Integral Equations of the Problem of Thermoelasticity in Cracked Isotropic Plate with Inclusion

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Abstract

A plane stress thermoelasticity problem, in cracked isotropic plate, is examined. The interaction between inclusion and cracks, in an infinite isotropic plate, under the influence of thermal field is studied. Based on the method of complex potentials, a system of singular integral equations is derived. The aforementioned equations, constitute the solution of the problem, namely the evaluation of the distribution of the temperature and the stress-strain state at any point of the plate.

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1 Introduction

The most common characteristic of materials is the appearance of discontinuities such as cracks, holes and several types of inclusions that ultimately determine their behavior. The appearance of discontinuities in combination with the presence of mechanical and thermal stresses, may affect the strength properties of the structures, and their resistance to loads. For this reason, it is important to study the stress-deformation state of the body under the influence of a thermal field. Numerous works have investigated diversified problems on the theory of the thermoelasticity [1-4]. This paper studies the interaction between an inclusion and two cracks, inside an infinite isotropic plate, subjected to a thermal field. It extends previous work [5] in solving the referred problem via the method of singular integral equations. The proposed method, has been adopted by many researches in solving plane problems in the theory of
elasticity, heat conduction and thermoelasticity, for cracked bodies [6-11]. The interaction between a crack and an elastic inclusion has also been investigated [12,13].

2 Statement of the Problem

We consider an infinite isotropic plate $S$ containing a single inclusion $\gamma_o$ of arbitrary shape, from which a section $\gamma$ is separated, and two cracks $l_1, l_2$. The plate is submitted to the normal stresses $N_1, N_2$ at infinity, and it is under the influence of homogeneous thermal flow $q_\infty$. The isotropic elastic inclusion contains a heat source $q_o$ at $z_o$, and the contact along the boundary $\gamma_o$ of the two bodies $S_1$ and $S_2$ is assumed to be thermally ideal. That means that along the following conditions are valid along the boundary:

$$T^+ = T^-$$

$$\lambda_1 \frac{\partial T^+}{\partial n} = \lambda_2 \frac{\partial T^-}{\partial n}$$

where $\lambda_1$ and $\lambda_2$ are the coefficients of thermal conductivity of the inclusion and the plate, respectively.

Furthermore we assume that:

i) the stresses $\sigma_n^\pm - i\sigma_t^\pm$ and the temperature $T^\pm$ along the part $\gamma$ of the boundary are known. (we use the symbol (+) for quantities referred to the inclusion $S_1$, whereas the symbol (-) correspond to the plate $S_2$).

ii) the stresses on the boundary $\gamma_1$ of the inclusion are equal for both bodies:

$$\sigma_n^+(t) - i\sigma_t^+(t) = \sigma_n^-(t) - i\sigma_t^-(t)$$

also, the displacement jump is given by:

$$g(t) = 2\mu_1 \frac{d}{dt} \left[ - \left( u^+(t) - u^-(t) \right) + i \left( v^+(t) - v^-(t) \right) \right]$$

where $u$ and $v$ are the components of the displacement vector.

iii) the temperature $T_1^+, T_1^-$ and the normal and shear stresses $\sigma_n^\pm_{l_1} - i\sigma_t^\pm_{l_1}$ on the edges of $l_1$ crack , are considered to be known. (+ for the upper edge and - for the lower).

iv) the temperature $T_2^+, T_2^-$, and the displacement components: $u_2^\pm(t) + iv_2^\pm(t)$ at the crack $l_2$, are given.

3 Formation of the State Equations

For the solution of the problem of thermoconductivity of a multiconnected cracked body, the temperature is expressed by:

$$T(x, y) = T_o(x, y) + T_s(x, y)$$
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Figure 1: Infinite isotropic plate containing an inclusion and cracks

where: \(T_o(x, y)\) is the submitted thermal field which is considered as known, and \(T_s(x, y)\) is the sought after thermal field which appears due to the existence of discontinuities at the body.

The derivation of the singular integral equations is based on the method of complex potentials. The thermal potential \(f(z)\), \([T(x, y) = 2Re f(z)]\), describing the thermal field \(T(x, y)\) is defined as:

\[
f^+(z) = \frac{1}{2\pi i} \int_{\gamma_o} \phi_o(\tau) d\tau - \frac{q}{2\pi \lambda_1} \ln(z - z_o), \quad z \in S_1 \tag{6}
\]

\[
f^-(z) = \frac{1}{2\pi i} \int_{\gamma_o} \phi_o(\tau) d\tau + \frac{q_\infty}{2} z e^{-i\beta_o} + \frac{1}{2\pi i} \int_{l_1} g_1(\tau) d\tau + \frac{1}{2\pi i} \int_{l_2} g_2(\tau) d\tau, \quad z \in S_2 \tag{7}
\]

where:

\[
\phi_o(t) = \begin{cases} 
\phi(t) & t \in \gamma \smallskip 
\phi_o(t) & t \in \gamma_1 \end{cases}
\]

\[
\phi(t) = \phi^{(1)}(t) + i\phi^{(2)}(t)
\]

\[
\phi_o(t) = \phi_o^{(1)}(t) + i\phi_o^{(2)}(t)
\]

with \(\phi, \phi_o, g_1, g_2\) the densities of the Cauchy integrals along the boundaries \(\gamma, \gamma_1, l_1, l_2\) respectively.
The Muskhelishvili complex potential for this problem, is given by:

\[
\Phi_+^o(z) = A_o \ln(z - z_0) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - z} d\tau, \quad z \in S_1
\]

\[
\Phi_-^o(z) = \Gamma + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - z} d\tau + \frac{1}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - z} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - z} d\tau,
\]

\[
\Psi_+^o(z) = -\frac{A_o z_0}{z - z_0} + \Psi^+(z), \quad z \in S_2
\]

\[
\Psi_-^o(z) = \Gamma' + \Psi^-(z), \quad z \in S_2
\]

where \(G_o, \phi_1, \phi_2\) denote the densities on \(\gamma_o, l_1, l_2\) respectively, and

\[
G_o = \begin{cases} G(t) & t \in \gamma, \\ G_1(t) & t \in \gamma_1 \end{cases}
\]

\[
A_o = -\frac{\beta m_o}{1 + k}, \quad \text{with} \quad m_o = -\frac{q_o}{4\pi \lambda_1}
\]

\[
\Gamma = \frac{N_1 + N_2}{4}, \quad \Gamma' = -\frac{N_1 - N_2}{2}
\]

the constants \(k\) and \(\beta\), for the isotropic material in the case of plane stress, are given by:

\[
k = \frac{3 - \nu}{1 + \nu}, \quad \beta = \frac{aE}{1 + \nu}
\]

The limit values of the function \(f(z)\) along the boundary \(\gamma\) according to Sohotsky- Plemelj formulae are:

\[z \rightarrow t^+ \in \gamma:\]

\[
f^+(t) = \frac{1}{2} \phi(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_0(\tau)}{\tau - t} d\tau - \frac{q}{2\pi \lambda_1} \ln(t - z_0)
\]

\[z \rightarrow t^- \in \gamma:\]

\[
f^-(t) = -\frac{1}{2} \phi(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_0(\tau)}{\tau - t} d\tau + \frac{q_\infty}{2} te^{-i\beta_o} + \frac{1}{2\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau
\]

\[
T^+(t) = f^+(t) + \overline{f^+(t)} \implies
\]

\[
T^+(t) = \frac{1}{2} \phi(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_0(\tau)}{\tau - t} d\tau - \frac{q}{2\pi \lambda_1} \ln(t - z_0) + \frac{1}{2} \phi(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_0(\tau)}{\tau - t} d\tau
\]

\[= -\frac{q}{2\pi \lambda_1} \ln(\bar{t} - z_0)
\]
\[ T^-(t) = f^-(t) + f^-(t) \implies \]
\[ T^-(t) = \text{Re} \left( q_\infty t e^{-i\beta_o} \right) - \frac{1}{2} \phi(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi(\tau)}{\tau - t} d\tau - \frac{1}{2} \overline{\phi(t)} - \frac{1}{2\pi i} \int_{\gamma_o} \frac{\overline{\phi(\tau)}}{\overline{\tau} - \overline{t}} d\tau \]

\[ - \frac{1}{2\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau - \frac{1}{2\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \]  

Subtracting relations (14), (15) by parts we obtain:

\[ T^+(t) - T^-(t) = \text{Re} \left( q_\infty t e^{-i\beta_o} \right) + \phi(t) + \overline{\phi(t)} - \text{Re} \frac{q}{\pi \lambda_1} \ln(t - z_o) + \]

\[ \text{Re} \left[ \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \right] \]  

Adding relations (14) and (15) by parts, we have:

\[ T^+(t) + T^-(t) = \text{Re} \left[ q_\infty t e^{-i\beta_o} - \frac{q}{\pi \lambda_1} \ln(t - z_o) + \frac{2}{\pi i} \int_{\gamma_o} \frac{\phi(\tau)}{\tau - t} d\tau + \right. \]

\[ \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \]  

when \( z \rightarrow t^+ \in \gamma_1 \): the limit values of \( f(z) \) are:

\[ f^+(t) = \frac{1}{2} \phi_{o1}(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_o(\tau)}{\tau - t} d\tau - \frac{q}{2\pi \lambda_1} \ln(t - z_o) \]  

and when \( z \rightarrow t^- \in \gamma_1 \):

\[ f^-(t) = -\frac{1}{2} \phi_{o1}(t) + \frac{q_\infty}{2} te^{-i\beta_o} + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_o(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \]  

therefore:

\[ T^+(t) = \frac{1}{2} \phi_{o1}(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{\phi_o(\tau)}{\tau - t} d\tau - \frac{q}{2\pi \lambda_1} \ln(t - z_o) \]  

and

\[ T^-(t) = \text{Re} \left( q_\infty t e^{-i\beta_o} \right) - \frac{1}{2} \left( \phi_{o1}(t) + \overline{\phi_{o1}(t)} \right) + \text{Re} \left[ \frac{1}{\pi i} \int_{\gamma_o} \frac{\phi(\tau)}{\tau - t} d\tau \right. \]

\[ + \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \]  

taking into consideration the condition: \( T^+(t) = T^-(t) \) we have:

\[ 2\phi_{o1}^{(1)}(t) = \text{Re} \left[ \frac{q}{\pi \lambda_1} \ln(t - z_o) + q_\infty t e^{-i\beta_o} + \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \right] \]  

(22)
while the condition $\lambda_1 \frac{\partial r^+}{\partial n} = \lambda_2 \frac{\partial r^-}{\partial n}$ becomes:

$$
\lambda_1 \frac{\partial}{\partial t} \phi^{(l)}_o(t) + \lambda_1 \frac{\partial}{\partial t} \left( \text{Re} \left( \frac{1}{\pi i} \int_{\gamma_0} \frac{\phi_o(\tau)}{\tau - t} d\tau - \frac{q}{\pi \lambda_1} \ln(t - z_o) \right) \right) = -\lambda_2 \frac{\partial}{\partial t} \phi^{(l)}_o(t) +
$$

$$
\lambda_2 \frac{\partial}{\partial t} \text{Re} \left( q_{\infty} t e^{-i\beta_o} + \frac{1}{\pi i} \int_{\gamma_0} \frac{\phi_o(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \right) \tag{23}
$$

when $z \rightarrow t \in l_1$ or $z \rightarrow t \in l_2$ according Plemelj formulae we have:

$$
f^\pm(t) = \pm \frac{1}{2} g_j(t) + \frac{q_{\infty}}{2} t e^{-i\beta_o} + \frac{1}{2\pi i} \int_{\gamma_0} \frac{\phi_o(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \tag{24}
$$

therefore

$$
T_j^+(t) = \pm \frac{1}{2} (g_j(t) + \bar{g}_j(t)) + \text{Re} \left( q_{\infty} t e^{-i\beta_o} \right)
$$

$$
+ \text{Re} \left[ \frac{1}{\pi i} \int_{\gamma_0} \frac{\phi_o(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \right] \tag{25}
$$

Subtracting relations (25) by parts we obtain:

$$
T_j^+(t) - T_j^-(t) = g_j(t) + \bar{g}_j(t) \Rightarrow
$$

$$
T_j^+(t) - T_j^-(t) = 2 g_j^{(1)}(t), \quad j = 1, 2 \tag{26}
$$

Adding relations (25) by parts we have:

$$
T_j^+(t) + T_j^-(t) = 2 \text{Re} \left( q_{\infty} t e^{-i\beta_o} \right) + \text{Re} \left[ \frac{1}{\pi i} \int_{\gamma_0} \frac{\phi_o(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_1} \frac{g_1(\tau)}{\tau - t} d\tau + \right.
$$

$$
\left. \frac{1}{\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \right] \quad j = 1, 2 \tag{27}
$$

For the definition of the stress-strain field, we use the complex potentials $\Phi^\pm_o(z), \Psi^\pm_o(z)$. The boundary stress conditions along the boundary $\gamma$ and along the lips of the crack $l_1$ are [3]:

$$
\sigma_m^\pm(t) - i \sigma_t^\pm(t) = \Phi_o^\pm(t) + \Phi_o^\pm(t) + \frac{dt}{dt} \left[ \tilde{T} \Phi_o^\pm(t) + \Psi_o^\pm(t) \right], \quad t \in \gamma \tag{28}
$$

$$
\sigma_n^\pm(t) - i \sigma_t^\pm(t) = \Phi_o^\pm(t) + \Phi_o^\pm(t) + \frac{dt}{dt} \left[ \tilde{T} \Phi_o^\pm(t) + \Psi_o^\pm(t) \right], \quad t \in l_1 \tag{29}
$$

Along the lips of the crack $l_2$, where the discontinuity of the displacements is known, the boundary conditions are:

$$
-2\mu_2 \frac{d}{dt} \left( u_2^\pm(t) + i v_2^\pm(t) \right) = \Phi_o^\pm(t) - k_2 \Phi_o^\pm(t) + \frac{dt}{dt} \left[ \tilde{T} \Phi_o^\pm(t) + \Psi_o^\pm(t) \right] + \beta \tilde{f}^\pm(t)
$$

$$
t \in l_2 \tag{30}
$$
The limited values of the defined function $\Phi_o(z)$ take the following form:

When $z \rightarrow t \in \gamma$:

$$
\Phi_o^+(t) = A_o \ln(t - z_o) + \frac{1}{2} G(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau
$$

$$
\Phi_o^-(t) = \Gamma - \frac{1}{2} G(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau
$$

(31)

(32)

When $z \rightarrow t \in \gamma_1$:

$$
\Phi_o^+(t) = A_o \ln(t - z_o) + \frac{1}{2} G_1(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau
$$

$$
\Phi_o^-(t) = \Gamma - \frac{1}{2} G_1(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau
$$

(33)

(34)

and on the cracks: $(t \in l_j, j = 1, 2)$

$$
\Phi_o^-(t) = \Gamma + \frac{1}{2} \phi_j(t) + \frac{1}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau
$$

(35)

Subtracting relations (28) by parts we obtain:

$$
\left[\Phi_o^+(t) - \Phi_o^-(t)\right] + \left[\Phi_o^+(t) - \Phi_o^-(t)\right] + \frac{\partial t}{\partial t} \left[\Gamma \left(\Phi_o^+(t) - \Phi_o^-(t)\right)
\right]
$$

$$
\left[\Psi_o^+(t) - \Psi_o^-(t)\right] = 2g(t), \quad t \in \gamma
$$

(36)

where: $2q(t) = \left(\sigma_o^+ + \sigma_o^-(t)\right) + i \left(\sigma_o^+(t) - \sigma_o^-(t)\right)$

Using relations (31) and (32), the last one becomes:

$$
\frac{\partial t}{\partial t} \left[\Gamma + \frac{A_o}{t - z_o} \int \frac{\phi_1(\tau)}{\tau - t} d\tau - \frac{1}{2\pi i} \int \frac{\phi_2(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int \frac{\phi_2(\tau)}{\tau - t} d\tau
$$

$$
- \frac{A_o z_o}{t - z_o} \Gamma + \Psi_o^+(t) - \Psi_o^-(t)\right] = 2g(t) \quad \Rightarrow
$$

$$
\Psi_o^+(t) - \Psi_o^-(t) = \frac{\partial t}{\partial t} \left[2g(t) - G(t) - G(t) - A(t) - A(t) - t \left[G'(t) + A'(t) - B(t), \quad t \in \gamma
\right]
$$

(37)
where: \[ A(t) = A_o \ln(t - z_o) - \Gamma - \frac{1}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau - \frac{1}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau \]

and \[ B(t) = -\frac{A_o z_o}{t - z_o} - \Gamma' \]

On the boundary \( \gamma_1 \) exists:

\[ \sigma_n^+(t) - i\sigma_n^-(t) = \sigma_n^+(t) - i\sigma_n^-(t) \Rightarrow \]

\[ \Phi_o^+(t) + \Phi_o^-(t) + \frac{dt}{dt} \left[ i\Phi_o^+(t) + \Psi_o^+(t) \right] = \Phi_o^-(t) + \Phi_o^-(t) + \frac{dt}{dt} \left[ i\Phi_o^-(t) + \Psi_o^-(t) \right] \Rightarrow \]

\[ \left[ \Phi_o^+(t) - \Phi_o^-(t) \right] + \left[ \Phi_o^+(t) - \Phi_o^-(t) \right] + \frac{dt}{dt} \left[ i\left( \Phi_o^+(t) - \Phi_o^+(t) \right) \right] \]

\[ = 0 \]

By substitution of (33) and (34) to the last relation, we have:

\[ G_1(t) + A_o \ln(t - z_o) - \Gamma - \frac{1}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau - \frac{1}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau + \]

\[ \overline{G_1(t)} + A_o \ln(\overline{t} - z_o) - \Gamma + \frac{1}{2\pi i} \int_{l_1} \frac{\overline{\phi_1(\tau)}}{\overline{\tau} - t} d\overline{\tau} + \frac{1}{2\pi i} \int_{l_2} \frac{\overline{\phi_2(\tau)}}{\overline{\tau} - \overline{t}} d\overline{\tau} + \]

\[ \frac{dt}{dt} \left[ i \left( G_1'(t) - \Gamma' - \overline{G_1(t)} - \overline{A(t)} - \overline{A(t)} \right) - \overline{G_1'(t) + A'(t)} \right] = 0 \Rightarrow \]

\[ \Psi^+(t) - \Psi^-(t) = \frac{dt}{dt} \left[ -G_1(t) - \overline{G_1(t)} - A(t) - \overline{A(t)} \right] - \overline{G_1'(t) + A'(t)} \]

Subtracting relations (29) by parts we obtain:

\[ \left[ \Phi_o^+(t) - \Phi_o^-(t) \right] + \left[ \Phi_o^-(t) - \Phi_o^+(t) \right] + \frac{dt}{dt} \left[ i\left( \Phi_o^+(t) - \Phi_o^+(t) \right) \right] \]

\[ \left( \Psi_o^+(t) - \Psi_o^-(t) \right) = 2q_1(t), \quad t \in l_1 \]

where: \[ 2q_1(t) = \left( \sigma_{n1}^+(t) - \sigma_{n1}^-(t) \right) + i \left( \sigma_{i1}^+(t) - \sigma_{i1}^-(t) \right) \]

also, subtracting relations (31) by parts, we have:

\[ \left[ \Phi_o^+(t) - \Phi_o^-(t) \right] - k_2 \left[ \Phi_o^+(t) - \Phi_o^-(t) \right] + \frac{dt}{dt} \left[ i\left( \Phi_o^+(t) - \Phi_o^+(t) \right) \right] \]

\[ + \left( \Psi_o^+(t) - \Psi_o^-(t) \right) + \beta \left( f^+(t) - f^-(t) \right) = 2q_2(t), \quad t \in l_2 \]
where: \[ 2q_2(t) = -2\mu_2 \left[ \left( \frac{du_2^+(t)}{dt} - \frac{du_2^-(t)}{dt} \right) + i \left( \frac{dv_2^+(t)}{dt} - \frac{dv_2^-(t)}{dt} \right) \right] \]

Solving equations (39), (40) in terms of \( \Psi^+(t) - \Psi^-(t) \), and taking into consideration relations (37) and (38) and Plemelj formulae for the function \( \Phi_0^+, \Phi_0^- \), we derive the following expression for \( \Psi_o(z) \):

\[
\Psi^+_o(z) = -\frac{A_o z_o}{z - z_o} + K \\
\Psi^-_o(z) = \Gamma' + K
\]

where: \[ K = \frac{1}{\pi i} \int_{\gamma} \frac{G(\tau)}{\tau - z} d\tau - \frac{1}{2\pi i} \int_{\gamma} \frac{G(\tau)}{\tau - z} d\tau - \frac{1}{2\pi i} \int_{\gamma} \frac{G(\tau)}{(\tau - z)^2} d\tau \]

Adding relations (28) by parts we obtain:

\[
\left[ \Phi^+_o(t) + \Phi^-_o(t) \right] + \left[ \Phi^+_o(t) + \Phi^-_o(t) \right] + \frac{dt}{dt} \left[ \Phi^+_o(\Psi^+_o(t) + \Phi^-_o(t)) \right] + \left( \Psi^+_o(t) + \Psi^-_o(t) \right) = 2\rho(t), \quad t \in \gamma
\]
Subtracting the above expressions by parts we obtain:

\[ + \frac{1}{\pi i} \int_{\gamma_1} \frac{\tau - t}{(\tau - t)^2} A(\tau) d\tau + \frac{1}{\pi i} \int_{\gamma_1} B(\tau) \frac{d\tau}{\tau - t} + \frac{1}{\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \]

\[ \frac{1}{\pi i} \int_{l_1} \frac{\tau - t}{(\tau - t)^2} \phi_1(\tau) d\tau + \frac{k_2}{\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_2} \frac{\tau - t}{(\tau - t)^2} \phi_2(\tau) d\tau + \]

\[ + \frac{\beta}{2\pi i} \int_{l_2} \frac{g_2(\tau)}{\tau - t} d\tau \]

\[ = 2p(t) - \frac{dt}{dt} \left[ \frac{2}{\pi i} \int_{\gamma} \frac{\phi(t)}{\tau - t} d\tau \right] + \frac{2}{\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau, \quad t \in \gamma \]  

(44)

where:

\[ 2p(t) = \left( \sigma_+(t) + \sigma_-(t) \right) - i \left( \sigma_+(t) + \sigma_-(t) \right) \]

\[ - A_o \ln(t - z_o) - \left( \Gamma + \Gamma' \right) + \frac{dt}{dt} \frac{A_o \omega_o}{t - z_o} - \frac{dt}{dt} \Gamma' \]

when \( t \in \gamma_1 \) exists:

\[ 2\mu_1 \frac{d}{dt} \left[ - \left( u^+(t) - iv^+(t) \right) \right] = \Phi_+(t) - k_1 \Phi_+(t) + \frac{dt}{dt} \left[ i \Phi_+(t) + \Psi_+(t) \right] - \beta_1 f^+(t) \]

\[ 2\mu_2 \frac{d}{dt} \left[ - \left( u^-(t) - iv^-(t) \right) \right] = \Phi_-(t) - k_2 \Phi_-(t) + \frac{dt}{dt} \left[ i \Phi_-(t) + \Psi_-(t) \right] - \beta_2 f^-(t) \]

Subtracting the above expressions by parts we obtain:

\[ - \Phi_+(t) - k_1 \Phi_+(t) + \frac{dt}{dt} \left[ i \Phi_+(t) + \Psi_+(t) \right] - \Gamma_o \frac{dt}{dt} \left[ i \Phi_+(t) + \Psi_+(t) \right] \]

\[ - \beta_1 f^+(t) + \beta_2 f^-(t) \Rightarrow \]

\[ \left[ \Phi^+(t) - \Gamma_o \Phi_+(t) \right] - \left[ k_1 \Phi_+(t) - \Gamma_o k_2 \Phi_-(t) \right] + \frac{dt}{dt} \left[ \Gamma \left( \Phi^+ - \Gamma_o \Phi^-(t) \right) + \Psi^+(t) - \Gamma_o \Psi^-(t) \right] = g_o(t) \]

where:

\[ g_o(t) = \left( \sigma_+(t) + \sigma_-(t) \right) - i \left( \sigma_+(t) + \sigma_-(t) \right) \]

\[ - A_o \ln(t - z_o) - \left( \Gamma + \Gamma' \right) + \frac{dt}{dt} \frac{A_o \omega_o}{t - z_o} - \frac{dt}{dt} \Gamma' \]

\[ \Gamma_o = \frac{\mu_1}{\mu_2} \]

Using the limited values of functions \( \Phi_o(z) \) when \( z \rightarrow t \in \gamma_1 \) along with the expressions (41) and (42), the previous equation becomes:

\[ \frac{1 + \Gamma_o}{2} G_1(t) + \frac{1}{2\pi i} \int_{\gamma_1} \frac{G_o(\tau)}{\tau - t} d\tau - \Gamma_o \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau - \frac{\Gamma_o}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau \]
Adding relations (29) and (30) by parts, and taking into consideration (35),

\[
- \frac{k_1 + \Gamma_o k_2}{2} G_1(t) - \frac{k_1 - \Gamma_o k_2}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau + \frac{\Gamma_o k_2}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau + \frac{dt}{dt} \left[ \frac{1 + \Gamma_o}{2} G_1'(t) - \frac{1 - \Gamma_o}{2\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau \right] - \frac{1 - \Gamma_o}{2\pi i} \int_{\gamma_o} \frac{A_o(\tau)}{\tau - t} d\tau - \frac{1 - \Gamma_o}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{\Gamma_o}{2\pi i} \int_{l_1} \frac{\tau - \tilde{t}}{(\tau - t)^2} G_o(\tau) d\tau
\]

\[
- \frac{1 - \Gamma_o}{2\pi i} \int_{\gamma_o} \frac{\tau A(\tau)}{(\tau - t)^2} d\tau - \frac{1 - \Gamma_o}{2\pi i} \int_{l_1} \frac{\phi_1(\tau)}{(\tau - t)^2} d\tau + \frac{\Gamma_o}{2\pi i} \int_{l_1} \frac{\tau - \tilde{t}}{(\tau - t)^2} \phi_1(\tau) d\tau
\]

\[
- \frac{1}{2\pi i} \int_{l_1} \frac{\tau \phi_1(\tau)}{(\tau - t)^2} d\tau + \frac{k_2(1 - \Gamma_o)}{2\pi i} \int_{l_2} \frac{\phi_2(\tau)}{\tau - t} d\tau + \frac{\Gamma_o}{2\pi i} \int_{l_2} \frac{\tau - \tilde{t}}{(\tau - t)^2} \phi_2(\tau) d\tau + \frac{dt}{dt} \left[ \frac{1 - \Gamma_o}{2\pi i} \int_{\gamma_o} \frac{\tau - \tilde{t}}{(\tau - t)^2} G_o(\tau) d\tau - \frac{1 - \Gamma_o}{2\pi i} \int_{\gamma_o} \frac{A(\tau)}{(\tau - t)^2} d\tau + \frac{\Gamma_o}{2\pi i} \int_{\gamma_o} \frac{\tau - \tilde{t}}{(\tau - t)^2} A(\tau) d\tau + \frac{1}{\pi i} \int_{\gamma_o} \frac{B(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_1} \frac{\tau - \tilde{t}}{(\tau - t)^2} \phi_1(\tau) d\tau + \frac{k_2}{\pi i} \int_{l_2} \frac{\tau - \tilde{t}}{(\tau - t)^2} \phi_2(\tau) d\tau + \frac{\beta}{2\pi i} \int_{l_2} \frac{\tau - \tilde{t}}{(\tau - t)^2} \phi_2(\tau) d\tau \right]
\]

\[
= \frac{2p_1(t)}{\pi i} - \frac{dt}{dt} \left[ \frac{2}{\pi i} \int_{\gamma} \frac{q(\tau)}{\tau - t} d\tau + \frac{2}{\pi i} \int_{l_1} \frac{q_1(\tau)}{\tau - t} d\tau + \frac{2}{\pi i} \int_{l_2} \frac{q_2(\tau)}{\tau - t} d\tau \right] + \frac{dt}{dt} \Gamma + \frac{dt}{dt} \Gamma',
\]

where:

\[
2p_1(t) = (\sigma_{n1}^+(t) + \sigma_{n1}^-(t)) - i(\sigma_{t1}^+(t) + \sigma_{t1}^-(t)) - 2(\Gamma + \Gamma') + \frac{dt}{dt} A_{o\omega_o} + \frac{dt}{dt} \Gamma + \frac{dt}{dt} \Gamma'
\]

On the crack \(l_2\)

\[
\frac{1}{\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau + \frac{k_2}{\pi i} \int_{\gamma_o} \frac{G_o(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{l_1} \frac{\phi_1(\tau)}{\tau - t} d\tau + \frac{k_2}{\pi i} \int_{l_2} \frac{\phi_1(\tau)}{\tau - t} d\tau
\]
where: \( 2p_2(t) = -2\mu_2 \left[ \left( \frac{dw_1^+(t)}{dt} + \frac{dw_2^-(t)}{dt} \right) - i \left( \frac{dw_1^+(t)}{dt} + \frac{dw_2^-(t)}{dt} \right) \right] - \left( \Gamma - k_2\Gamma \right) - \frac{dt}{dt} \Gamma + \frac{dt}{dt} A_o\zeta_0 - \frac{dt}{dt} \Gamma' \)

\[ \text{4 Concluding Remarks} \]

The interaction between cracks and inclusion, under the influence of a thermal field, in an isotropic loaded plate was examined. Expressions (17), (23), (27) and (44), (45), (46), (47) describe the stress-strain as well as thermal fields. The derived system of equations, when quantitatively evaluated, allow the calculation of not only the distribution of stresses and displacements, but also the temperature distribution at every point of the body.

\[ \text{References} \]


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