

On Formula to Compute Primes and the n^{th} Prime

Issam Kaddoura

Lebanese International University
Faculty of Arts and Sciences, Lebanon
issam.kaddoura@liu.edu.lb

Samih Abdul-Nabi

Lebanese International University
School of Engineering, Lebanon
samih.abdulnabi@liu.edu.lb

Abstract

In this paper, we propose a new primality test, and then we employ this test to find a formula for π that computes the number of primes within any interval. We finally propose a new formula that computes the n th prime number as well as the next prime for any given number.

Keywords: prime, congruence, primality test, Euclidean algorithm, sieve of Eratosthenes

1. Introduction

Since Euclid [12], primes and prime generation were a challenge of interest for number theory researchers. Primes are used in many fields, one just need to mention for example the importance of primes in networking and certificate generation [1]. Securing communication between two devices is achieved using primes since primes are the hardest to decipher [5].

The search for prime numbers is a continuous task for researchers. Some like [4] are looking for twin primes others like [11] are looking for large scaled prime numbers. The prime counting function is a function that gives the number of primes that are less than or equal to a given number.

Many like [6] and others have presented the formula to compute the number of primes between 1 and a given integer n .

This paper is divided as follow: in section 2 we present the primality test. In section 3, we introduce the prime counting function that we will use in section 5 to find the next prime to any given number. In section 4, we conduct some results and build the n^{th} prime function. In section 6, we will use the primality test to compare our results with some recent results in the literature and conclude this paper.

2. Primality test

In the paper, we employ the Euclidian algorithm, Sieve of Eratosthenes and the fact that every prime is of the form $6k \pm 1$ where k an integer.

Let x be a real number, the floor of x , denoted by $\lfloor x \rfloor$ is the largest integer that is less or equal to x . To test the primality of x it is enough to test the divisibility of x by all primes $\leq \lfloor \sqrt{x} \rfloor$.

Let $S_1(x)$ be of the form

$$S_1(x) = \frac{(-1)^{\left\lfloor \frac{\lfloor \sqrt{x} \rfloor}{6} \right\rfloor + 1}}{\left\lfloor \frac{\lfloor \sqrt{x} \rfloor}{6} \right\rfloor + 1} \sum_{k=1}^{\left\lfloor \frac{\lfloor \sqrt{x} \rfloor}{6} \right\rfloor + 1} \left[\left\lfloor \frac{x}{6k+1} \right\rfloor - \frac{x}{6k+1} \right] \quad (2.1)$$

Similarly, let $S_2(x)$ be of the form

$$S_2(x) = \frac{(-1)^{\left\lfloor \frac{\lfloor \sqrt{x} \rfloor}{6} \right\rfloor + 1}}{\left\lfloor \frac{\lfloor \sqrt{x} \rfloor}{6} \right\rfloor + 1} \sum_{k=1}^{\left\lfloor \frac{\lfloor \sqrt{x} \rfloor}{6} \right\rfloor + 1} \left[\left\lfloor \frac{x}{6k-1} \right\rfloor - \frac{x}{6k-1} \right] \quad (2.2)$$

Theorem 1: If x is any integer such that $\text{g.c.d}(x, 6) = 1$

and $S(x) = \frac{S_1(x) + S_2(x)}{2}$, then

- (i) x is prime if and only if $S(x) = 1$
- (ii) x is composite if and only if $0 \leq S(x) < 1$

Proof: x is prime $\Rightarrow \text{gcd}(x, i) = 1 \quad \forall \quad 1 \leq i \leq x-1$

$$\Rightarrow x \not\equiv 0 \pmod{i} \quad \forall \quad 1 \leq i \leq x-1 \Rightarrow \left[\left\lfloor \frac{x}{6k+1} \right\rfloor - \frac{x}{6k+1} \right] = -1,$$

$$\left[\left\lfloor \frac{x}{6k-1} \right\rfloor - \frac{x}{6k-1} \right] = -1 \quad \forall \quad k \text{ within the range of the summation in the}$$

formulas of $S_1(x)$ and $S_2(x)$.

$$\Rightarrow S_1(x) = \frac{(-1)^{\left\lfloor \frac{\sqrt{x}}{6} \right\rfloor + 1}}{\left\lfloor \frac{\sqrt{x}}{6} \right\rfloor + 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{x}}{6} \right\rfloor + 1} (-1) = 1$$

Similarly $S_2(x) = 1$ and consequently $S(x) = 1$

The proof of the second part of the theorem is obvious.

3. Prime Counting Function

The prime counting function, denoted by the Greek letter $\pi(n)$, is the number of primes less than or equal to a given number n . Computing the primes is one of the most fundamental problems in number theory. You can see [10] for the latest works regarding prime counting functions.

Using the previous primality test, we define the following new form of the prime counting function π .

Recall that

$$\lfloor S(i) \rfloor = \begin{cases} 1 & \text{if } i \text{ is prime} \\ 0 & \text{if } i \text{ is composite} \end{cases} \tag{3.1}$$

Then $\sum_{i=m}^n \lfloor S(i) \rfloor$ counts the primes between m and n where $m \leq n$.

And we can write a formula for $\pi(n)$ as follows :

$$\pi(n) = 4 + \sum_{i=7}^n \lfloor S(i) \rfloor \tag{3.2}$$

The size of this summation can be dramatically reduced by considering only i of the form $6j+5$ or $6j+7$.

$$\pi(x) = 4 + \sum_{j=1}^{\left\lfloor \frac{x-1}{6} \right\rfloor} \lfloor S(6j+1) \rfloor + \sum_{j=1}^{\left\lfloor \frac{x+1}{6} \right\rfloor} \lfloor S(6j-1) \rfloor \tag{3.3}$$

Thus the following theorem is already proved.

Theorem 5: $\forall x \geq 7$, $\pi(x)$ gives the number of primes $\leq x$.

4. The n^{th} Prime Function

We are now ready to introduce our new formula to find the n^{th} prime. The n^{th} prime number is denoted by p_n with $p_1=2, p_2=3, p_3=5$ and so on.

First we introduce $f_n(x)$ as follows

$$f_n(x) = 1 - \left\lfloor \frac{x}{n} \right\rfloor \quad \text{For } n = 1, 2, 3 \dots \text{ and } x = 0, 1, 2 \dots \tag{4.1}$$

Or

$$f_n(x) = \left\lfloor \frac{2n}{x+n+1} \right\rfloor \quad \text{For } n = 1, 2, 3 \dots \text{ and } x = 0, 1, 2 \dots \tag{4.2}$$

These functions have the property that

$$f_n(x) = \begin{cases} 1 & \text{for } x < n \\ 0 & \text{for } x \geq n \end{cases} \tag{4.3}$$

It is well known that $P_n \leq 2(\lfloor n \log n \rfloor + 1)$; see [8] and [2] for more details.

Using the following formula combined with the above formula for π

$$P_n = 7 + \sum_{x=7}^{2(\lfloor n \log n \rfloor + 1)} f_n(\pi(x)) \tag{4.4}$$

We use $f_n(x)$ as in (4.1) to obtain the following formula for n^{th} prime in full:

$$P_n = 7 + \sum_{x=7}^{2(\lfloor n \log n \rfloor + 1)} \left(1 - \left\lfloor \frac{4}{n} + \frac{1}{n} \left(\sum_{j=1}^{\lfloor \frac{x-1}{6} \rfloor} \lfloor S(6j+1) \rfloor + \sum_{j=1}^{\lfloor \frac{x+1}{6} \rfloor} \lfloor S(6j-1) \rfloor \right) \right\rfloor \right) \tag{4.5}$$

$$P_n = 3 + 2\lfloor n \log n \rfloor - \sum_{x=7}^{2(\lfloor n \log n \rfloor + 1)} \left(\left\lfloor \frac{1}{n} \left(4 + \left(\sum_{j=1}^{\lfloor \frac{x-1}{6} \rfloor} \lfloor S(6j+1) \rfloor + \sum_{j=1}^{\lfloor \frac{x+1}{6} \rfloor} \lfloor S(6j-1) \rfloor \right) \right) \right\rfloor \right)$$

Or using $f_n(x)$ as in (4.2) to obtain the formula for n^{th} prime in full:

$$P_n = 7 + \sum_{x=7}^{2(\lfloor n \log n \rfloor + 1)} \left\lfloor \frac{2n}{\pi(x) + n + 1} \right\rfloor = \sum_{x=7}^{2(\lfloor n \log n \rfloor + 1)} \left\lfloor \frac{2n}{5 + n + \sum_{j=1}^{\lfloor \frac{x-1}{6} \rfloor} \lfloor S(6j+1) \rfloor + \sum_{j=1}^{\lfloor \frac{x+1}{6} \rfloor} \lfloor S(6j-1) \rfloor} \right\rfloor \tag{4.6}$$

These formulas are in terms of n alone and we do not need to know any of the previous primes.

See [2] for formulas of the same nature.

The Wolfram Mathematica implementation of P_n as in (4.5) is as follow:

$$\begin{aligned}
 A[x_] &:= (-1/(\text{Floor}[(\text{Floor}[\text{Sqrt}[x]])/6] + 1)) * \text{Sum}[\text{Floor}[\text{Floor}[(x/(6 k + 1))] - (x/(6 k + 1))], \\
 &\quad \{k, 1, \text{Floor}[(\text{Floor}[\text{Sqrt}[x]])/6] + 1\}] \\
 SB[x_] &:= (-1/(\text{Floor}[(\text{Floor}[\text{Sqrt}[x]])/6] + 1)) * \text{Sum}[\text{Floor}[\text{Floor}[(x/(6 k - 1))] - (x/(6 k - 1))], \\
 &\quad \{k, 1, \text{Floor}[(\text{Floor}[\text{Sqrt}[x]])/6] + 1\}] \\
 SS[x_] &:= (SA[x] + SB[x])/2 \\
 PN[x_] &:= 4 + \text{Sum}[\text{Floor}[\text{SS}[6 j + 1]], \{j, 1, \text{Floor}[(x - 1)/6]\}] + \text{Sum}[\text{Floor}[\text{SS}[6 j - 1]], \\
 &\quad \{j, 1, \text{Floor}[(x + 1)/6]\}] \\
 PT[x_] &:= 3 + 2 (\text{Floor}[x * \text{Log}[x]]) - \text{Sum}[\text{Floor}[(1/x) * (4 + \\
 &\quad (\text{Sum}[\text{Floor}[\text{SS}[6 j + 1]], \{j, 1, \text{Floor}[(i - 1)/6]\}] + \\
 &\quad \text{Sum}[\text{Floor}[\text{SS}[6 j - 1]], \{j, 1, \text{Floor}[(i + 1)/6]\}]), \\
 &\quad \{i, 7, 2 (\text{Floor}[x * \text{Log}[x]] + 1)\}]
 \end{aligned}$$

5. Next Prime

The function $nextp(n)$ finds the first prime number that is greater than a given number n . As in [9] and using $S(x)$ as defined in section 2, it is clear that:

$$\prod_{x=n+1}^{n+i} (1 - \lfloor S(x) \rfloor) = 1 \quad \forall 1 \leq i \leq nextp(n) - n - 1$$

and

$$\prod_{x=n+1}^{n+i} (1 - \lfloor S(x) \rfloor) = 0 \quad \forall i \text{ such that } nextp(n) - n \leq i \leq 2n$$

now consider the summation

$$\begin{aligned}
 \sum_{i=1}^n \prod_{x=n+1}^{n+i} (1 - \lfloor S(x) \rfloor) &= \sum_{i=1}^{nextp(n)-n} \prod_{x=n+1}^{n+i} (1 - \lfloor S(x) \rfloor) + \sum_{i=nextp(n)-n+1}^n \prod_{x=n+1}^{n+i} (1 - \lfloor S(x) \rfloor) \\
 &= \sum_{i=1}^{nextp(n)-n} (1) + \sum_{i=nextp(n)-n+1}^n (0) = nextp(n) - n
 \end{aligned}$$

finally we obtain

$$nextp(n) = n + \sum_{i=1}^n \prod_{x=n+1}^{n+i} (1 - \lfloor S(x) \rfloor) \tag{5.1}$$

We used the proposed primality test to implement $nextp(n)$ as follow:

- 1) Set $k = \left\lceil \frac{n-1}{6} \right\rceil$
- 2) Set $m = 6k + 1$
- 3) If $S(m) = 1$ then go to step 8
- 4) Set $m = 6k + 5$
- 5) If $S(m) = 1$ then go to step 8
- 6) $k = k + 1$
- 7) Go to step 2
- 8) Output the value of m

The Wolfram Mathematica implementation of $nextp(n)$ is as follow:

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A[x_] := (-1/(Floor[(Floor[Sqrt[x]])/6] + 1))* Sum[Floor[Floor[(x/(6 k + 1))] - (x/(6 k + 1))],
{ k, 1, Floor[(Floor[Sqrt[x]])/6] + 1}
SB[x_] := (-1/(Floor[(Floor[Sqrt[x]])/6] + 1))* Sum[Floor[Floor[(x/(6 k - 1))] - (x/(6 k - 1))],
{ k, 1, Floor[(Floor[Sqrt[x]])/6] + 1}
SS[x_] := (SA[x] + SB[x])/2
n=Input["Input a number:"];
k=Ceiling[(n-1)/6];
m=0;
While[True,
  m=6k+1;
  If[SS[m]==1,Break[]];
  m=6k+5;
  If[SS[m]==1,Break[]];
  k=k+1;]

```

6. Experimental Results

n th prime		
n	P(n)	Value
50	4.1s	229
100	25.58s	541
200	162.19s	1223
250	286.4s	1583

Table 1: nth prime

We implemented our algorithm using Wolfram Mathematica version 8. Table 1 shows the results for the nth prime while table 2 shows the results for the next prime. Those experimental results show the complexity of our primality test

Next prime		
Next to	nextp(n)	Value
10 ⁸	0.04s	100000007
10 ⁹	0.187s	1000000007
10 ¹⁰	2.012s	10000000019
10 ¹¹	1.061s	100000000003
10 ¹²	43.68s	1000000000039
10 ¹³	132.242	10000000000037

Table 2: Next prime

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Received: February, 2012