Some Topological Indices of Spider’s Web Planar Graph

Mohamed Essalih

LRIT associated unit to CNRST (URAC 29), Faculty of Sciences, Mohammed V-Agdal University, BP 1014, Rabat, Morocco
essalih.mohamed@yahoo.fr

Mohamed El Marraki

LRIT associated unit to CNRST (URAC 29), Faculty of Sciences, Mohammed V-Agdal University, BP 1014, Rabat, Morocco
marraki@fsr.ac.ma

Gabr Al hagri

Department of Computer Sciences, Faculty of Sciences
Mohammed V-Agdal University, BP 1014, Rabat, Morocco
chima_gapr@yahoo.fr

Abstract

The theory of graphs, with its diverse applications in natural (Chemistry, Biology) and social sciences in general and in theoretical computer science in particular, is becoming an important component of the mathematics curriculum in colleges and universities all over the world. This paper presents some topological indices, like the index Wiener $W(G)$, the degree distance index $DD(G)$ and the hyper-Wiener index $WW(G)$ of a graph $G$; for the Spider’s Web planar graph $R_n$.

Mathematics Subject Classification: 05C12, 05C05

Keywords: Cobweb planar graph, Grid planar graph, Graph, Hyper-Wiener index, Wiener index, degree distance index, first Zagreb index
1 Introduction

Many structures involving real-world situations can be conveniently represented on paper by means of a diagram consisting of a set of points usually dots together with lines joining some or all pairs of these points. For example, the points in a diagram could represent different cities in a country of communication, and a line joining two points that does not pass through a third point may indicate that there is direct air service between the two cities represented by those two points. A mathematical abstraction of such structures involving points and lines leads us to the concept of graph.

A graph $G$ is a triplet consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. We denote $|V(G)| = n$ is the vertex number of $G$ and $|E(G)| = m$ is the edges number of $G$ (see Figure 1). A graph which contains neither multiple edges nor loops is called a simple graph. A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A graph $G$ is connected if each pair of the vertices in $G$ belongs to a path. A graph which represents in the plan is called a planar graph.

![Figure 1: The butterfly planar graph](image)

The degree of vertex $v$ in a graph $G$, which is written $\text{deg}(v)$, is the number of edges incident to $v$, except that each loop (the edge $uv$ with $u = v$) at $v$ counts twice, and we call distance between two distinct vertices of graph $G$, $u$ and $v$, the smallest length of path between $u$ and $v$ in $G$. The diameter of $G$, denoted by $D(G)$, is defined as the maximum distance between any two vertices of $G$, that is, $D(G) = \max\{d(u, v) : \forall(u, v) \in V(G)^2\}$ [8][9]. In the following, we consider only the simple planar connected graphs. Let $d_G(k)$ be the number of pairs of vertices of $G$ that are at distance $k$, $\lambda$ a real number, and $W_\lambda(G) = \sum_{k \geq 1} d_G(k)k^\lambda$. $W_\lambda(G)$ is called the Wiener-type invariant of $G$ associated to the real number $\lambda$. Note that $d_G(0)$ and $d_G(1)$ represent the number of vertices and edges, respectively [3]. The Wiener index of the graph $G$ equals to the sum of distances between all pairs of vertices of the respective molecular graph, i.e, $W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v)$ (The case of $\lambda = 1$ of the
$W_{\lambda}(G)$, and we define the index Wiener of a vertex $u$ in the graph $G$ as $w(u, G) = \sum_{v \in V(G)} d(u, v)$ [5][4]. The hyper-Wiener index $WW$ is defined as $WW(G) = \frac{1}{2} (W_1(G) + W_2(G))$ [3]. The degree distance index, is defined as $DD(G) = \sum_{\{u,v\} \subseteq V(G)} (\text{deg}(u) + \text{deg}(v)) \ d(u, v)$, see [1] for details. It has been demonstrated that $DD(G)$ and $W(G)$ are closely mutually related for certain classes of molecular graphs [6]. The Zagreb indices are defined as $M_1(G) = \sum_{u \in V(G)} \text{deg}(u)^2$ [6][3][7].

2 The main Result

We give in this section some theoretic results about the Wiener index $W(G)$, the Hyper-Wiener index $WW(G)$ and the degree distance index $DD(G)$, according to $d_G(k)$ ( The number of pairs of vertices of $G$ that are at distance $k$ ), and the diameter of $G$.

**Theorem 2.1** ([2]) Let $G$ be a connected finite undirected graph without loops or multiple edges, with $n$ vertices, $m$ edges, and with $D(G) \geq 2$, we have:

$$W(G) = n(n - 1) - m + d_G(3) + 2d_G(4) + \ldots + (D - 2)d_G(D) \quad (1)$$

**Corollary 2.2** ([2]) Let $G$ be a Graph with $n$ vertices, $m$ edges and with $D(G) = 2$, then :

$$W(G) = n(n - 1) - m. \quad (2)$$

**Theorem 2.3** Let $G$ be a connected finite undirected graph without loops or multiple edges, with $n$ vertices, $m$ edges, and with $D(G) \geq 2$, we have, then:

$$DD(G) = \sum_{u \in V(G)} w(u, G)\text{deg}(u). \quad (3)$$

**Proofs:**

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} (\text{deg}(u) + \text{deg}(v)) d(u, v)$$

$$= \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} (\text{deg}(u) + \text{deg}(v)) d(u, v)$$

we have:

$$\sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)\text{deg}(u) = \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)\text{deg}(v)$$
then:

$$DD(G) = \sum_{u \in V(G)} \sum_{v \in V(G)} d(u,v) \text{deg}(u)$$

$$= \sum_{u \in V(G)} w(u,G) \text{deg}(u). \Box$$

**Corollary 2.4** Let $G$ be a graph with $n$ vertices, $m$ edges and with $D(G) = 2$, then:

$$DD(G) = 4(n-1)m - M_1(G). \quad (4)$$

**Proofs:** we use theorem 2.3 we have:

$$DD(G) = \sum_{u \in V(G)} w(u,G) \text{deg}(u).$$

we have $D(G) = 2$ (see [2] for details) then:

$$w(u,G) = \sum_{v \in V(G)} d(u,v) + \sum_{v \in V(G)} d(u,v)$$

$$n = \sum_{v \in V(G)} d(u,v) + n_2 + 1$$

with $n_2$ is the number of vertex $v$ with $d(u,v) = 2$ then:

$$w(u,G) = \text{deg}(u) + 2((n-1) - \text{deg}(u))$$

$$DD(G) = \sum_{u \in V(G)} (\text{deg}(u) + 2((n-1) - \text{deg}(u)) \text{deg}(u)$$

$$= 4(n-1)m - M_1 \Box$$

**Theorem 2.5** ([2]) Let $G$ be a connected finite undirected graph without loops or multiple edges, with $n$ vertices, $m$ edges, and with $D(G) \geq 2$, we have:

$$WW(G) = \frac{1}{2} (3n(n-1) - 4m + (3^2 - 3)d_G(3) + ... + (D^2 + D - 6)d_G(D)). \quad (5)$$

**Corollary 2.6** ([2]) Let $G$ be a graph with $n$ vertices, $m$ edges and with $D(G) = 2$, then:

$$WW(G) = \frac{3}{2} n(n-1) - 2m. \quad (6)$$
3 Application

We are going to apply in this section the corollaries and theorems of the precedent section for some graphs with diameter equals two, with diameter equals four, some particular graphs with diameter greater than four, and we shall apply also the Theorem 2.3 about the Cobweb planar graph.

3.1 The graphs of diameter two

Lemma 3.1 \( W_n \) is a Wheel planar graph with the number of vertices \( n \) and the number of edges \( m = 2n - 2 \) we have:

\[
M_1(W_n) = n^2 + 7n - 8, \quad \text{for} \quad (n \geq 3) \tag{7}
\]

Proof: By calculation.

![Figure 2: The Wheel planar graph \( W_n \)](image)

Theorem 3.2 \( W_n \) is a Wheel planar graph with the number of vertices \( n \) and the number of edges \( m \) we have:

\[
W(W_n) = n^2 - 3n + 2, \quad \text{for} \quad (n \geq 5) \tag{8}
\]
\[
DD(W_n) = 7n^2 - 23n + 16, \quad \text{for} \quad (n \geq 4) \tag{9}
\]
\[
WW(W_n) = \frac{3}{2}n^2 - \frac{11}{2}n, \quad \text{for} \quad (n \geq 3) \tag{10}
\]

Proof: We have just applied the corollaries 2.2, 2.4, 2.6, and using the Lemma 3.1.

3.2 The graphs of diameter four

The sunflower planar graph \( S_n \) is a graph which always has an odd number of vertices \( n \) and a number of edges \( m = 2(n - 1) \). The central vertex \( v_0 \) has a degree \( deg(v_0) = \frac{n-1}{2} \), the odd index vertices \( v_1, v_3, ... v_{n-2} \) have a degree 5 and the even index vertices \( v_2, v_4, ... v_{n-1} \) have a degree 2.
Lemma 3.3 \( S_n \) is a Sunflower planar graph \( n \geq 11 \), with the number of vertices \( n \) and the number of edges \( m \).

- For \( i = 1, 2, \ldots, n - 1 \), we have:

\[
w(v_i, S_n) = \begin{cases} 
\frac{7}{2}n - \frac{35}{2}, & \text{if } i \text{ is even and } i \neq 0 \\
\frac{3}{2}n - \frac{9}{2}, & \text{if } i = 0 \\
\frac{5}{2}n - \frac{23}{2}, & \text{if } i \text{ is odd and } i \neq 0 
\end{cases}
\]  \hspace{1cm} (11)

- For \( k = 1, 2, 3, 4 = D(S_n) \), we have:

\[
d_{S_n}(k) = \begin{cases} 
2n - 2, & \text{if } k = 1 \\
\frac{1}{8}n^2 + n - \frac{9}{8}, & \text{if } k = 2 \\
\frac{1}{8}n^2 - 2n + \frac{7}{8}, & \text{if } k = 3 \\
\frac{1}{8}n^2 - \frac{3}{2}n + \frac{11}{8}, & \text{if } k = 4 
\end{cases}
\]  \hspace{1cm} (12)

**Proof:** By calculation.

Theorem 3.4 \( S_n \) is a Sunflower planar graph, with the number of vertices \( n \) and the number of edges \( m \) we have:

\[
W(S_n) = \frac{3}{2}n^2 - 8n + \frac{13}{2}, \quad \text{for } (n \geq 11)
\]  \hspace{1cm} (13)

\[
DD(S_n) = \frac{21}{2}n^2 - \frac{115}{2}n + 47, \quad \text{for } (n \geq 11)
\]  \hspace{1cm} (14)

\[
WW(S_n) = \frac{25}{8}n^2 - \frac{176}{8}n + \frac{151}{8}, \quad \text{for } (n \geq 11)
\]  \hspace{1cm} (15)

**Proof:** We have just applied the Theorems 2.1, 2.3, 2.5, and using the Lemma 3.3.
3.3 The graphs with diameter greater than four

In this section, we shall be studying the graphs $G$ of $D(G) \geq 4$, as, Grid planar graph $G_n$, the cycle planar graph $C_n$.

- The Cycle planar graph

**Lemma 3.5** $C_n$ is a Cycle planar graph $n \geq 2$, with number of vertices $n$ and number of edges $m$ ($m = n$).

- For $i = 1, 2, ..., n$, we have:

$$w(v_i, C_n) = \begin{cases} \frac{1}{4}n^2, & \text{if } n \text{ even} \\ \frac{1}{4}n^2 - \frac{1}{4}, & \text{if } n \text{ odd} \end{cases}$$

- For $k = 1, 2, ..., \frac{n}{2} = D(C_n)$, we have:

$$d_{C_n}(k) = \begin{cases} n, & \text{if } n \text{ is even and } 1 \leq k < \frac{n}{2} \\ \frac{n}{2}, & \text{if } n \text{ is even and } k = \frac{n}{2} \\ n, & \text{if } n \text{ is odd and } 1 \leq k \leq \frac{n-1}{2} \end{cases}$$

**Proof:** By calculation.

![Figure 4: The Cycle planar graph $C_n$](image)

**Theorem 3.6** $C_n$ is a Cycle planar graph $n \geq 2$, with the number of vertices $n$ and the number of edges $m$.

$$W(C_n) = \begin{cases} \frac{1}{8}n^3, & \text{if } n \text{ is even} \\ \frac{7}{8}n^3 - \frac{1}{8}n, & \text{if } n \text{ is odd} \end{cases}$$

$$DD(C_n) = \begin{cases} \frac{1}{2}n^3, & \text{if } n \text{ is even} \\ \frac{1}{2}n^3 - \frac{1}{2}n, & \text{if } n \text{ is odd} \end{cases}$$

$$WW(C_n) = \begin{cases} \frac{n^2(n+2)(n+1)}{48}, & \text{if } n \text{ is even} \\ \frac{n(n+3)(n^2-1)}{48}, & \text{if } n \text{ is odd} \end{cases}$$
Proof: We have just applied the Theorems 2.1, 2.3, 2.5, and using the Lemma 3.5.

- The Grid planar graph

**Lemma 3.7** \( G_n \) is a grid planar graph \( n \geq 2 \), with the number of vertices \( n \) and the number of edges \( m \) (\( m = \frac{3}{2} n - 2 \)).

- For \( i = 1, 2, \ldots, \frac{n}{2} \), we have:

\[
w(u_{2i-1}, G_n) = w(u_{2i}, G_n) = \frac{(n - 2i)^2}{4} + \frac{(2i - 2)^2}{4} + n - 1. \tag{21}\]

- For \( k = 1, 2, \ldots, \frac{n}{2} = D(G_n) \), we have:

\[
d_{G_n}(k) = \begin{cases} 
n, & \text{if } k = 0 \\
m, & \text{if } k = 1 \\
2n - 4k + 2, & \text{if } k = 2, \ldots, D(G_n) \end{cases} \tag{22}\]

Proof: by calculation.

![Figure 5: The Grid planar graph \( G_n \).](image)

**Theorem 3.8** \( G_n \) is a grid planar graph (see Figure 5) with number of vertices \( n \) and number of edges \( m \) we have:

\[
W(G_n) = \frac{n^3}{12} + \frac{n^2}{4} - \frac{n}{3} \quad \text{for } (n \geq 3) \tag{23}\]

\[
DD(G_n) = \frac{n^3}{2} + \frac{n^2}{2} - 2n \quad \text{for } (n \geq 2) \tag{24}\]

\[
WW(G_n) = \frac{n^4}{96} + \frac{n^3}{12} + \frac{5n^2}{24} + \frac{17}{3}n - 24 \quad \text{for } (n \geq 4) \tag{25}\]

Proof: We have just applied the Theorems 2.1, 2.3, 2.5, and using the Lemma 3.7.
• The Cobweb planar graph

The spider’s web or Cobweb planar graph, is a graph composed with \( L \) levels; in level \( l \), we have a cycle graph of \( p \) vertices. The number of edges \( m = Lp + (L - 1)p \), and number of vertices \( n = Lp \), (see Figure 6).

**Lemma 3.9** \( \mathcal{R}_n \) is a Cobweb planar graph with number of vertices \( n \) and number of edges \( m \). The wiener index of vertex \( i \), in level \( l \) \( u_{i,l} \) is:

- For \( i = 1, \ldots, p \), and \( l = 1, \ldots, L \)

\[
 w(u_{i,l}, \mathcal{R}_n) = \begin{cases} 
 pl(l - 1) + \frac{L}{4}((p + 1)^2 + 2Lp - 4lp - 2), & \text{if } p \text{ is odd} \\
 pl(l - 1) + \frac{L}{4}((p + 1)^2 + 2Lp - 4lp - 1), & \text{if } p \text{ is even}
\end{cases}
\]

(26)

**Proof:** by calculation.

![Figure 6: The Cobweb planar graph \( \mathcal{R}_n \)](image)

**Corollary 3.10** \( \mathcal{R}_n \) is a Cobweb planar graph (see Figure 6) of number of vertices \( n \), number of edges \( m \) and \( L \) levels we have:

\[
 W(\mathcal{R}_n) = \begin{cases} 
 \frac{3p^3L^2 + 4p^2L^3 - 4p^3L - 3pL^2}{24}, & \text{if } p \text{ is odd} \\
 \frac{3p^3L^2 + 4p^2L^3 - 4p^3L}{24}, & \text{if } p \text{ is even}
\end{cases}
\]

(27)
Proof: we've just proved the case where \( p \) is odd, the other case is similar.

\[
W(\mathcal{R}_n) = \frac{1}{2} \sum_{u \in V(\mathcal{R}_n)} w(u, \mathcal{R}_n); \text{ We use the precedent lemma we have;}
\]

\[
W(\mathcal{R}_n) = P \sum_{l=1}^{L} (pl(l-1) + \frac{L(2lp - 4l^2 + (p+1)^2 - 2)}{4})
\]

\[
= \frac{1}{24}(6L(2pL^2 + Lp^2 + 2Lp - L) + 4pL + 4pL(3L + 2L^2) - 12L(pL^2 + 2pL + p))
\]

\[
= \frac{1}{12}(3p^3L^2 - 3pL^2 + 4p^2L^3 - 4p^2L)\square
\]

**Theorem 3.11** \( \mathcal{R}_n \) is a Cobweb planar graph (see Figure 6) of number of vertices \( n \) and number of edges \( m \) we have:

\[
DD(\mathcal{R}_n) = 8W(\mathcal{R}_n) - pw(u_{1,1}, \mathcal{R}_n) - pw(u_{1,L}, \mathcal{R}_n)\quad (28)
\]

with \( C_i \) is the cycle planar graph in the level \( l_i \), and \( u \) is a vertex on cycle.

Proof: \( DD(\mathcal{R}_n) = \sum_{u \in V(\mathcal{R}_n)} w(u, \mathcal{R}_n) \text{ deg}(u) \)

\[
= w(u_{1,1}, \mathcal{R}_n) \text{ deg}(u_{1,1}) + w(u_{1,2}, \mathcal{R}_n) \text{ deg}(u_{1,2}) + ... + w(u_{i,L-1}, \mathcal{R}_n) \text{ deg}(u_{i,L-1}) + w(u_{i,L}, \mathcal{R}_n) \text{ deg}(u_{i,L})
\]

\[
= 3( w(u_{1,1}, \mathcal{R}_n) + w(u_{1,L}, \mathcal{R}_n) ) + 4( \sum_{i=2}^{L-1} w(u_{i,L}, \mathcal{R}_n) )
\]

\[
= 3( pw(u_{1,1}, \mathcal{R}_n) + pw(u_{1,L}, \mathcal{R}_n) ) + 4( p \sum_{i=2}^{L-1} w(u_{1,L}, \mathcal{R}_n) )
\]

\[
= 4p( \sum_{i=1}^{L} w(u_{1,L}, \mathcal{R}_n) ) - pw(u_{1,1}, \mathcal{R}_n) - pw(u_{1,L}, \mathcal{R}_n)\square
\]

4 conclusion

We have mentioned here some theoretical results about the Wiener index \( W(G) \), degree distance index \( DD(G) \) and The hyper-Wiener index.
Topological indices of Spider’s web planar graph

WW(G) of a simple planar connected graphs, relating to the $d_G(k)$, and the diameter of $G$. We have finished our work by giving some graphs with like diameter, Wheel planar graph $W_n$, Grid planar graph $G_n$, Cycle planar graph $C_n$, and the Sunflower planar graph $S_n$. we have also given the degree distance index of the Cobweb planar graph $R_n$, using proposed theorem.

References


