Applied Mathematical Sciences, Vol. 6, 2012, no. 67, 3339 - 3345

Optimization of the Difference of Increasing and Radiant Functions

H. Mohebi

Department of Mathematics
Shahid Bahonar University of Kerman
and Islamic Azad University, Jiroft Branch, Iran
hmohebi@uk.ac.ir

M. Esmaeili

Department of Mathematics
Islamic Azad University
Jiroft Branch, Jiroft, Iran
esmaeili1355@gmail.com

Abstract

In this paper, we present necessary and sufficient conditions for the global minimum of the difference of non-positive strictly increasing and radiant functions. Also, a characterization of the dual problem for this class of functions is presented.

Mathematics Subject Classification: 90C46, 26A48

Keywords: Global optimization, monotonic analysis, abstract convexity, increasing and radiant function, support set, subdifferential

1 Introduction

Recently many authors have discussed the theoretical development of optimality conditions for certain classes of global optimization problems (see [3, 4, 7]). One of the most important global optimization problems is that of minimizing a DC function (difference of two convex functions) that is

$$\text{minimize } h(x) \text{ subject to } x \in X,$$

where $h(x) = f(x) - g(x)$ and $f,\ g$ are convex functions. In a general case, DC functions can be replaced by DAC functions (difference of two abstract convex functions), in particular, the minimizing of the difference of two increasing and co-radiant (ICR) functions and also the minimizing of the difference of two increasing and convex along rays functions (see, for example, [2, 5, 6]). In this paper, we replace f and g by non-positive increasing and radiant functions (IR functions) and we present a necessary and sufficient condition for the global minimum of h. We also outline a dual approach to the study of the global optimization problem for these functions. Our approach is based on the Toland-Singer formula and some results were obtained in [1].

2 Preliminaries

Let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$.
A function $f : X \longrightarrow [-\infty, +\infty]$ is called radiant if $f(\lambda x) \leq \lambda f(x)$ for all $x \in X$ and all $\lambda \in (0, 1]$. It is easy to see that f is radiant if $f(\lambda x) \geq \lambda f(x)$ for all $x \in X$ and all $\lambda \geq 1$. The function f is called increasing if $x \geq y \implies f(x) \geq f(y)$. A function $f : X \longrightarrow [-\infty, +\infty]$ is called IR if f is increasing and radiant.
Let X be a set and L be a set of real valued functions $l : X \longrightarrow \mathbb{R}$, which will be called abstract linear. For each $l \in L$ and $c \in \mathbb{R}$, consider the shift $h_{l,c}$ of l on the constant $c : h_{l,c}(x) := l(x) - c, \quad x \in X$. The function $h_{l,c}$ is called L-affine. Recall (see [9]) that the set L is called a set of abstract linear functions if $h_{l,c} \notin L$ for all $l \in L$ and all $c \in \mathbb{R} \setminus \{0\}$. The set of all L-affine functions will be denoted by H_L.
A function $f : X \longrightarrow (-\infty, +\infty]$ is called proper if $\operatorname{dom} f \neq \emptyset$, where $\operatorname{dom} f$ is defined by $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. Let $\mathcal{F}(X)$ be the union of all functions $f : X \longrightarrow (-\infty, +\infty]$ and the function $-\infty$.
Recall (see [9]) that a function $f \in \mathcal{F}(X)$ is called H-convex ($H = L$, *or* $H = H_L$) if $f(x) = \sup\{h(x) : h \in \text{supp } (f, H)\}$ for all $x \in X$, where $\text{supp } (f, H) := \{h \in H : h \leq f\}$ is called the support set of the function f, and $h \leq f$ if and only if $h(x) \leq f(x)$ for all $x \in X$. For a function $f \in \mathcal{F}(X)$, define the Fenchel-Moreau L-conjugate f^* of f (see [9]) by $f^*(l) := \sup_{x \in X}[l(x) - f(x)], \ (l \in L)$.
Let $f : X \longrightarrow (-\infty, +\infty]$ be a function and $x_0 \in dom f$. Recall (see [9]) that an element $l \in L$ is called an L-subgradient of f at x_0 if $f(x) \geq f(x_0) + l(x) - l(x_0)$ for all $x \in X$. The set $\partial_L f(x_0)$ of all L-subgradients of f at x_0 is called L-subdifferential of f at x_0.
Now, consider the function $u : X \times X \times (-\infty, 0) \longrightarrow [-\infty, 0]$ is defined by:

$$u(x, y, \beta) := \sup\{\lambda \leq \beta : \lambda y \geq -x\}, \quad (x,\ y \in X;\ \beta \in (-\infty, 0)), \tag{2.1}$$

(we use the convention $\sup \emptyset = -\infty$).

The function u was introduced and examined in [8]. In the sequel, for each $y \in X$ and each $\beta < 0$, we consider the function $u_{(y,\beta)} : X \to [-\infty, 0]$ is defined by $u_{(y,\beta)}(x) := u(x, y, \beta)$ for all $x \in X$, and set $L := \{u_{(y,\beta)} \; : \; y \in X, \; \beta < 0\}$. It is easy to check that L is a set of IR functions. In the following, we gather some results for non-positive IR functions which will be used later.

Proposition 2.1. *([8], Proposition 4.1) Let $f : X \to [-\infty, 0]$ be an IR function. Then*

$$supp\ (f, L) = \{u_{(y,\beta)} \in L \; : \; f(-\beta y) \geq \beta\}. \tag{2.2}$$

Theorem 2.1. *([8], Theorem 4.1) Let $f : X \to [-\infty, 0]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \neq -\infty$. Then*

$$D := \{u_{(y,\beta)} \in L : f(x_0) \geq u_{(y,\beta)}(x_0), \beta - u_{(y,\beta)}(x_0) \leq f(-\beta y) - f(x_0)\} \subset \partial_L f(x_0). \tag{2.3}$$

Moreover, the equality holds if and only if $f(x_0) = 0$.

3 Dual Optimality Conditions

Let $f, \; g : X \to [-\infty, +\infty]$ be proper functions. Let $h := f - g$. Now, consider the following extremal problem:

$$h(x) \to min \;\text{ subject to } x \in X. \tag{3.1}$$

Clearly, if $\inf_{x \in X} h(x) = -\infty$, then (3.1) is trivial. Therefore, we consider $\inf_{x \in X} h(x) > -\infty$. Now, consider the following problem:

$$g^*(l) - f^*(l) \to min \quad \text{subject to } l \in \; \operatorname{dom} f^*. \tag{3.2}$$

The problem defined by (3.2) is called the dual problem with respect to (3.1). In the following, we give the well-known Toland-Singer formula (see [10, 11]).

Theorem 3.1. *Let Z be a set and U be a set of real valued abstract linear functions defined on Z. Let $f, \; g : Z \to (-\infty, +\infty]$ be proper H_U-convex functions such that* $\inf_{x \in Z}[f(x) - g(x)] > -\infty$. *Then,* $\inf\{f(x) - g(x) \; : \; x \in Z\} = \inf\{g^*(l) - f^*(l) \; : \; l \in U\}$.

So the following result can be obtained directly from Theorem 3.1.

Proposition 3.1. *Let $f, \; g : X \to [-\infty, 0]$ be proper IR functions such that $inf_{x \in X}[f(x) - g(x)] > -\infty$. Then,* $\inf\{f(x) - g(x) \; : \; x \in X\} = \inf\{g^*(u_{(y,\beta)}) - f^*(u_{(y,\beta)}) \; : \; u_{(y,\beta)} \in L\}$.

Lemma 3.1. *Let $f,\ g : X \to [-\infty, 0]$ be IR functions, $x \in X$ be such that $f(x) + g(x) > -\infty$, and let $\epsilon < \min\{f(x),\ g(x)\}$ be arbitrary. Then, $u_{(\frac{x}{-\epsilon},\epsilon)} \in \partial_L f(x) \cap \partial_L g(x)$. Moreover, $g^*(u_{(\frac{x}{-\epsilon},\epsilon)}) - f^*(u_{(\frac{x}{-\epsilon},\epsilon)}) = f(x) - g(x)$.*

Proof: Since $f(x) > \epsilon$ and by the definition of $u_{(y,\beta)}$ one has $u_{(\frac{x}{-\epsilon},\epsilon)}(x) = \epsilon$, then in view of Theorem 2.1 we obtain $u_{(\frac{x}{-\epsilon},\epsilon)} \in \partial_L f(x)$. By a similar argument we have $u_{(\frac{x}{-\epsilon},\epsilon)} \in \partial_L g(x)$. Now, since $u_{(\frac{x}{-\epsilon},\epsilon)} \in \partial_L f(x)$, then by Fenchel-Young equality we have $f^*(u_{(\frac{x}{-\epsilon},\epsilon)}) = u_{(\frac{x}{-\epsilon},\epsilon)}(x) - f(x)$. Therefore, we conclude that

$$\begin{aligned} g^*(u_{(\frac{x}{-\epsilon},\epsilon)}) - f^*(u_{(\frac{x}{-\epsilon},\epsilon)}) &= u_{(\frac{x}{-\epsilon},\epsilon)}(x) - g(x) - u_{(\frac{x}{-\epsilon},\epsilon)}(x) + f(x) \\ &= f(x) - g(x), \end{aligned}$$

which completes the proof. ■

Theorem 3.2. *Let $f,\ g : X \to [-\infty, 0]$ be proper IR functions such that $\inf_{x\in X}[f(x) - g(x)] > -\infty$. Let $x_0 \in X$ and $\varepsilon < \min\{f(x_0),\ g(x_0)\}$. Then, x_0 is a global minimizer of the problem (3.1) if and only if $u_{(\frac{x_0}{-\varepsilon},\varepsilon)}$ is a global minimizer of the problem (3.2).*

Proof: Suppose that x_0 is a global minimizer of the problem (3.1). Since $\varepsilon < \min\{f(x_0),\ g(x_0)\}$, then, by Lemma 3.1 and Proposition 3.1 we get

$$\begin{aligned} g^*(u_{(\frac{x_0}{-\varepsilon},\varepsilon)}) - f^*(u_{(\frac{x_0}{-\varepsilon},\varepsilon)}) &= f(x_0) - g(x_0) \\ &= \inf_{x\in X}[f(x) - g(x)] \\ &= \inf\{g^*(u_{(y,\beta)}) - f^*(u_{(y,\beta)})\ :\ u_{(y,\beta)} \in L\}. \end{aligned}$$

Hence $u_{(\frac{x_0}{-\varepsilon},\varepsilon)}$ is a global minimizer of the problem (3.2). Conversely, assume that $u_{(\frac{x_0}{-\varepsilon},\varepsilon)}$ is a global minimizer of the problem (3.2). Then, it follows from Lemma 3.1 and Proposition 3.1 that x_0 is a global minimizer of the problem (3.1). ■

4 Necessary and Sufficient Conditions

In this section, we present necessary and sufficient conditions for the global minimum of the difference of strictly non-positive IR functions. First, consider the function $h := g - f$, where $f,\ g : X \to [-\infty, +\infty]$ are proper functions. Let $\eta := inf_{x\in X} h(x) > -\infty$. This implies that $f(x) \leq g(x) - \eta,\quad \forall\ x \in X$. Let $\tilde{g}(x) := g(x) - \eta$. It is easy to see that $f(x) \leq \tilde{g}(x)$ for all $x \in X$ if and only if supp $(f, L) \subset$ supp $(\tilde{g}, L)$, or equivalently, x_0 is a global minimizer of the function h if and only if

$$\text{supp } (f, L) \subset \text{supp } (\tilde{g}, L). \tag{4.1}$$

Now, consider a set U of functions defined on a set Z. We assume that U is equipped with the natural (pointwise) order relation. Recall (see [9]) that a function f is called a maximal element of the set U, if $f \in U$ and $\bar{f} \in U,\ \bar{f}(x) \geq f(x)\ for\ all\ x \in Z \implies \bar{f} = f$. We now concentrate on the support set of IR functions and we obtain some results which will be used later.

Proposition 4.1. *Let $f : X \to [-\infty, 0]$ be an IR function and let $u_{(y,\beta)} \in$ supp (f, L). Assume that $u_{(y,\beta)}$ is a maximal element of supp (f, L). Then, $f(-\beta y) = \beta$.*

Proof: Let $u_{(y,\beta)} \in$ supp (f, L). Then, by Proposition 2.1, we have $f(-\beta y) \geq \beta$. Consider $u_{(\frac{-\beta y}{-f(-\beta y)}, f(-\beta y))} \in L$. Then, in view of the definition of $u_{(y,\beta)}$ we conclude that $u_{(\frac{-\beta y}{-f(-\beta y)}, f(-\beta y))}(-\beta y) = f(-\beta y)$. Since $f(-\beta y) \geq \beta$, it follows from Proposition 2.1 that $u_{(\frac{-\beta y}{-f(-\beta y)}, f(-\beta y))} \in$ supp (f, L). Also, by using $f(-\beta y) \geq \beta$ and the definition of $u_{(y,\beta)}$ one has

$$u_{(y,\beta)}(x) \leq u_{(\frac{-\beta y}{-f(-\beta y)}, f(-\beta y))}(x), \quad \forall\ x \in X. \tag{4.2}$$

Since $u_{(y,\beta)}$ is a maximal element of supp (f, L), then by (4.2) we obtain

$$u_{(y,\beta)}(x) = u_{(\frac{-\beta y}{-f(-\beta y)}, f(-\beta y))}(x), \quad \forall\ x \in X. \tag{4.3}$$

Put $x := -\beta y$ in (4.3), we get $f(-\beta y) = \beta$. ■

Remark 4.1. *The converse statement of Proposition 4.1 is not valid. Consider IR function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in \mathbb{R}$. It follows from Proposition 2.1 that $u_{(-1,\beta)} \in$ supp (f, L) and $f(\beta) = \beta$ for all $\beta < 0$. But the maximal element of the support set of f dose not exist.*

In the following, we show by extra conditions that the converse statement of Proposition 4.1 holds. The proof of the following result is similar to the one of Proposition 4.2 in [2], and so we omit it.

Proposition 4.2. *Let $f : X \to [-\infty, 0]$ be a strictly IR function. Let $y \in X$ be such that $\varepsilon_y := \max\{\beta < 0\ :\ f(-\beta y) \geq \beta\} < 0$. Then, $u_{(y,\varepsilon_y)}$ is a maximal element of the support set of f if and only if $f(-\varepsilon_y y) = \varepsilon_y$.*

Corollary 4.1. *Let $f : X \to [-\infty, 0]$ be a strictly IR function such that $\varepsilon_y := \max\{\beta < 0\ :\ f(-\beta y) \geq \beta\} < 0$ $(y \in X)$. Then for each $u_{(y,\beta)} \in$ supp (f, L) there exists a maximal element $u_{(\tilde{y},\tilde{\beta})}$ of the support set of f such that $u_{(y,\beta)} \leq u_{(\tilde{y},\tilde{\beta})}$. In this case, we have $\tilde{y} = \frac{-\varepsilon_y y}{-f(-\varepsilon_y y)}$, and $\tilde{\beta} = f(-\varepsilon_y y)$.*

Proof: This is an immediate consequence of Proposition 4.1 and Proposition 4.2. ■

Theorem 4.1. *Let $f,\ g : X \to [-\infty, 0]$ be strictly IR functions such that $\varepsilon_y := \max\{\beta < 0 \ : \ f(-\beta y) \geq \beta\} < 0$, and $\eta_z := \max\{\beta < 0 \ : \ g(-\beta z) \geq \beta\} < 0$ $(y,\ z \in X)$. Then the following assertions are equivalent:*
(i) supp $(f, L) \subset$ supp (g, L).
(ii) For each maximal element $u_{(y,\varepsilon_y)}$ of supp (f, L) there exists a maximal element $u_{(z,\eta_z)}$ of supp (g, L) such that $u_{(y,\varepsilon_y)}(x) \leq u_{(z,\eta_z)}(x)$ for all $x \in X$.

Proof: $(i) \Rightarrow (ii)$. Suppose that supp $(f, L) \subset$ supp (g, L). Let $u_{(y,\varepsilon_y)}$ be a maximal element of supp (f, L), then $u_{(y,\varepsilon_y)} \in$ supp (g, L). Thus, by Corollary 4.1 there exists a maximal element $u_{(z,\eta_z)}$ of supp (g, L) such that $u_{(y,\varepsilon_y)}(x) \leq u_{(z,\eta_z)}(x)$ for all $x \in X$.
$(ii) \Rightarrow (i)$. Let $u \in$ supp (f, L) be arbitrary. Then by Corollary 4.1 there exists a maximal element $u_{(y,\varepsilon_y)}$ of supp (f, L) such that $u \leq u_{(y,\varepsilon_y)}$. Therefore, by the hypothesis (ii) there exists a maximal element $u_{(z,\eta_z)} \in$ supp (g, L) such that $u_{(y,\varepsilon_y)} \leq u_{(z,\eta_z)}$. Then, $u_{(z,\eta_z)} \geq u$, and hence $u \in$ supp (g, L). This completes the proof. ∎

In the following, we present necessary and sufficient conditions for the global minimum of the difference of strictly IR functions.

Theorem 4.2. *Let $f,\ g : X \to [-\infty, 0]$ be strictly IR functions such that $f(x) \leq g(x)$ for all $x \in X$. Then, $x_0 \in X$ is a global minimizer of the function $h := g - f$ if and only if for each $y \in X$ with $f(-\varepsilon_y y) = \varepsilon_y$ there exists $z \in X$ with $\tilde{g}(-\eta_z z) = \eta_z$ such that $u_{(y,\varepsilon_y)} \leq u_{(z,\eta_z)}$, where $\tilde{g}(x) := g(x) - h(x_0)$ for all $x \in X$, $\varepsilon_y := \max\{\beta < 0 \ : \ f(-\beta y) \geq \beta\} < 0$ and $\eta_z := \max\{\beta < 0 \ : \ \tilde{g}(-\beta z) \geq \beta\} < 0$ $(y,\ z \in X)$. It is worth noting that since $h(x_0) \geq 0$, then one has $\tilde{g}$ is a strictly IR function.*

Proof: Due to (4.1), x_0 is a global minimizer of the function h if and only if supp $(f, L) \subset$ supp $(\tilde{g}, L)$. Now, the result follows from Proposition 4.2 and Theorem 4.1. ∎

Acknowledgment. This research was supported by Islamic Azad University, Jiroft Branch.

References

[1] A. R. Doagooei and H. Mohebi, *Monotonic analysis over ordered topological vector spaces: IV*, Journal of Global Optimization, 45 (2009), 355-369.

[2] A. R. Doagooei and H. Mohebi, *Optimization of the difference of ICR functions,* Nonlinear Analysis, 71 (2009), 4493-4499.

[3] B. M. Glover and V. Jeyakumar, *Nonlinear extensions of Farkas' lemma with applications to global optimization and least squares,* Mathematics of Operations Research, 20 (1995), 818-837.

[4] B. M. Glover, Y. Ishizuka, V. Jeyakumar and A. M. Rubinov, *Inequality systems and global optimization,* Journal of Mathmatical Analysis and Applications, 202 (1996), 900-919.

[5] B. M. Glover and A. M. Rubinov, *Increasing convex along rays functions with applications to global optimization,* Journal of Optimization Theory and Applications, 102 (1999), 615-642.

[6] B. M. Glover and A. M. Rubinov, *Toland-Singer formula cannot distinguish a global minimizer from a choice of stationary points,* Numerical Functional Analysis and Optimization, 20 (1999), 99-119.

[7] J. B. Hiriart, *From convex to nonconvex minimization: necessary and sufficient conditions for global optimality,* in Nonsmooth Optimization and Related Topics, Plenum, New York (1989), 219-240.

[8] H. Mohebi, *Abstract convexity of radiant functions with applications,* to appear.

[9] A. M. Rubinov, *Abstract convexity and global optimization*, Kluwer Academic Publishers, Boston, Dordrecht, London, 2000.

[10] I. Singer, *Abstract convex analysis,* Wiley-Interscience, New York, 1997.

[11] J. F. Toland, *Duality in nonconvex optimization,* Journal of Mathmatical Analysis and Applications, 66 (1978), 399-415.

Received: February, 2012