which is called the *spacelike surface of revolution of elliptic type* in $\mathbb{R}^4_1$. We will assume $f' \neq 0, g' \neq 0$ and $\rho' \neq 0$. The system of equations (1) yields $\mathfrak{l}_1^{\pm} = (\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, 1)$, where

$$\mathfrak{l}_1 = \cos v \frac{g'\rho' \pm f'}{(f')^2 + (\rho')^2}, \ \mathfrak{l}_2 = \sin v \frac{g'\rho' \pm f'}{(f')^2 + (\rho')^2},$$

$$\mathfrak{l}_3 = \frac{g'}{f'} - \frac{\rho'}{f'}\frac{g'\rho' \pm f'}{(f')^2 + (\rho')^2} = \frac{g'f' \pm \rho'}{(f')^2 + (\rho')^2}.$$

We are able to define the coefficients of the second fundamental form of M associated to $\mathfrak{l}_r^{\pm}$-Gauss maps.

$$b_{11}^{\mathfrak{l}_1^{\pm}} = \langle \mathfrak{l}_1^{\pm}, X_{uu}\rangle = \rho''\frac{g'\rho' \pm f'}{(f')^2 + (\rho')^2} + f''\frac{g'f' \mp \rho'}{(f')^2 + (\rho')^2} - g'',$$

$$b_{12}^{\mathfrak{l}_1^{\pm}} = 0,\ b_{22}^{\mathfrak{l}_1^{\pm}} = \langle \mathfrak{l}_1^{\pm}, X_{vv}\rangle = -\rho\frac{g'\rho' \pm \rho'}{(f')^2 + (\rho')^2}.$$

Solving the equation $\det\left[(b_{ij}^{\mathfrak{l}_1^{\pm}}) - k(g_{ij})\right] = 0$, we obtain the following principal curvatures of surface in terms of $\mathfrak{l}_1^{\pm}$-Gauss map, respectively

$$k_1^{\mathfrak{l}_1^{+}} = \rho''\frac{g'\rho' + f'}{(f')^2 + (\rho')^2} + f''\frac{g'f' - \rho'}{(f')^2 + (\rho')^2} - g'', \tag{9}$$

$$k_1^{\mathfrak{l}_1^{-}} = \rho''\frac{g'\rho' - f'}{(f')^2 + (\rho')^2} + f''\frac{g'f' + \rho'}{(f')^2 + (\rho')^2} - g'', \tag{10}$$

$$k_2^{\mathfrak{l}_1^{+}} = -\frac{1}{\rho}\frac{g'\rho' + f'}{(f')^2 + (\rho')^2};\quad k_2^{\mathfrak{l}_1^{-}} = -\frac{1}{\rho}\frac{g'\rho' - f'}{(f')^2 + (\rho')^2}. \tag{11}$$

Theorem 4.8. *If the surface difined by (8) is totally umbilic, then*

$$\rho(u) = \pm\frac{1}{\sqrt{C_2}}\sinh\left(\sqrt{C_2}(u - C_1)\right),$$

$$g(u) = \pm\frac{C}{\sqrt{C_2}}\cosh\left(\sqrt{C_2}(u - C_1)\right),$$

$$f(u) = \pm\frac{\sqrt{C+1}}{\sqrt{C_2}}\cosh\left(\sqrt{C_2}(u - C_1)\right),$$

where C, C_1, C_2 are constants.

Proof. The conditions for M to be totally umbilic,

$$k_1^{\mathfrak{l}_1^{+}} = k_2^{\mathfrak{l}_1^{+}} \text{ and } k_1^{\mathfrak{l}_1^{-}} = k_2^{\mathfrak{l}_1^{-}},$$

are equivalent to the following system of equations

$$\begin{cases} (f')^2 + (\rho')^2 - (g')^2 = 1, & (a) \\ \rho g'' = \rho' g', & (b) \\ \rho(\rho'' f' - f''\rho') = -f'. & (c) \end{cases} \tag{12}$$

It follows from (b) that

$$g' = C\rho,\ C = const. \tag{13}$$

Differentiating both sides of (a), we obtain

$$f'f'' = g'g'' - \rho'\rho''. \tag{14}$$

After multiplying both sides of (c) by f', we have

$$\rho(\rho''(f')^2 - f'f''\rho') = -(f')^2. \tag{15}$$

Substitute both (13) and (14) to (15), we get a diffirential equation in terms of the function ρ

$$\rho(u).\rho''(u) - (\rho'(u))^2 + 1 = 0.$$

The claim of the theorem is deduced from Remark 4.3. □

Remark 4.9. *By changing the parameters*

$$t = \sqrt{C_2}(u - C_1),\ v = v,$$

we get the new parametrization of the totally umbilical spacelike surface of revolution defined by (8)

$$X(t, v) = (B\sinh t \cos v, B\sinh t \sin v, A\cosh t, CB\cosh t),$$

where $A = \pm\frac{\sqrt{C+1}}{\sqrt{C_2}}, B = \pm\frac{1}{\sqrt{C_2}}$ *are constants.*

Theorem 4.10. *If the surface defined by (8) is maximal, then*

$$\rho(u) = \pm\sqrt{(u - C_1)^2 - C},$$

$$g(u) = \frac{C_2}{\sqrt{C}}\operatorname{arccosh}\left(\frac{u - C_1}{\sqrt{C}}\right) + m,$$

$$f(u) = \frac{\sqrt{C_2^2 - C}}{\sqrt{C}}\operatorname{arccosh}\left(\frac{u - C_1}{\sqrt{C}}\right) + k,$$

where C, C_1, C_2, m, k *are constants.*

Proof. It follows from (9), (10) and (11) that the conditions for the maximality of M are described by the following equations

$$k_1^{\xi_1^+} = -k_2^{\xi_1^+} \text{ and } k_1^{\xi_1^-} = -k_2^{\xi_1^-}.$$

Therefore, we have a system of equations defining condition of maximal surfaces

$$\begin{cases} (f')^2 + (\rho')^2 - (g')^2 = 1, & (a') \\ \rho g'' = -\rho' g', & (b') \\ \rho(\rho'' f' - f''\rho') = f'. & (c') \end{cases} \tag{16}$$

By using the similar method for the solving the system of equations (12) we get an equation in terms of the function ρ

$$\rho(u).\rho''(u) + (\rho'(u))^2 - 1 = 0.$$

Remark 4.3 yields the result of the theorem. $\square$

Remark 4.11. *By changing the parameters*

$$t = \operatorname{arccosh}\left(\frac{u - C_1}{\sqrt{C}}\right),\ v = v,$$

we get a new parametrization of the maximal spacelike surface of revolution of elliptic type defined by (8)

$$X(t, v) = (A \sinh t \cos v, A \sinh t \sin v, Bt + m, Dt + k),$$

where $A = \pm\frac{1}{\sqrt{C}}, B = \pm\frac{C_2}{\sqrt{C}}, D = \frac{\sqrt{C_2^2 - C}}{\sqrt{C}}$ *are constants.*

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