Applied Mathematical Sciences, Vol. 6, 2012, no. 64, 3177 - 3183

New Technique For Solving System of First Order Linear Differential Equations

Karwan H. F. Jwamer and Aram M. Rashid

University of Sulaimani
Faculty of Science and Science Education
School of Science, Department of Mathematics
Sulaimani, Iraq
jwameri1973@gmail.com
arammr.maths@gmail.com

Abstract

In this paper, we used new technique for finding a general solution of (2×2) and (3×3) system of first order nonhomogeneous linear differential equations.

Mathematics Subject Classifications: 34A30, 34A34

Keywords: System of linear differential equations, variation of parameters, general solutions

1 INTRODUCTION

The study of differential equations is a wide field in pure and applied mathematics, physics, meteorology, and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods [3,5].

Mathematicians also study weak solutions (relying on weak derivatives), which are types of solutions that do not have to be differentiable everywhere.

This extension is often necessary for solutions to exist, and it also results in more physically reasonable properties of solutions, such as possible presence of shocks for equations of hyperbolic type. Several authors studied the solutions of linear differential equations and system of differential equation but they used different methods, see[$1-5$] .In the present work we find the solution of (2×2) and (3×3) systems of first order nonhomogeneous linear differential equations using new technique which defined in section two .

2 DESCRIPTION OF THE METHOD

Consider the following system of first order nonhomogeneous linear differential

$$\begin{cases} x' = a_{11}x + a_{12}y + f_1(t) \\ \\ y' = a_{21}x + a_{22}y + f_2(t) \end{cases} \tag{1}$$

Where $f_1(t)$ and $f_2(t)$ are continuous functions of the variable t on the interval I, by the following technique, we can reduce the given system to a non-homogeneous second order linear differential equation with constant coefficients:

The matrix form of system (1) is :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Now let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \implies \begin{cases} tr(A) = a_{11} + a_{22} \\ det(A) = a_{11}a_{22} - a_{12}a_{21} \end{cases}$

Next, from system (1), we have : $y' = a_{21}x + a_{22}y + f_2(t)$

Differentiating both sides with respect to t, then we obtain :

$y'' = a_{21}x' + a_{22}y' + f_2'(t)$ and as $x' = a_{11}x + a_{12}y + f_1(t)$

So

$$\begin{aligned} y'' &= a_{21}(a_{11}x + a_{12}y + f_1(t)) + a_{22}y' + f_2'(t) \\ &= a_{11}(a_{21}x) + a_{12}a_{21}y + a_{21}f_1(t) + a_{22}y' + f_2'(t) \end{aligned} \tag{2}$$

Also since $a_{21}x = y' - a_{22}y - f_2(t)$ or $x = \frac{1}{a_{21}}(y' - a_{22}y - f_2(t))$

From (2), we have

$$\begin{aligned} y'' &= a_{11}(y' - a_{22}y - f_2(t)) + a_{12}a_{21}y + a_{21}f_1(t) + a_{22}y' + f_2'(t) \\ &= (a_{11} + a_{22})y' - (a_{11}a_{22} - a_{12}a_{21})y + (f_2'(t) + a_{21}f_1(t) - a_{11}f_2(t)) \\ &= tr(A)y' - det(A)y + (f_2'(t) + a_{21}f_1(t) - a_{11}f_2(t)). \end{aligned}$$

Therefore,

$$y'' - tr(A)y' + det(A)y = g(t), \tag{3}$$

where $g(t) = f_2'(t) + a_{21}f_1(t) - a_{11}f_2(t)$.

We see that eq.(3) is a non-homogeneous second order differential equation with constant coefficients.

Now we try to solve eq.(3) by using variation of parameters, suppose that the complementary solution of (3) is:

$$y_c = c_1y_1 + c_2y_2 \tag{4}$$

The crucial idea is to replace the constants c_1 and c_2 in (3) by functions $v_1(t)$ and $v_2(t)$ respectively, this gives the particular solution,

$$y_p = v_1(t)y_1 + v_2(t)y_2 \tag{5}$$

By putting eq.(5) in (3), then we get a system of two linear algebraic equations for derivatives $v_1'(t)$ and $v_2'(t)$ of the unknown functions:

$$\begin{cases} v_1'(t)y_1 + v_2'(t)y_2 = 0 \\ \\ v_1'(t)y_1' + v_2'(t)y_2' = 0 \end{cases} \tag{6}$$

Since y_1 and y_2 are fundamental set of solutions, then the Wronskian of y_1 and y_2 does not equal to zero, that is:

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$ Thus by Cramer's rule we have:

$$v_1'(t) = \frac{-y_2g(t)}{w(y_1, y_2)} \quad \text{and} \quad v_2'(t) = \frac{y_1g(t)}{w(y_1, y_2)}.$$

Hence

$$v_1(t) = \int \frac{-y_2g(t)}{w(y_1, y_2)}dt \quad and \quad v_2(t) = \int \frac{y_1g(t)}{w(y_1, y_2)}dt \tag{7}$$

By substituting (7) in (5) we get the particular solution of (3), and then the general solution of eq. (3) is :

$$y = y_c + y_p \tag{8}$$

Thus

$$x = \frac{1}{a_{21}}[y'_c + y'_p - a_{22}(y_c + y_p) - f_2(t)], a_{21} \neq 0. \tag{9}$$

Therefore, the equations (8) and (9) are solutions of the system (1).

Proposition : Consider the following system

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + f_1(t) \\ y' = a_{21}x + a_{22}y + a_{23}z + f_2(t) \\ z' = a_{31}x + a_{32}y + a_{33}z + f_3(t) \end{cases} \tag{10}$$

where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous functions and if $a_{11} + a_{22} = 0$ or $a_{22} + a_{12}a_{31}^2 = a_{11} + a_{21}a_{32}^2$, then the system (10) can be reducing to a non-homogeneous third order linear differential equation with constant coefficients.

Proof: First we take $a_{11} + a_{22} = 0$. The matrix form of system (10) is

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

In system (10), since $z' = a_{31}x + a_{32}y + a_{33}z + f_3(t)$, then we start the previous technique to get the requirement
$z'' = a_{31}x' + a_{32}y' + a_{33}z' + f'_3(t)$

$$z''' = a_{31}x'' + a_{32}y'' + a_{33}z'' + f''_3(t) \tag{11}$$

But

$x'' = a_{11}x' + a_{12}y' + a_{13}z' + f'_1(t)$
$y'' = a_{21}x' + a_{22}y' + a_{23}z' + f'_2(t)$

So eq.(11) becomes

$$\begin{aligned} z''' &= a_{31}[a_{11}x' + a_{12}y' + a_{13}z' + f'_1(t)] + a_{32}[a_{21}x' + a_{22}y' + a_{23}z'' + f'_2(t)], \\ &+ a_{33}z' + f''_3(t) \end{aligned}$$

$$\begin{aligned} \therefore z''' &= (a_{11}a_{31} + a_{21}a_{32})x' + (a_{12}a_{31} + a_{22}a_{32})y' + (a_{13}a_{31} + a_{23}a_{32})z' \\ &+ a_{33}z'' + [a_{31}f'_1(t) + a_{32}f'_2(t) + f''_3(t)]. \end{aligned}$$

Again since

$$x' = a_{11}x + a_{12}y + a_{13}z + f_1(t)$$
$$y' = a_{21}x + a_{22}y + a_{23}z + f_2(t),$$

Then we have:

$$\begin{aligned} z''' &= (a_{11}a_{31} + a_{21}a_{32})[a_{11}x + a_{12}y + a_{13}z + f_1(t)] \\ &+ (a_{12}a_{31} + a_{22}a_{32})[a_{21}x + a_{22}y + a_{23}z + f_2(t)] \\ &+ (a_{13}a_{31} + a_{23}a_{32})z' + a_{33}z'' + [a_{31}f_1'(t) + a_{32}f_2'(t) + f_3''(t)], \end{aligned}$$

$$\begin{aligned} z''' &= [a_{11}(a_{11}a_{31} + a_{21}a_{32}) + a_{21}(a_{12}a_{31} + a_{22}a_{32})]x \\ &+ [a_{12}(a_{11}a_{31} + a_{21}a_{32}) + a_{22}(a_{12}a_{31} + a_{22}a_{32})]y \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32})]z \\ &+ (a_{13}a_{31} + a_{23}a_{32})z' + a_{33}z'' + [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) + f_3''(t)], \end{aligned}$$

$$\begin{aligned} z''' &= [a_{31}(a_{11}^2 + a_{12}a_{21}) + a_{21}a_{32}(a_{11} + a_{22})]x + a_{32}[(a_{22}^2 + a_{12}a_{21}) + a_{12}a_{31}(a_{11} + a_{22})]y \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32})]z \\ &+ (a_{13}a_{31} + a_{23}a_{32})z' + a_{33}z'' + [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) + f_3''(t)], \end{aligned}$$

By hypothesis $a_{11} + a_{22} = 0$, then we have

$$\begin{aligned} z''' &= a_{31}(a_{11}^2 + a_{12}a_{21})x + a_{32}(a_{11}^2 + a_{12}a_{21})y \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32})]z \\ &+ (a_{13}a_{31} + a_{23}a_{32})z' + a_{33}z'' + [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) + f_3''(t)], \end{aligned}$$

$$\begin{aligned} z''' &= (a_{11}^2 + a_{12}a_{21})(a_{31}x + a_{32}y) + (a_{13}a_{31} + a_{23}a_{32})z' + a_{33}z'' \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32})]z \\ &+ [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) + f_3''(t)], \end{aligned}$$

But $a_{31}x + a_{32}y = z' - a_{33}z - f_3(t)$, then we have

$$\begin{aligned} z''' &= (a_{11}^2 + a_{12}a_{21})(z' - a_{33}z - f_3(t)) + (a_{13}a_{31} + a_{23}a_{32})z' + a_{33}z'' \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32})]z \\ &+ [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) + f_3''(t)], \end{aligned}$$

$$\begin{aligned} z''' &= a_{33}z'' + (a_{11}^2 + a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32})z' \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32}) - a_{33}(a_{11}^2 + a_{12}a_{21})]z \\ &+ [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) - (a_{11}^2 + a_{12}a_{21})f_3(t) + f_3''(t)], \end{aligned}$$

$$\begin{aligned} z''' &= a_{33}z'' + (a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32})z' \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32}) - a_{33}(a_{11}^2 + a_{12}a_{21})]z \\ &+ [(a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} \\ &+ a_{22}a_{32})f_2(t) + a_{31}f_1'(t) + a_{32}f_2'(t) - (a_{11}^2 + a_{12}a_{21})f_3(t) + f_3''(t)], \end{aligned}$$

Now let

$$\begin{aligned} g(t) &= (a_{11}a_{31} + a_{21}a_{32})f_1(t) + (a_{12}a_{31} + a_{22}a_{32})f_2(t) \\ &+ a_{31}f_1'(t) + a_{32}f_2'(t) - (a_{11}^2 + a_{12}a_{21})f_3(t) + f_3''(t)], \end{aligned}$$

$$\begin{aligned} z''' &= a_{33}z'' + (a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32})z' \\ &+ [a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32}) - a_{33}(a_{11}^2 + a_{12}a_{21})]z + g(t). \end{aligned}$$

Since

$$\begin{aligned} det(A) &= a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \\ &= a_{13}(a_{11}a_{31} + a_{21}a_{32}) + a_{23}(a_{12}a_{31} + a_{22}a_{32}) - a_{33}(a_{11}^2 + a_{12}a_{21}), \end{aligned}$$

Hence we obtain

$$z''' = a_{33}z'' + (a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32})z' + det(A)z + g(t),$$

or

$$z''' - a_{33}z'' - (a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32})z' - det(A)z = g(t).$$

This is a non-homogeneous third order linear differential equation with constant coefficients we can solve it by variation of parameter and obtain $z(t)$. Substitute $z(t)$ in the first and second equation in system (10) and use the technique for (2×2) we obtain $y(t)$ and $x(t)$.

3 Conclusion

In this paper , we conclude that (2×2) and (3×3) systems of first order nonhogeneous linear differential equations can solved by new technique which defined as before , also in the future we want developed the same technique for (n×n) system .

References

[1] C.G. Cullen, Linear Algebra and Differential Equations, 2^{nd} ed., Pws-Kent, Boston, 1991.

[2] E.A.Cddington and N.Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[3] H.C. Saxena, Finite Differences and Numerical Analysis, 14^{th} Rev.Edn.,India, ISBN:81-219-0339-4,1998 .

[4] M.A. Raisinghania, Ordinary and Partial Differential Equations, S.Chand Company Ltd.India , ISBN:81-219-0892-2, 2008

[5] S.R.K. Lyengar and R.K. Jain, Numerical Methods, New Age International (p) Limited,India, ISBN: 978-81-224-2707-3, 2009.

Received: May, 2012