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Incorporating Uncertainty in Financial Models

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Abstract

Financial problems are often based on a rigorous mathematical structure and are strongly dependent on the imprecision of the input data. In this paper we show how it is possible to model the uncertainty by assuming that the fundamental parameters are fuzzy numbers. The massive computation effort involved in computations, can be successfully handled due to a flexible and precise parametric representation of the lower and upper branches of the membership functions of the fuzzy numbers.

Mathematics Subject Classification: 08A72, 90C31

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1 Introduction

Finance became, in the last forty years, one of the most prolific applied mathematical sciences, maintaining also a bright interest in the modelling of uncertainty that is the real central aspect of all financial models. The probabilistic approach has always been considered the most important, but it has often been shown that it can involve problems that may be difficult to handle.

A different approach is practicable. Many studies show that fuzzy numbers may represent an important theoretical and practical tool to tackle uncertainty in several financial problems. By this way, it is possible to model the uncertainty about some parameters across intervals of values. Intervals are built with differentiated levels of uncertainty; given a crisp value, the levels produce an asymmetric or nonlinear shape depending on subjective beliefs and available information of the decision maker. It follows that fuzzy parameters describe the possible uncertainty that starts gradually from a null variation to the greatest variation. The study of the propagation of uncertainty in the model dynamics is a central issue.

Figure 1: The cover back of Taleb's book.

Nassim Nicholas Taleb published in 2007 the book entitled: The Black Swan: The Impact of the Highly Improbable. It is everyplace considered one of the twelve most influential books since the second world war. Taleb also argues that the bell curve is a great intellectual fraud because the Gaussian model of real world induces to think that black swans, for example financial crises, are rare events and consequently he stigmatizes the use of probability theory in social sciences.

Fuzzy logic posits a world in which absolutes, such as those implied in the words "true" and "false", are less important and interesting than the matters of degree between them. In other words we could say that gray swans are more interesting than white and black swans.

In order to support this approach, Zadeh in [15] states that fuzziness in real world exists in many fields and, especially in human sciences like economics, fuzzy mathematics can provide rigorous models and Hryniewicz in [8] suggests that a proper description of vagueness and imprecision can be based on the mathematical theory of fuzzy numbers.

However we have to underline that fuzzy logic does not have to substitute the probability structure, and in many case it can be complementary to it: random variables and fuzzy parameters can coexist without changing each other their original nature.

About three decades ago, Dubois and Prade developed the arithmetical structure of fuzzy numbers, introduced by Zadeh in [14], and they introduced the well known LR model and the corresponding formulas for the fuzzy arithmetics (see [2] and the references therein).

The robust and efficient parametric representation studied in [12] gives the possibility to model the shape of the uncertain parameters assumed as fuzzy numbers.

In this paper we give some evidence about the capability of fuzzy numbers to capture the uncertainty intrinsic in well known financial models and we argue about the robustness of the fuzzy mathematics. We also show a fuzzy calculator that has been designed for the numerical applications. After the introduction, in the second section we explain the foundations of the fuzzy mathematics useful for a better reading of the whole paper. In the third section the fuzzy financial models become to be described: first of all, the capital budgeting techniques where the variables do not have a stochastic nature. The fourth section shows the so called fuzzy-stochastic approach (introduced in [16]) in the valuation of option prices. The last section concludes and discusses about the relevance of the fuzzy approach to model uncertainty in social sciences.

2 Foundations

A fuzzy number $u = \langle a^-, a, a^+ \rangle_{L,R}$ is defined by its core $a \in \mathbb{R}$ and by its membership function $\mu : \mathbb{R} \longrightarrow [0,1]$ with support in $[a^-, a^+]$ defined as

$$\mu(x) = \begin{cases} L(x) & if & a^- \leq x \leq a \\ R(x) & if & a \leq x \leq a^+ \\ 0 & otherwise & \end{cases} \quad \text{for} \quad x \in \mathbb{R} \tag{1}$$

where $L(x)$ is an increasing function with $L(a^-) = 0$, $L(a) = 1$ and $R(x)$ is a decreasing function with $R(a) = 1$, $R(a^+) = 0$. The left $L(.)$ and right $R(.)$ shape functions of u are assumed to be differentiable and , given, $\alpha \in]0,1]$, the $\alpha - cuts$ are the compact nested intervals $[u]_\alpha = \{x | \mu(x) \geq \alpha\}$.

A flexible Lower Upper parametrization of fuzzy numbers is obtained by representing the lower and upper branches u_α^- and u_α^+ of u on the trivial decomposition of interval $[0,1]$, with $N = 1$ and $\alpha_0 = 0, \alpha_1 = 1$; u can be represented by a vector of 8 components:

$$u = (u_0^-, \delta u_0^-, u_0^+, \delta u_0^+; u_1^-, \delta u_1^-, u_1^+, \delta u_1^+) \tag{2}$$

where $u_0^-, \delta u_0^-, u_1^-, \delta u_1^-$ define the lower branch u_α^-, and $u_0^+, \delta u_0^+, u_1^+, \delta u_1^+$ define the upper one u_α^+.

The LR and the LU representations of fuzzy numbers require to use appropriate (monotonic) shape functions to model either the left and right branches of the membership function or the lower and upper branches of the $\alpha - cuts$.

In [12] we describe how the fuzzy extension principle can be applied to fuzzy function calculation associated to the LU representation of the fuzzy quantities; we show that the proposed parametrization approximates the shapes with acceptable small errors and with the relevant advantage of producing good approximations of the arithmetic operations between ant kind of LU or LR fuzzy numbers.

In all the applications we are going to describe in the next sections, the fundamental step is the computation of fuzzy-valued functions. Let $v = f(u_1, u_2, ..., u_n)$ denote the fuzzy extension of a differentiable function f in n variables and its LU representation as a fuzzy number is

$$v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,...,N} \tag{3}$$

where

$$v_i^- = f(\widehat{x}_{1,i}^-, \widehat{x}_{2,i}^-, ..., \widehat{x}_{n,i}^-) \text{ and } v_i^+ = f(\widehat{x}_{1,i}^+, \widehat{x}_{2,i}^+, ..., \widehat{x}_{n,i}^+) \tag{4}$$

and the slopes δv_i^-, δv_i^+ are computed by

$$\begin{aligned}
\delta v_i^- &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^- = u_{k,i}^-}}^{n} \frac{\partial f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)}{\partial x_k} \delta u_{k,i}^- + \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^- = u_{k,i}^+}}^{n} \frac{\partial f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)}{\partial x_k} \delta u_{k,i}^+ \\
\delta v_i^+ &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^+ = u_{k,i}^-}}^{n} \frac{\partial f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)}{\partial x_k} \delta u_{k,i}^- + \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^+ = u_{k,i}^+}}^{n} \frac{\partial f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)}{\partial x_k} \delta u_{k,i}^+ .
\end{aligned} \tag{5}$$

The $\alpha - cuts$ $[v_\alpha^-, v_\alpha^+]$ of v are the images of the $\alpha - cuts$ of $(u_1, u_2, ..., u_n)$ and are obtained by solving the box-constrained optimization problems

$$(EP)_\alpha : \begin{cases} v_\alpha^- = \min\left\{f(x_1, x_2, ..., x_n) | x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+],\ k = 1, 2, ..., n\right\} \\ v_\alpha^+ = \max\left\{f(x_1, x_2, ..., x_n) | x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+],\ k = 1, 2, ..., n\right\} . \end{cases} \tag{6}$$

Except for simple elementary cases for which the optimization problems above can be solved analytically, the direct application of (EP) may present some difficulties, that may be avoided with a multiple population differential evolution algorithm that can take into account any possible nonlinearity.

The differential evolution algorithm is designed to compute the values (4) and the slopes (5) of the LU representation of $v = f(u_1, u_2, ..., u_n)$. The basic idea is to find the *min* or *max* of $\{f(u_1, ..., u_n)|(u_1, ..., u_n)\in\mathbb{A} \subset \mathbb{R}^n\}$ by starting with an initial population of p feasible points for each generation and to obtain a new set of points by randomly combining the individuals of the current population and by selecting the best generated elements to continue in the next generation. The initial population is chosen randomly and should try to cover uniformly the entire parameter space.

A useful package thought as a windows-based frame similar to a standard hand-calculator is useful in order to show the peculiarities of the parametric representation.

Figure 2 shows a complete view of the calculator; from left to right we can see the grids of the fuzzy numbers X, Y and Z. Z is the result of the operations while X and/or Y are the operands. For each element $u \in \{X, Y, Z\}$the grid contains the values α_i, u_i^-, δu_i^-, u_i^+ and δu_i^+ respectively in the LU View Mode or x, $\mu(x)$, $\delta\mu(x)$ in the LR Mode. To start the calculations, we have implemented a set of predefined types, including triangular, trapezoidal, exponential, gamma, etc. For a given type, it is possible to define the number N of subintervals ($N + 1$ points) in the uniform $\alpha-$decomposition.

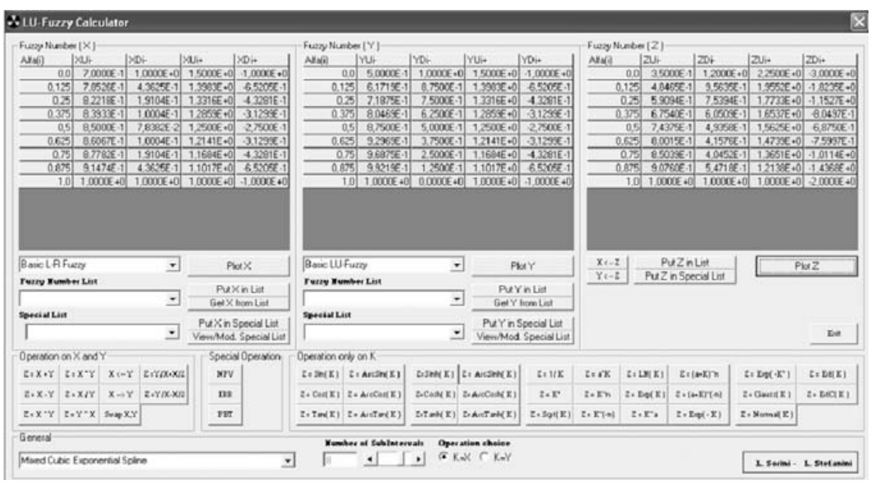

Figure 2: General window of the LU-fuzzy calculator.

The calculations are performed by clicking the button of the corresponding operation. The left group of buttons involves the binary operations (see Figure 3)

It is possible to save a given (X, Y or Z) temporary result into a stored list (Put in List button), by assigning a name to it; a saved fuzzy number can

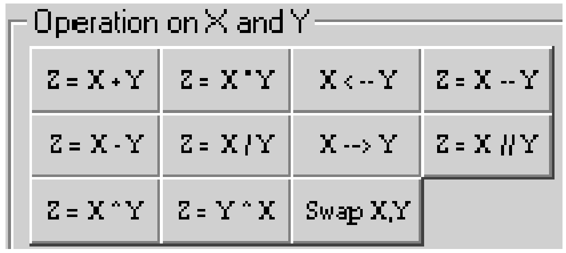

Figure 3: Binary operations and assignments.

be reloaded either in X or Y for further use (Get from List button). The Plot button opens a popup window with the graph of the membership function of the corresponding fuzzy number and is possible select a fuzzy number from a list of predefined types; (View LR,View LU button) allow to switch between LR or LU fuzzy representation.

Figure 4: Fuzzy Number Control Panel.

To obtain the graphs or other representations, one of the models can be selected (Figure 5).

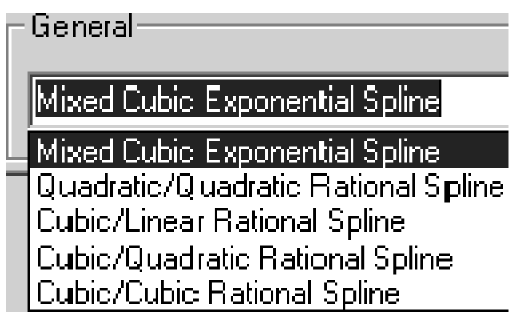

Figure 5: Choosing the monotonic spline model.

The inputs can be chosen from a pre-defined list or introduced directly; fuzzy inputs and output can be visualized in a graphic in order to interpret their meaning in terms of asymmetry, nonlinearities, convexity or concavity of the shape.

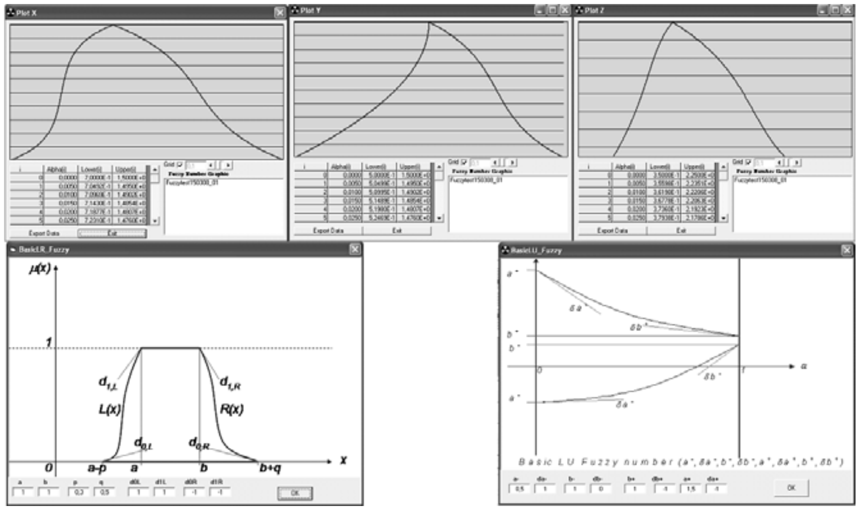

Figure 6: Possible shapes in of the LU-fuzzy calculator.

For example it is possible to select a trapezoidal fuzzy number with four sub intervals and five $\alpha - cuts$.

If the selection is loaded into the X-area, the corresponding grid appears and it is possible to plot and to switch in LR View mode immediately.
A second fuzzy number can be loaded into Y and the button corresponding to the operation Z=XˆY is activated. The Z-grid is calculated by the rules of the fuzzy calculus (see Figure 8).

To see the graphical representation of X, Y and/or Z, click the corresponding Plot button and the popup windows appear; note that the fuzzy numbers X and Z are represented in LR mode instead Y is in LU mode.

We believe that the desktop calculator represents a powerful tool in order to perform applications of fuzzy numbers because it enables the interpretation of shapes as inputs and outputs. Most of all, it enables the use of LR or LU parametric representations.

3 Fuzzy capital budgeting

We first approach the well known financial model called capital budgeting, analyzed in a preliminary version in [5].

Some contributions in this area can be found in [9] where the classical capital budgeting methods are generalized to evaluate and compare projects in which the cash flows, duration time and required rate of return are given in the form of a fuzzy number.

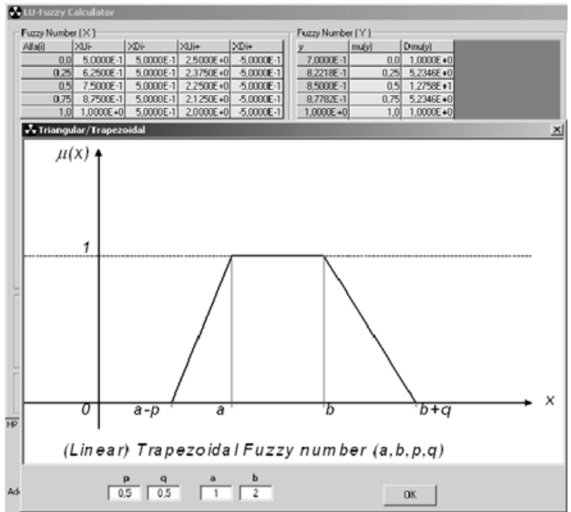

Figure 7: Example for trapezoidal fuzzy number loaded in X

In [10] the fuzzy binomial valuation approach to evaluate investment projects with embedded real options is developed and applied in practice to evaluate an investment project. In [13] the classical discounted cash flow model is generalized in the case of vague cash flows and imprecise discount rate and it is shown that the fuzzy discounted cash flow model should be more suitable to capture the elements of valuation than non-fuzzy models.

A powerful method to introduce uncertainty through fuzzy numbers into numerical methods is in [1], and it opens new directions in the research of fuzzy numerical methods useful in finance.

The Net Present Value (hereafter NPV) criteria consists in choosing the investment that presents the biggest value of the discounted cash flows:

$$NPV = \sum_{k=1}^{K} \frac{F_k}{(1+R)^k} - F_0$$

The uncertainty about the future cash flows and the interest rate enters when these key variables are fuzzified and expressed through the LU parametrization:

$$F_i = \left(F_i^-, \delta F_i^-, F_i^+, \delta \mathrm{F}_i^+\right) \quad i = 0, 1, ..., K$$

$$R = \left(R^-, \delta R^-, R^+, \delta R^+\right)$$

The main criticism to NPV criteria is in the fact that the choice on the interest rate R depends on the person which evaluates the investment. On the other hand this criticism can become an added value because the evaluation depends

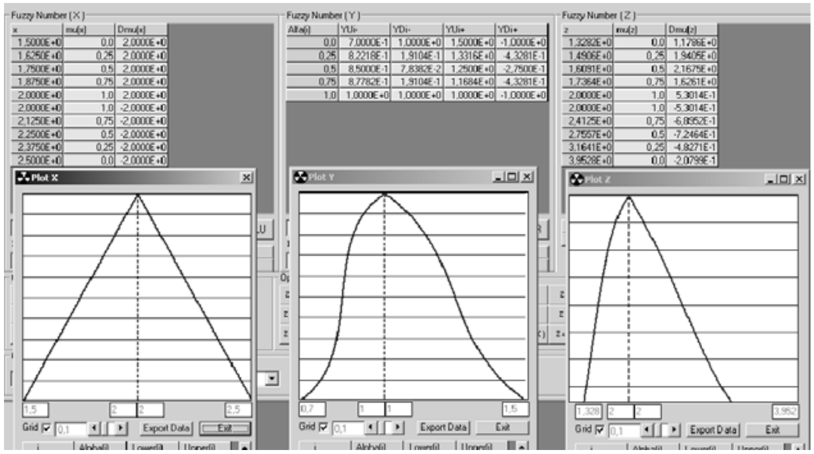

Figure 8: An example of a fuzzy Z=XˆY.

on the financial experience, on the personal feelings and on the risk propensity of the single individual: these aspects are well represented trough a fuzzy number.

The function $NPV = f(R)$ has partial derivatives defined as follows

$$\frac{\partial f}{\partial R}^{\pm} = -\sum_{k=1}^{K} F_k^{\pm} k \left(1 + R^{\mp}\right)^{-k-1}$$

where $[R]_\alpha = [R_\alpha^-, R_\alpha^+]$ and supposing the first cash flow $F_0 < 0$ and $F_i \geq 0$ $\forall i = 1, ..., K$ we have

$$\frac{\partial f}{\partial F_i} = \frac{1}{(1+R)^i} > 0, \ \frac{\partial f}{\partial F_0} = -1, \ \frac{\partial f}{\partial R} < 0$$

and the LU representation for the NPV, validated by the extension principle, becomes

$$\begin{aligned} NPV^- &= \sum_{k=1}^{K} \frac{F_k^-}{(1+R^+)^k} - F_0^+ \\ NPV^+ &= \sum_{k=1}^{K} \frac{F_k^+}{(1+R^-)^k} - F_0^- \end{aligned}$$

$$\delta NPV^- = \sum_{k=1}^{K} \frac{1}{(1+R^+)^k} \delta F_k^- - \delta F_0^+ + \frac{\partial f}{\partial R}^- \delta R^+$$

$$\delta NPV^{+}=\sum_{k=1}^{K}\frac{1}{(1+R^{-})^{k}}\delta F_{k}^{+}-\delta F_{0}^{-}+\frac{\partial f}{\partial R}^{+}\delta R^{-}$$

Example We consider the same example as in [9] where temporal horizon is composed of three periods, the fuzzy risk-free interest rate is (0.09, 0.1, 0.11) and the four fuzzy cash flows are

$$\begin{aligned} F_0 &= \quad (-900, 1000, 1100) \\ F_1 &= \quad (-90, 100, 110) \\ F_2 &= \quad (-180, 200, 220) \\ F_3 &= \quad (-1800, 2000, 2200) \end{aligned}$$

Figure 9 shows the calculation of the fuzzy NPV.

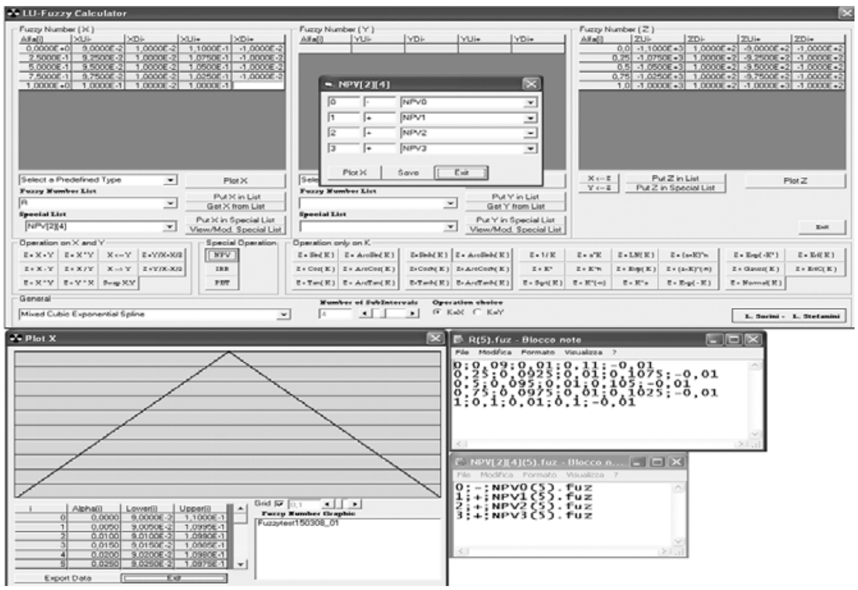

Figure 9: Computation of fuzzy NPV.

In the table we report the computational result for the NPV, in particular we show that it exists an asymmetry produced by computations even if the inputs of the problem are symmetric: the asymmetry analysis can be carried on because of the properties of the LU representation.

α	NPV$^-$	NPV$^+$	δNPV$^-$	δNPV$^+$	Asym
1	758.8	320.6	758.8	-320.69	0%
0.75	678.9	318.1	839.3	-323.3	0,83%
0.5	599.7	315.5	920.5	-326.1	1,66%
0.25	521.2	312.9	1002.3	-328.7	2,5%
0	443.31	310.4	1084.9	-331.5	3,34%

The comparison between various projects is fundamental for a good decision making and it is based on the concept of ranking functions. In particular, the ordering of fuzzy numbers can be approached in many ways; in most cases, the fuzzy numbers u are transformed into real numbers. By the use of the LU-fuzzy representation, the ranking functions can be computed, numerically or analytically, in terms of the parameters $u_i^{\pm}$ and $\delta u_i^{\pm}$ that define u.

The slopes in the LU-representation are the sensitivity parameters of NPV with respect of the input data. This aspect strengthen the interpretation of fuzzy models as extension of the interval analysis and as a guide to modulate the sensitivity.

We test the fuzzy version of the three criterion on some simulated situations and the first relevant aspect is that the fuzziness of the interest rate can overcome the problems of a fixed rate that put in agreement everybody involved in the decision process.

Considered the weakness of the second criteria, the elaboration of the fuzzy internal rate is postponed for a future work in order to find a robust methodology in the crisp case.

4 Fuzzy option pricing

Concerning the financial models for option pricing, we extensively analyze the introduction of uncertainty through fuzzy numbers in [7] and [6], where many references to several analogous studies can be found. In the fundamental Black and Scholes option pricing model, the underlying stock price process $\{S_t\}_{t\geq 0}$ evolves in a stochastic way as:

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where the risk-free interest rate r and the volatility σ are constant, the strike price X and the time to maturity T are fixed. The call option price is defined as:

$$C(S, r, \sigma, X, T) = SN(d_1) - Xe^{-rT}N(d_2) \tag{7}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + rT}{\sigma\sqrt{T}} + \frac{\sigma}{2}\sqrt{T} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

and $N(x)$ is the cumulated normal distribution function.

It is widespread known that real facts contradict the possibility to have constant volatility and interest rates; that is why in the recent years many models have been introduced to define stochastic behaviors. The practical application of some models, however, can presents technicalities sometimes difficult to solve. The approach we choose to model the volatility and interest rate uncertainty is the fuzzy mathematics.

In the function C three parameters are handled as fuzzy numbers due to the uncertainty about their true values; they are the underlying stock price, the volatility and the risk-free interest rate:

$$S = \left(S_i^-, \delta S_i^-, S_i^+, \delta S_i^+\right)_{i=0,1,\ldots,N}$$

$$\sigma = \left(\sigma_i^-, \delta\sigma_i^-, \sigma_i^+, \delta\sigma_i^+\right)_{i=0,1,\ldots,N}$$

$$r = \left(r_i^-, \delta r_i^-, r_i^+, \delta r_i^+\right)_{i=0,1,\ldots,N}$$

and the fuzzy extension of C has the form:

$$C = \left(C_i^-, \delta C_i^-, C_i^+, \delta C_i^+\right)_{i=0,1,\ldots,N}$$

based on the LU-parametric representation.

The fuzzy option pricing model can compute call prices for all the possible values of volatility, interest rate, underlying price, strike price and time to maturity. The values can be obtained from real data or by simulation.

The flexibility of the fuzzy numbers to describe uncertainty is evident; in fact, subjective opinions can be reflected in the shape of fuzzy numbers. The present financial situation, for example, can persuade the investor that the future interest rates will not move away from today's value or that the future volatility will possibly have smaller values.

Many experiments show that the shape of the resulting fuzzy option price is greatly influenced by the underlying price: if the perceived uncertainty about the behavior of the underlying price is large then the same uncertainty will reflect in the call option price.

Regarding the asymmetry of the shape of an input fuzzy number, Figure 11 shows a possible situation in which the investor believes that variations in volatility have a higher degree of possibility for bigger values and a lower one for smaller values.

The fuzzy call option price is weakly influenced by the asymmetry in volatility and strongly influenced by the shape of the underlying and interest rate. So it seems excessive only to focus the research on the volatility structure, rather then on the remaining parameters.

In [3] and more extensively in [4] we model the parameters in the Heston stochastic volatility model as fuzzy numbers. In particular we show that the parameter that gives the biggest contribution in the uncertainty propagation is the initial level of the variance of the stochastic volatility. It implies that, when finding an option price, a careful study must be devoted to a good valuation of the initial volatility.

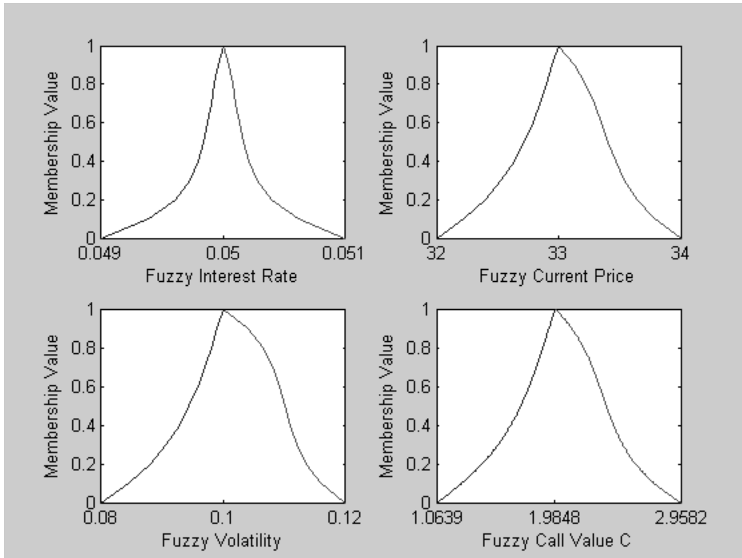

Figure 10: Computation of the fuzzy call option price.

5 Conclusions

Incorporating uncertainty in financial models is a primary requirement in order to validate the hypothesis with real data. The well known probability approach is one of the possible choice but the stochastic nature of some variables may produce some technicalities or difficulties that may be hard to solve. We believe that a major interest could be devoted to an alternative way to model uncertainty based on the definition of parameters or variables that vary in intervals or that are defined as fuzzy numbers. The fuzzy mathematics introduced in [14] has a long history and it has been investigated properly in many directions; it follows that a rigorous framework can be the scenario for future in-depth examination. The strength of fuzzy numbers in social sciences is stressed when one or more sources of uncertainty have to be encompassed in a model. When the value of a parameter is not known you can estimate it, choosing a statistical approach, or you can model it as a fuzzy number that varies its values depending of increasing orders of possibility; different values and shapes of the fuzzy number can reflect the uncertainty about its value by including, however, subjective ideas about the fuzzy number.

One of our nearest research area is the risk management and in particular the determination of fuzzy classes for the modelling of ratings and the definition of a fuzzy Value at Risk to measure the portfolio risk.

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