An Abstract Version of Majorana’s
Uniqueness Theorem

M. M. Alghanmi, M. Sh. Alhuthali and Ezzat R. Hassan

Department of Mathematics
Faculty of Science, King Abdul Aziz University
P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract. This paper provides a necessary condition for non-uniqueness of solutions for the Cauchy problem (CP)

\[ \hat{x}' = \hat{f}(t, \hat{x}), \quad \hat{x}(0) = \hat{0}, \]  

where \( \hat{f} : [0, 1] \times B \rightarrow B \), \( B \) is a real or complex Banach space. As a consequence new uniqueness criteria are deduced. In particular, the present work is an abstract version of a result which is due to Majorana [9].

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1 Introduction

In literature there are several nonuniqueness results which are derived by an inversion of uniqueness criteria of Kamke-type [1, 3, 4, 5, 6, 10, 11]. Recently, reversing to this approach, Majorana [9] established first nonuniqueness result and then uniqueness criteria has been deduced. He proved, with the aid of an auxiliary scalar equation ASE of the form,

\[ u = tf(t, u), \]  

1ezzat1611@yahoo.com
a uniqueness result for the following Cauchy problem:

\[ x' = f(t, x), \quad x(0) = 0, \]  

(1.2)

where \( x \) and 0 are reals. One of the major advantages of Majorana’s result consists in the fact that any of the standard Lipschitz condition types do not apply, i.e., those conditions imposed that the difference \( f(t, x) - f(t, y) \) is bounded. In order to put our results into context, let us start by formulating the classical theorem of Majorana:

**Theorem 1.1**[9]. Let the function \( f(t, x) \) be defined in \([0, a] \times \mathbb{R}\), continuous with respect to \( t \) and such that \( f(t, 0) = 0 \) for every \( t \in [0, a] (a < 1) \). Further let Cauchy problem (1.2) have two different classical solutions defined in \( t \in [0, \alpha] \). Then for every \( \varepsilon > 0 \), there exists \( t \in (0, \alpha] \) such that (1.1) has at least two different roots \( u \) with \( |u| < \varepsilon \).

An immediate consequence of this latter Theorem is the following uniqueness criterion:

**Theorem 2.1**[9]. Let the function \( f(t, x) \) be defined in \([0, a] \times \mathbb{R}\), continuous with respect to \( t \), such that \( f(t, 0) = 0 \) for every \( t \in [0, a] (a < 1) \). Further let there exist \( \varepsilon > 0 \) such that \( u = 0 \) is the only root of (1.1) with \( |u| < \varepsilon \) for every \( t \in [0, \alpha] \). Then, for (1.2), \( x(t) = 0 \) is the only classical solution defined in \( t \in [0, \alpha] \).

Therefore we have, in \( \mathbb{R} \), a very close relation between (1.1) and (1.2). It is one of the goals of this work to retain this relation in a suitable generalized sense. However, Majorana’s results are not directly extendable to an arbitrary abstract space as the following example shows.

**Example 1.1**[2]. Let us consider the Cauchy problem:

\[ \hat{x}' = \hat{f}(t, \hat{x}), \quad \hat{x}(0) = \hat{0}, \]  

(1.3)

where

\[ \hat{f}(t, \hat{x}) = \begin{cases} \left( \frac{2}{\sqrt{\|\hat{r}\|}}(x_1 + x_2), \frac{2}{\sqrt{\|\hat{r}\|}}(x_2 - x_1) \right) & \text{if } \hat{x} \neq \hat{0}, \\ 0 & \text{if } \hat{x} = \hat{0}. \end{cases} \]

In polar coordinates (1.3) becomes

\[ r' = 2\sqrt{r}, \quad \theta' = \frac{-2}{\sqrt{r}}. \]

Thus, besides the trivial solution \( \hat{x} = \hat{0} \), there is at least one another given by

\[ \hat{x} = \begin{cases} \left( t^2 \cos \frac{1}{t^2}, t^2 \sin \frac{1}{t^2} \right) & \text{if } t \neq 0 \\ \hat{0} & \text{if } t = 0. \end{cases} \]
We conclude that (1.3) satisfies the assumptions of Theorem 1.1[9], however, one can’t find more than the trivial root to the auxiliary scalar equation ASE corresponding to (1.3), namely,

\[ u_1 = \lambda(u_1 + u_2), \quad u_2 = \lambda(u_2 - u_1), \]

where \( \lambda \) is an arbitrary scalar.

A way to provide a version of Majorana’s uniqueness theorem in Banach space consists in replacing (1.1) with a suitable one. Our main concern in this work is the classical Cauchy problem (0.1), where \( \hat{f} \) takes values in a real or complex Banach space \( B \) with \( \dim B < \infty \), \( \hat{x} \) and \( \hat{0} \) are in \( B \).

Before starting the main work we shall introduce some concepts. Throughout the following \( \| \cdot \| \) stands for the norm in \( B \). Let \( B^* \) be the anti-dual of \( B \), i.e., the space of all continuous anti-linear (or conjugate linear) functionals on \( B \). The image of \( \hat{x} \in B \) under \( \hat{x}^* \in B^* \) will be denoted by \( \langle \hat{x}, \hat{x}^* \rangle \). For each \( \hat{x} \in B \), \( K(\hat{x}) \) denotes the nonempty set of all \( \hat{x}^* \in B^* \) for which \( \langle \hat{x}, \hat{x}^* \rangle = \| \hat{x} \|^2 = \| \hat{x}^* \|^2 \). A duality mapping of \( B \) is a function \( k : B \to 2^{B^*} \) such that \( k(\hat{x}) \in K(\hat{x}) \) for each \( \hat{x} \in B \). We shall rely on the following simple observation due to [7, 12].

Suppose that (C1), (C2) below are fulfilled

(C1) \( \hat{\varphi}(t) \) has a weak derivative \( \hat{\varphi}'(t) \in B \),

(C2) \( \| \hat{\varphi}(t) \| \) is differentiable. Then

\[ \| \hat{\varphi}(t) \| \frac{d}{dt} \| \hat{\varphi}(t) \| = \Re \langle \hat{\varphi}'(t), \hat{\varphi}^*_t \rangle, \quad (1.4) \]

for each \( \hat{\varphi}^*_t \in K(\hat{\varphi}(t)) \). The underlying idea to provide counterpart to (1.1) is based on duality mappings concept as shown in the next sections. If \( \hat{f}(t, \hat{x}) \) is continuous on \( [0, 1] \times B \), it is well known that (0.1) has at least one \( C^1 \)-solution if either \( \dim B < \infty \) or if \( \hat{f} \) is Lipschitz. The continuity of \( \hat{f} \) is natural in the literature and then it is assumed here. Throughout this work, whenever we speak of a solution to (0.1), we will mean a strongly continuous, once weakly continuous differentiable function \( x \) on some interval \( [0, a] (a < 1) \).

## 2 Main Result

**Theorem 1.2.** Let the abstract function \( \hat{f}(t, \hat{x}) \) be continuous on \( [0, 1] \times B \) and satisfy \( \hat{f}(t, \hat{0}) = \hat{0} \) for every \( t \in [0, 1] \). Further let (0.1) admit two different solutions defined in \( [0, a] (a < 1) \). Then, for every \( \varepsilon > 0 \), and for every duality
mapping \( k : B \to 2^{B^*} \), there exists \( t \in [0, a] \) such that the following scalar equation

\[
\text{Re} < \hat{f}(t, \hat{u}), k(\hat{u}) > = \frac{1}{2t} \| \hat{u} \|^2,
\]

(2.1)

has at least two different roots \( \hat{u} \) with \( \| \hat{u} \| < \varepsilon \).

**Proof.** It follows, by the assumption \( f(t, 0) = 0 \), that (0.1) has the zero solution, so we assume that (0.1) has the solution \( \hat{\varphi}(t) \neq 0 \).

Let \( \varepsilon > 0 \) and \( k : B \to 2^{B^*} \) be given. Since \( \hat{u} = 0 \) is a root of (2.1) for every \( t \in [0, a] \), it is sufficient to show that there exists \( t \in [0, a] \) for which (2.1) is satisfied by some \( \hat{u} \neq 0 \) with \( \| \hat{u} \| < \varepsilon \). Let a real-valued function \( A_k(t) \) be defined by setting

\[
A_k(t) = \begin{cases} \\
\frac{\| \hat{\varphi}(t) \|^2}{t} & t \neq 0 \\
0 & t = 0,
\end{cases}
\]

\( t \in [0, a] \). Of course \( A_k(t) \) is continuous in \([0, a]\), differentiable in \((0, a)\), and for every \( t \) in \((0, a)\) and for every \( k(\hat{\varphi}(t)) \in K(\hat{\varphi}(t)) \), using (1.4), we have

\[
A_k'(t) = \frac{1}{f_2} [\text{Re} < 2t \hat{f}(t, \hat{\varphi}(t)), k(\hat{\varphi}(t)) > - \| \hat{\varphi}(t) \|^2].
\]

(2.2)

Now, fix \( t_2 \in (0, a) \) with \( \hat{\varphi}(t_2) \neq 0 \) such that \( \| \hat{\varphi}(t) \| < \varepsilon \) for every \( t \in [0, t_2] \). Denote \( t_1 = \sup \{ t \in [0, t_2] : A_k(t) = 0 \} \). Clearly \( \hat{\varphi}(t_1) = 0 \) and \( \hat{\varphi}(t) \neq 0 \) for every \( t \in (t_1, t_2) \). At this point there are just two possibilities:

P1: If there exists an \( t \in (t_1, t_2) \) such that \( A_k'(t) = 0 \), then from (2.2) for such a \( t \), (2.1) is satisfied by \( \hat{u} = \hat{\varphi}(t) \). Hence the proof is accomplished just taking these \( t \) and \( \hat{u} = \hat{\varphi}(t) \).

P2: Otherwise if \( A_k'(t) \neq 0 \) for every \( t \in (t_1, t_2) \). According to Darboux property \( A_k'(t) \) has a constant sign in \((t_1, t_2)\), then we take \( \hat{u} = \hat{\varphi}(t_2)(\neq 0) \), \( k(\hat{\varphi}(t_2)) \in K(\hat{\varphi}(t_2)) \), and define

\[
G(t) = \text{Re} < 2t \hat{f}(t, \hat{\varphi}(t_2)), k(\hat{\varphi}(t_2)) > - \| \hat{\varphi}(t_2) \|^2, \quad \text{for every } t \in [0, a].
\]

Now let us suppose that \( A_k'(t) > 0 \) for every \( t \in (t_1, t_2) \). \( G(0) = - \| \hat{\varphi}(t_2) \|^2 < 0 \). On the other hand, we have \( G(t_2) = t_2^2 A_k'(t_2) > 0 \). It follows, by continuity of \( G \), that there exists an \( t \in (0, t_2) \) such that \( G(t) = 0 \). It remains to show that the case \( A_k'(t) < 0 \) is impossible. Assume that \( A_k'(t) < 0 \), for every \( t \in (t_1, t_2) \), using (2.2),

\[
t \frac{d}{dt} \| \hat{\varphi}(t) \|^2 = 2 \text{Re} < t \hat{\varphi}'(t), k(\hat{\varphi}(t)) > \leq \| \hat{\varphi}(t) \|^2,
\]
Majorana’s uniqueness theorem

\[ t \frac{d}{dt} \| \hat{\varphi}(t) \|^2 - \| \hat{\varphi}(t) \|^2 \leq 0, \]

\[ \frac{d}{dt} t \| \hat{\varphi}(t) \|^2 \leq 0, \]

for \( t > t_1 \). Moreover,

\[ \lim_{t \to t_1} \| \hat{\varphi}(t) \|^2 = 0, \]

because of continuity of \( \hat{\varphi} \), and \( \hat{\varphi}(t_1) = \hat{0} \). Thus, the continuous non-negative function \( \| \hat{\varphi}(t) \|^2 \) is non-increasing for \( t > t_1 \), while its limit, as \( t \) approaches \( t_1 \) from the right, is zero. Consequently, one must have \( \| \hat{\varphi}(t) \|^2 = 0 \) for \( t > t_1 \), which contradicts the definition \( t_1 = \sup \{ t \in [0, t_2] : A_k(t) = 0 \} \), and the proof will thus be accomplished.

An immediate consequence of Theorem 1.2 is the following uniqueness criterion.

**Theorem 2.2.** Let the abstract function \( \hat{f}(t, \hat{x}) \) be continuous and satisfy \( \hat{f}(t, \hat{0}) = \hat{0} \) for every \( t \in [0, 1] \). Assume further that there exist \( \varepsilon > 0 \), duality mapping \( k : B \to 2^{B^*} \) and \( t_0 \in (0, 1] \) such that \( \hat{u} = \hat{0} \) is the only root of the auxiliary scalar equation (2.1) with \( \| \hat{u} \| < \varepsilon \) for every \( t \in [0, t_0] \). Then the (0.1) admits in the interval \( [0, t_0] \) only the zero solution.

As it was pointed out by Majorana the crucial point in Theorems 2.1, 2.2 is the assumption that the (0.1) has the zero solution. We follow Majorana’s procedure to remove this restriction. If we know a solution \( \hat{\varphi} \) of (0.1), then, by means of change of variables \( \hat{x} = \hat{p} + \hat{\varphi}(t) \) (0.1) becomes

\[ \begin{cases}
\hat{p}' = \hat{F}(t, \hat{p}), \\
\hat{p}(0) = \hat{0},
\end{cases} \tag{2.3} \]

where \( \hat{F}(t, \hat{p}) = \hat{f}(t, \hat{p} + \hat{\varphi}(t)) - \hat{f}(t, \hat{\varphi}(t)) \). It is clear that any two different solutions of (0.1) are mapped to different solutions of (2.3). Moreover (2.3) admit the zero solution \( \hat{0} \), which corresponds to the solution \( \hat{\varphi} \) of (0.1). We thus get the counterpart to (2.1).

\[ Re < \hat{F}(t, \hat{u}), k(\hat{u}) > = \frac{1}{2t} \| \hat{u} \|^2. \tag{2.4} \]

We can restate Theorems 2.1, 2.2 involving the nontrivial solution \( \hat{\varphi} \) instead of \( \hat{x} = \hat{0} \).

**Theorem 2.3.** Let the abstract function \( \hat{f}(t, \hat{x}) \) be continuous on \([0, 1] \times B \)
and let \( \hat{\varphi} \) be a solution of (0.1). Further let (0.1) admit two different solutions defined in \([0, a]\) \((a < 1)\). Then, for every \( \varepsilon > 0 \), and for every duality mapping \( k : B \rightarrow 2^{B^*} \), there exists \( t \in [0, a] \) such that (2.4) has at least two different roots \( \hat{u} \) with \( \| \hat{u} \| < \varepsilon \).

**Theorem 2.4.** Let the abstract function \( \tilde{f}(t, \tilde{x}) \) be continuous and let \( \hat{\varphi} \) be a solution of (0.1). Assume further that there exist \( \varepsilon > 0 \), duality mapping \( k : B \rightarrow 2^{B^*} \) and \( t_0 \in (0, 1] \) such that \( \hat{u} = \hat{0} \) is the only root of the auxiliary scalar equation (2.4) with \( \| \hat{u} \| < \varepsilon \) for every \( t \in [0, t_0] \). Then the (0.1) admits in the interval \([0, t_0]\) only the solution \( \hat{\varphi} \).

**References**


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