Commutative $\mathcal{N}$-Ideals Based on a Sub-$BCK$-Algebra of a $BCI$-Algebra

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Abstract
Based on a sub-$BCK$-algebra $K$ of a $BCI$-algebra $X$, the notion of commutative $\mathcal{N}$-ideals of $X$ is introduced. Relations between a commutative $\mathcal{N}$-ideal and an $\mathcal{N}$-ideal are investigated.

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1 Introduction
A (crisp) set $A$ in a universe $X$ can be defined in the form of its characteristic function $\mu_A : X \rightarrow \{0, 1\}$ yielding the value 1 for elements belonging to the set $A$ and the value 0 for elements excluded from the set $A$. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative

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information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [4] introduced a new function which is called negative-valued function, and constructed \( N \)-structures. They discussed \( N \)-subalgebras and \( N \)-ideals in BCK/BCI-algebras. Using a sub-\( BCK \)-algebra \( K \) of a \( BCI \)-algebra \( X \) and a number \( \rho \in [-1, 0] \), Jun et al. [3] introduced the notions of \( N(K, \rho) \)-subalgebras and \( N(K, \rho) \)-ideals in \( BCI \)-algebras. They investigated their properties, and showed that these two notions are independent each other by providing examples.

In this paper, we introduce the notion of commutative \( N(K, \rho) \)-ideal of a \( BCI \)-algebra \( X \), and discuss the relation between an \( N(K, \rho) \)-ideal and a commutative \( N(K, \rho) \)-ideal. Conditions for an \( N(K, \rho) \)-ideal to be a commutative \( N(K, \rho) \)-ideal are provided.

2 Preliminaries

Let \( K(\tau) \) be the class of all algebras with type \( \tau = (2, 0) \). By a \( BCI \)-algebra we mean a system \( X := (X, *, 0) \in K(\tau) \) in which the following axioms hold:

(a1) \( ((x * y) * (x * z)) * (z * y) = 0 \),

(a2) \( (x * (x * y)) * y = 0 \),

(a3) \( x * x = 0 \),

(a4) \( x * y = y * x = 0 \implies x = y \)

where \( x, y \) and \( z \) are elements of \( X \). If a \( BCI \)-algebra \( X \) satisfies \( 0 * x = 0 \) for all \( x \in X \), then we say that \( X \) is a \( BCK \)-algebra. We can define a partial ordering \( \preceq \) by

\[ (\forall x, y \in X) (x \preceq y \iff x * y = 0) \] .

In a \( BCK/BCI \)-algebra \( X \), the following hold:

(b1) \( x * 0 = x \),

(b2) \( (x * y) * z = (x * z) * y \),

where \( x, y \) and \( z \) are elements of \( X \). A \( BCK \)-algebra \( X \) is said to be commutative if it satisfies the following equality:

\[ x * (x * y) = y * (y * x) \tag{2.1} \]

where \( x \) and \( y \) are elements of \( X \).
A non-empty subset $S$ of a $BCK/BCI$-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$. A subset $A$ of a $BCK/BCI$-algebra $X$ is called an ideal of $X$ if it satisfies:

$$0 \in A,$$

$$x \ast y \in A, y \in A \Rightarrow x \in A \tag{2.2}$$

where $x$ and $y$ are elements of $X$. A subset $A$ of a BCK-algebra $X$ is called a commutative ideal of $X$ (see [5]) if it satisfies (2.2) and

$$(x \ast y) \ast z \in A, z \in A \Rightarrow x \ast (y \ast (y \ast x)) \in A \tag{2.3}$$

where $x, y$ and $z$ are elements of $X$.

We refer the reader to the books [1] and [5] for further information regarding $BCK/BCI$-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $F(X, [-1, 0])$ the collection of functions from a set $X$ to $[-1, 0]$. We say that an element of $F(X, [-1, 0])$ is a negative-valued function from $X$ to $[-1, 0]$ (briefly, $N$-function on $X$). By an $N$-structure we mean an ordered pair $(X, f)$ of $X$ and an $N$-function $f$ on $X$.

**Definition 2.1 ([4])**. By a subalgebra of a $BCK/BCI$-algebra $X$ based on $N$-function $f$ (briefly, $N$-subalgebra of $X$), we mean an $N$-structure $(X, f)$ in which $f$ satisfies the following condition:

$$f(x \ast y) \leq \bigvee\{f(x), f(y)\} \tag{2.5}$$

where $x$ and $y$ are elements of $X$.

**Definition 2.2 ([4])**. By an ideal of a $BCK/BCI$-algebra $X$ based on $N$-function $f$ (briefly, $N$-ideal of $X$), we mean an $N$-structure $(X, f)$ in which $f$ satisfies the following condition:

$$f(0) \leq f(x) \leq \bigvee\{f(x \ast y), f(y)\} \tag{2.6}$$

where $x$ and $y$ are elements of $X$. 
Table 1: BCK-operation

<table>
<thead>
<tr>
<th>*_X</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

For any \( \mathcal{N} \)-structure \((X, f)\) and \( \alpha \in [-1, 0] \), the set

\[
C(f; \alpha) := \{ x \in X \mid f(x) \leq \alpha \}
\]

is called the closed support of \((X, f)\) related to \( \alpha \).

Proposition 2.3 ([4]). An \( \mathcal{N} \)-structure \((X, f)\) is an \( \mathcal{N} \)-subalgebra (resp. ideal) of a BCK/BCI-algebra \( X \) if and only if every closed support of \((X, f)\) related to \( \alpha \) is a subalgebra (resp. ideal) of \( X \) for all \( \alpha \in [-1, 0] \).

For our convenience, the empty set \( \emptyset \) is regarded as a subalgebra (resp. ideal) of \( X \).

3 Commutative \( \mathcal{N} \)-ideals based on a sub-BCK-algebra

Definition 3.1 ([3]). Let \( \varrho \in [-1, 0] \) and let \( K \) be a sub-BCK-algebra of a BCI-algebra \( X \). An \( \mathcal{N} \)-structure \((X, f)\) is called an \( \mathcal{N} \)-ideal of \( X \) based on \( K \) and \( \varrho \) (briefly, \( \mathcal{N}(K, \varrho) \)-ideal of \( X \)) if it satisfies:

\[
(\forall x \in K) (\forall y \in X \setminus K) (f(0) \leq f(x) \leq \varrho \leq f(y)).
\]  
(3.1)

\[
(\forall x, y \in K) (f(x) \leq \vee \{f(x \ast y), f(y)\}).
\]  
(3.2)

Definition 3.2. Let \( \varrho \in [-1, 0] \) and let \( K \) be a sub-BCK-algebra of a BCI-algebra \( X \). An \( \mathcal{N} \)-structure \((X, f)\) is called a commutative \( \mathcal{N} \)-ideal of \( X \) based on \( K \) and \( \varrho \) (briefly, commutative \( \mathcal{N}(K, \varrho) \)-ideal of \( X \)) if it satisfies (3.1) and

\[
(\forall x, y, z \in K) (f(x \ast (y \ast (y \ast x))) \leq \vee \{f((x \ast y) \ast z), f(z)\}).
\]  
(3.3)

Example 3.3. Let \((X; \ast_X, 0)\) be the BCK-algebra whose elements are \( X = \{0, a, b, c\} \) and whose BCK-operation is given by the Cayley table (see Table 1). For any group \( G \) with identity \( e \), let \( Y = (G \setminus \{e\}) \cup X \). We define the operation \( \ast \) on \( Y \) by the following way:
Table 2: BCK-operation

<table>
<thead>
<tr>
<th>*ₖ</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
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<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) for \(x, y \in G \setminus \{e\}\), we put
\[
x \ast y = \begin{cases} \ x y \in G \setminus \{e\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}
\]

(2) for \(x, y \in X\), we put \(x \ast y = x \ast_X y\) in \(X\),

(3) for \(x \in G \setminus \{e\}\) and \(y \in X\), we put \(x \ast y = y \ast x = x\).

Then \((Y; \ast, 0)\) is a BCI-algebra (see [2]) and \((X; \ast, 0)\) is a sub-BCK-algebra of \(Y\). Let \((X, f)\) be an \(\mathcal{N}\)-structure in which \(f\) is defined by \(f(0) = -0.9\), \(f(a) = f(b) = -0.6\), \(f(c) = -0.8\) and \(f(x) = -0.4\) for all \(x \in G \setminus \{e\}\). Then \((X, f)\) is a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(Y\) for \(\varrho \in [-0.6, -0.4]\) where \(K = X\).

**Theorem 3.4.** Let \(\varrho \in [-1, 0]\) and let \(K\) be a sub-BCK-algebra of a BCI-algebra \(X\). Then every commutative \(\mathcal{N}(K, \varrho)\)-ideal is a \(\mathcal{N}(K, \varrho)\)-ideal.

**Proof.** Let \((X, f)\) be a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(X\). Taking \(y = 0\) in (3.3) induces (3.2), and so \((X, f)\) is an \(\mathcal{N}(K, \varrho)\)-ideal of \(X\). \(\square\)

The following example shows that the converse of Theorem 3.4 may not be true.

**Example 3.5.** Consider the BCK-algebra \(K = \{0, a, b, c\}\) with the operation \(*ₖ\) which is given by the Table 2. For any group \(G\) with identity \(e\), let \(Y = (G\setminus \{e\}) \cup X\) and define the operation \(\ast\) on \(Y\) by the similar way to Example 3.3. Then \((Y; \ast, 0)\) is a BCI-algebra and \((K; \ast, 0)\) is a sub-BCK-algebra of \(Y\). Let \((Y, f)\) be an \(\mathcal{N}\)-structure in which \(f\) is defined by \(f(0) = -0.8\), \(f(a) = -0.5\), \(f(b) = -0.4\), \(f(c) = -0.6\) and \(f(x) = -0.3\) for all \(x \in G \setminus \{e\}\). Then \((Y, f)\) is an \(\mathcal{N}(K, \varrho)\)-ideal of \(Y\) for \(\varrho \in [0.3, 0.4]\), but \((Y, f)\) is not a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(Y\) since \(f(a \ast (b \ast (b \ast a))) = -0.5 \not\geq -0.6 = \vee \{f((a \ast b) \ast c), f(c)\}\).

**Proposition 3.6.** Let \(\varrho \in [-1, 0]\) and let \(K\) be a sub-BCK-algebra of a BCI-algebra \(X\). Then every commutative \(\mathcal{N}(K, \varrho)\)-ideal \((X, f)\) of \(X\) satisfies the following inequality:
\[
(\forall x, y \in K) \quad f(x \ast (y \ast (y \ast x))) \leq f(x \ast y)). \tag{3.4}
\]
Proof. Let \((X, f)\) be a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(X\). Taking \(z = 0\) in (3.3) and using (3.1) and (b1) induces the desired result. \(\square\)

We provide a condition for an \(\mathcal{N}(K, \varrho)\)-ideal to be a commutative \(\mathcal{N}(K, \varrho)\)-ideal.

**Theorem 3.7.** Let \(\varrho \in [-1, 0]\) and let \(K\) be a sub-BCK-algebra of a BCI-algebra \(X\). If an \(\mathcal{N}(K, \varrho)\)-ideal \((X, f)\) satisfies the condition (3.4), then \((X, f)\) is a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(X\).

**Proof.** Let \((X, f)\) be an \(\mathcal{N}(K, \varrho)\)-ideal of \(X\) that satisfies the condition (3.4). Using (3.4) and (3.2), we have

\[ f(x \ast (y \ast (y \ast x))) \leq f(x \ast y) \leq \{f((x \ast y) \ast z), f(z)\} \]

for all \(x, y, z \in K\). Hence \((X, f)\) is a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(X\). \(\square\)

**Lemma 3.8 ([5]).** An ideal \(A\) of a BCK-algebra \(X\) is commutative if and only if it satisfies:

\[ (\forall x, y \in X) \ (x \ast y \in A \Rightarrow x \ast (y \ast (y \ast x)) \in A). \]

**Lemma 3.9 ([3]).** Let \(\varrho \in [0, 1]\) and let \(K\) be a sub-BCK-algebra of a BCI-algebra \(X\). If \((X, f)\) is an \(\mathcal{N}(K, \varrho)\)-ideal of \(X\), then

1. \(K \subseteq C(f; \varrho)\).
2. \((\forall \beta \in [-1, 0])(\beta < \varrho \Rightarrow C(f; \varrho)\) is an ideal of \(K\).

**Theorem 3.10.** Let \(\varrho \in [0, 1]\) and let \(K\) be a sub-BCK-algebra of a BCI-algebra \(X\). If \((X, f)\) is a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(X\), then \(K \subseteq C(f; \varrho)\) and \(C(f; t)\) is a commutative ideal of \(K\) for all \(t \in [-1, 0]\) with \(t > \varrho\).

**Proof.** If \((X, f)\) is a commutative \(\mathcal{N}(K, \varrho)\)-ideal of \(X\), then \((X, f)\) is an \(\mathcal{N}(K, \varrho)\)-ideal of \(X\) by Theorem 3.4. It follows from Lemma 3.9 that \(K \subseteq C(f; \varrho)\) and \(C(f; t)\) is an ideal of \(K\) for all \(t \in [-1, 0]\) with \(t > \varrho\). Let \(x, y \in K\) be such that \(x \ast y \in C(f; t)\). Then \(f(x \ast y) \leq t\), and so \(f(x \ast (y \ast (y \ast x))) \leq f(x \ast y) \leq t\) by Proposition 3.6. Hence \(x \ast (y \ast (y \ast x)) \in C(f; t)\). It follows from Lemma 3.8 that \(C(f; t)\) is a commutative ideal of \(K\) for all \(t \in [-1, 0]\) with \(t > \varrho\). \(\square\)

**Lemma 3.11.** Let \((X, f)\) be an \(\mathcal{N}(K, \varrho)\)-ideal of \(X\) where \(\varrho \in [0, 1]\) and \(K\) is a sub-BCK-algebra of a BCI-algebra \(X\). If the inequality \(x \ast y \leq z\) holds in \(K\), then \(f(x) \leq \{f(y), f(z)\}\) for all \(x, y, z \in K\).

**Proof.** Assume that the inequality \(x \ast y \leq z\) holds in \(K\). Then \((x \ast y) \ast z = 0\), and so \(f(x \ast y) \leq \{f((x \ast y) \ast z), f(z)\} = \{f(0), f(z)\} = f(z)\) by using (3.1) and (3.2). It follows from (3.2) that

\[ f(x) \leq \{f(x \ast y), f(y)\} \leq \{f(y), f(z)\} \]

for all \(x, y, z \in K\). \(\square\)
Theorem 3.12. Let \((X, f)\) be an \(N(K, \varrho)\)-ideal of \(X\) where \(\varrho \in [0, 1]\) and \(K\) is a sub-BCK-algebra of a BCI-algebra \(X\). If \(K\) is commutative, then \((X, f)\) is a commutative \(N(K, \varrho)\)-ideal of \(X\).

Proof. If \(K\) is commutative, then

\[
\begin{align*}
&((x * (y * (y * x))) * ((x * y) * z)) * z = ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\leq (x * (y * (y * x))) * (x * y) = (x * (x * y)) * (y * (y * x)) = 0,
\end{align*}
\]

that is, \((x * (y * (y * x))) * ((x * y) * z) \leq z\) for all \(x, y, z \in K\). It follows from Lemma 3.11 that \(f(x * (y * (y * x))) \leq \vee \{f((x * y) * z), f(z)\}\) for all \(x, y, z \in K\). Therefore \((X, f)\) is a commutative \(N(K, \varrho)\)-ideal of \(X\).

References


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